Optimization without Constraint : The Case of More than One Choice Variable IV

Pawin Siriprapanukul

CW Ch.11

Outline

- Economic Applications (Continued)
- Comparative-Static Aspects of Optimization

Economic Applications: Example 3: Price Discrimination

- A monopolist firm sells a single product in two or more separate markets. Assume three separate choice variables, Q_1, Q_2 and Q_3.
- Total revenue and total cost functions are as follows:

$$R = R_1(Q_1) + R_2(Q_2) + R_3(Q_3)$$
$$C = C(Q), \text{where } Q = Q_1 + Q_2 + Q_3.$$

- The profit function is:

$$\pi = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q).$$

- **FOC**:

$$\pi_1 = R_1'(Q_1) - C'(Q)\frac{\partial Q}{\partial Q_1} = R_1'(Q_1) - C'(Q) = 0$$
$$\pi_2 = R_2'(Q_2) - C'(Q)\frac{\partial Q}{\partial Q_2} = R_2'(Q_2) - C'(Q) = 0$$
$$\pi_3 = R_3'(Q_3) - C'(Q)\frac{\partial Q}{\partial Q_3} = R_3'(Q_3) - C'(Q) = 0$$

Economic Applications: Example 3: Price Discrimination

- We can conclude that at the critical point:

$$MR_1 = MR_2 = MR_3 = MC.$$

- For the monopolist, the total revenue in market i can be written as $R_i = P_i Q_i$, where P_i is a function of Q_i as well. Therefore:

$$MR_i = \frac{dR_i}{dQ_i} = P_i + Q_i \frac{dP_i}{dQ_i} = P_i \left(1 + \frac{Q_i}{P_i} \frac{dP_i}{dQ_i}\right) = P_i \left(1 + \frac{1}{\varepsilon_{di}}\right) = P_i \left(1 - \frac{1}{|\varepsilon_{di}|}\right),$$

where ε_{di} is the point elasticity of demand in market i.

- The optimal condition above can be rewritten as

$$P_1 \left(1 - \frac{1}{|\varepsilon_{d1}|}\right) = P_2 \left(1 - \frac{1}{|\varepsilon_{d2}|}\right) = P_3 \left(1 - \frac{1}{|\varepsilon_{d3}|}\right).$$

This means the *smaller* the value of $|\varepsilon_{di}|$, the *higher* the price charged in that market.

Economic Applications: Example 3: Price Discrimination

- **SOC**:

$$\pi_{11} = R_1''(Q_1) - C''(Q)\frac{\partial Q}{\partial Q_1} = R_1''(Q_1) - C''(Q)$$

$$\pi_{22} = R_2''(Q_2) - C''(Q)\frac{\partial Q}{\partial Q_2} = R_2''(Q_2) - C''(Q)$$

$$\pi_{33} = R_3''(Q_3) - C''(Q)\frac{\partial Q}{\partial Q_3} = R_3''(Q_3) - C''(Q)$$

$$\pi_{12} = \pi_{21} = \pi_{13} = \pi_{31} = \pi_{23} = \pi_{32} = -C''(Q).$$

We have

$$|H| = \begin{vmatrix} R_1'' - C'' & -C'' & -C'' \\ -C'' & R_2'' - C'' & -C'' \\ -C'' & -C'' & R_3'' - C'' \end{vmatrix}$$

Economic Applications:
Example 3: Price Discrimination

For profit-maximization, it must be the case that

1) $|H_1| = R_1'' - C'' < 0$. That is the slope of MR_1 is less than the slope of MC
2) $|H_2| = (R_1'' - C'')(R_2'' - C'') - (C'')^2 > 0$. (This implies $R_2'' - C'' < 0$ as well.)
3) $|H_3| = (R_1'' - C'')(R_2'' - C'')(R_3'' - C'') - (C'')^3 - (R_1'' - C'')(C'')^2 - (R_2'' - C'')(C'')^2 - (R_3'' - C'')(C'')^2 < 0.$

