Least Squares Estimation Methods

Ordinary Least Squares (OLS)

Generalized Least Squares (GLS)

Feasible Generalized Least Squares (FGLS)

- Weighted Least Squares (WLS)

  Heteroscedasticity

- Cochrane-Orcutt Technique

  Autocorrelation

Other Least Squares Methods

- Nonlinear Least Squares (NLS)
- System Equation Estimation Methods

OLS Matrix Approach

Scalar Notation

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i$$
$$where \quad i = 1, 2, 3, \ldots, n$$

Matrix Notation

$$\underset{n\times 1}{y} = \underset{n\times k}{X} \; \underset{k\times 1}{\beta} + \underset{n\times 1}{u}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2n} & x_{3n} & \cdots & x_{kn} \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

OLS Estimation

Estimated Regression Coefficients

3-variable scalar notation

$$\hat{\beta}_2 = \frac{\left(\sum y_i x_{2i}\right)\left(\sum x_{3i}^2\right) - \left(\sum y_i x_{3i}\right)\left(\sum x_{2i} x_{3i}\right)}{\left(\sum x_{2i}^2\right)\left(\sum x_{3i}^2\right) - \left(\sum x_{2i} x_{3i}\right)^2}$$

k-variable matrix notation

$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times k}{X'X})^{-1} \underset{k\times n}{X'} \underset{n\times 1}{y}$$

Assume normal distribution:

$$u \sim N(0, \sigma^2 I)$$

Variance-Covariance Matrix

Under OLS Assumptions

$$\sum_{n\times n} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \cdots & \sigma_{\mathrm{n}}^2 \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 I$$

Under Heteroscedasticity Problem

$$\sum_{n\times n} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

Weighted Least Squares (WLS)

Scalar $Y_i = \beta_1 + \beta_2 X_i + u_i$

$$\frac{Y_i}{\sigma_i} = \beta_1\left(\frac{1}{\sigma_i}\right) + \beta_2\left(\frac{X_i}{\sigma_i}\right) + \left(\frac{u_i}{\sigma_i}\right)$$

$$Y_i^* = \beta_1^* X_{0i}^* + \beta_2^* X_i^* + u_i^*$$

Matrix

$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times n}{X'} \underset{\substack{n\times n \\ k\times k}}{\Sigma^{-1}} \underset{n\times k}{X})^{-1} \underset{k\times n}{X'} \underset{\substack{n\times n \\ k\times 1}}{\Sigma^{-1}} \underset{n\times 1}{y}$$

Autocorrelation: Cochrane-Orcutt

Scalar $$Y_t = \beta_1 + \beta_2 X_t + u_t$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

1. Estimate model using OLS and obtain estimated u_t $\hat{u}_t = \hat{\rho}\hat{u}_{t-1} + \upsilon_t$
2. Compute
3. $\left(Y_t - \hat{\rho}Y_{t-1}\right) = \beta_1\left(1 - \hat{\rho}\right) + \beta_2\left(X_t - \hat{\rho}X_{t-1}\right) + \varepsilon_t$

$$Y_t^* = \beta_1^* + \beta_2^* X_t^* + \varepsilon_t^*$$

4. Iterative procedure to estimate ρ

Autocorrelation: Cochrane-Orcutt

Variance-Covariance Matrix $\underset{n\times n}{\Sigma} = \sigma^2 \underset{n\times n}{\Omega}$

where

$$\underset{\mathrm{n\times n}}{\Omega} = \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & \rho & 1 \end{bmatrix}$$

Matrix

$$\underset{k\times 1}{\hat{\beta}} = \underset{k\times k}{(\underset{k\times n}{X'}\,\underset{n\times n}{\hat{\Omega}^{-1}}\,\underset{n\times k}{X})^{-1}}\;\underset{k\times 1}{\underset{k\times n}{X'}\,\underset{n\times n}{\hat{\Omega}^{-1}}}\;\underset{n\times 1}{y}$$

OLS vs GLS vs FGLS

OLS Estimation

$$\underset{n\times n}{\Sigma} = \sigma^2 \underset{n\times n}{I}