Economic Applications: Example 4: Price Discrimination

- Now, suppose the monopolist has the specific average-revenue functions:

$$P_1 = 63 - 4Q_1$$

$$P_2 = 105 - 5Q_2$$

$$P_3 = 75 - 6Q_3,$$

and the cost function is:

$$C = 20 + 15Q, \text{where } Q = Q_1 + Q_2 + Q_3.$$

- The profit function is:

$$\pi = (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - 6Q_3)Q_3 - (20 + 15Q)$$

Economic Applications: Example 4: Price Discrimination

- **FOC:**

$$63 - 8Q_1 = 15$$

$$105 - 10Q_2 = 15$$

$$75 - 6Q_3 = 15,$$

and the optimal quantities are:

$Q_1^* = 6, Q_2^* = 9, Q_3^* = 5$, which means $Q^* = 20$

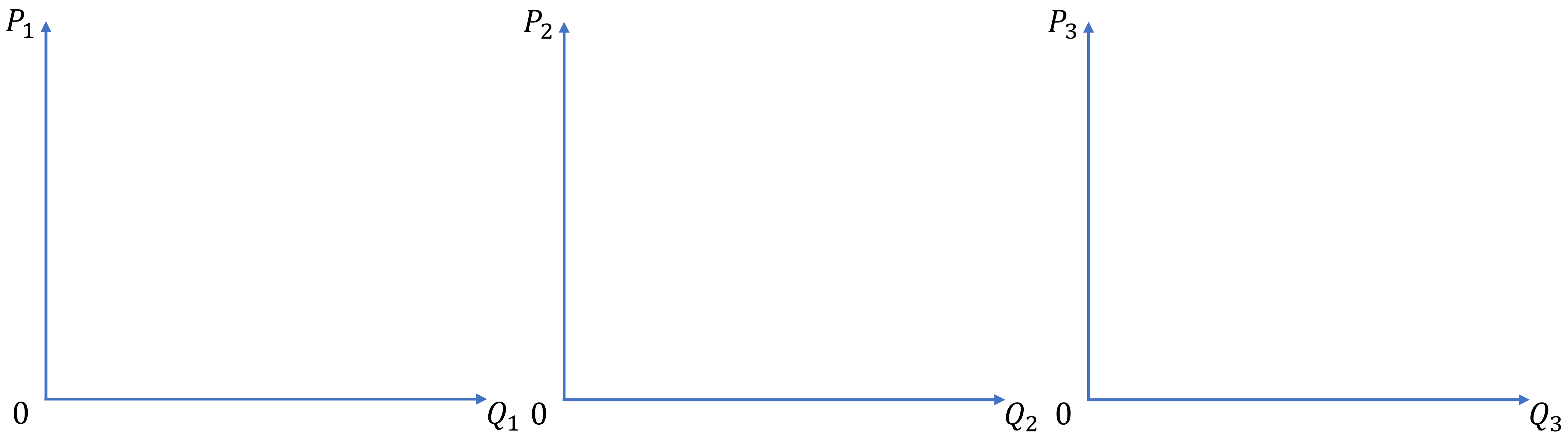
- **SOC:**

$$|H| = \begin{vmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -6 \end{vmatrix}$$

$|H_1| = -8 < 0, |H_2| = 80 > 0, |H_3| = -480 < 0$. Critical point is the maximum point.

Economic Applications:
Example 4: Price Discrimination

- What is the graph of this?



Example (Exercise 17.6 in S&B)

Dingbat Airlines has regular flights between Ypsilanti and Kalamazoo. It can treat business and pleasure travelers as separate markets by demanding advance purchase and Saturday night stay-over for pleasure travelers. Suppose that it notes a demand function of $Q_B = 16 - P_B$ for business travelers and a demand function of $Q_P = 10 - P_P$ for pleasure travelers and that it has a cost function for all travelers of $C(Q) = 10 + Q^2$, where $Q = Q_P + Q_B$. How much should it charge in each market to maximize its profit?

FOC & SOC:

Graph

Economic Applications: Example 5: Input Decisions of a Firm

- Consider a competitive firm with the following profit function:

$$\pi = R - C = PQ - wL - rK$$

where $P =$ price, $Q =$ output, $L =$ labor, $K =$ capital

$w, r =$ input prices for L and K, respectively.

- Now, assume that the production function $Q = Q(K, L)$ is a Cobb-Douglas function:

$$Q = L^{\alpha} K^{\beta},$$

where $0 < \alpha = \beta < \frac{1}{2}$.

- Substituting the production function into the profit function, we get:

$$\pi = P(L^{\alpha} K^{\alpha}) - wL - rK.$$

Economic Applications:
Example 5: Input Decisions of a Firm

- **FOC:**

$$\frac{\partial \pi}{\partial L} = P\alpha L^{\alpha-1}K^{\alpha} - w = 0$$

$$\frac{\partial \pi}{\partial K} = P\alpha L^{\alpha}K^{\alpha-1} - r = 0.$$

From the first equation, we have:

$$K = \left(\frac{w}{P\alpha}L^{1-\alpha}\right)^{1/\alpha}.$$

Substitute this into the second equation, we get:

$$L^* = \left(P\alpha w^{(\alpha-1)}r^{-\alpha}\right)^{1/(1-2\alpha)} \quad \text{and} \quad K^* = \left(P\alpha r^{(\alpha-1)}w^{-\alpha}\right)^{1/(1-2\alpha)}.$$

Substituting L^* and K^* into the production function, we get:

$$Q^* = \left(\frac{\alpha^2 P^2}{wr}\right)^{\alpha/(1-2\alpha)}.$$

Economic Applications: Example 5: Input Decisions of a Firm

- **SOC:**

$$|H| = \begin{vmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{vmatrix} = \begin{vmatrix} P\alpha(\alpha-1)L^{\alpha-2}K^{\alpha} & P\alpha^2 L^{\alpha-1}K^{\alpha-1} \\ P\alpha^2 L^{\alpha-1}K^{\alpha-1} & P\alpha(\alpha-1)L^{\alpha}K^{\alpha-2} \end{vmatrix}.$$

Then, we have

$$|H_1| = P\alpha(\alpha-1)L^{\alpha-2}K^{\alpha} < 0 \qquad \text{(Since } \alpha < 1/2\text{)}$$

$$|H_2| = P^2\alpha^2 L^{2\alpha-2}K^{2\alpha-2}(1-2\alpha) > 0 \qquad \text{(Since } \alpha < 1/2\text{).}$$

This indicates the case of maximization.

- Actually, the solution L^* and K^* are the firm's input demand functions.

Example (Exercise 17.4 in S&B)

- A firm uses two inputs to produce a single product. If its production function is $Q = x^{1/4}y^{1/4}$ and if it sells its output for \$1 a unit and buys each input for \$4 a unit, find its profit-maximizing input bundle. (Check the second-order conditions as well.)

Comparative-Static Aspects of Optimization: Reduced-Form Solutions

- In example 1, we consider the case of a competitive firm supplies two products to the markets. What will happen if price of the first product increases?
- Recall that the optimal output levels of the two-product firm are:

$$Q_1^* = \frac{4P_{10} - P_{20}}{15} \text{ and } Q_2^* = \frac{4P_{20} - P_{10}}{15}.$$

- These are reduced-form solutions and simple partial differentiation alone is sufficient to tell us all the comparative-static properties of the model. That is:

$$\frac{\partial Q_1^*}{\partial P_{10}} = \frac{4}{15} > 0 \text{ and } \frac{\partial Q_2^*}{\partial P_{10}} = -\frac{1}{15} < 0.$$

- Similarly, we can analyze the effect of changes in P_{20}, as follows:

$$\frac{\partial Q_1^*}{\partial P_{20}} = -\frac{1}{15} < 0 \text{ and } \frac{\partial Q_2^*}{\partial P_{10}} = \frac{4}{15} > 0.$$

- For maximum profit, each product should be produced in a larger quantity if its market price rises or if the market price of the other product falls.

Assignment

- Exercise 11.6 No.4

  If the cost function of Example 4 is changed to $C = 20 + 15Q + Q^2$

  (a) Find the new equilibrium quantities. (Use fractions.)

  (b) Find the new equilibrium prices.

  (c) Verify that the second-order sufficient condition is met.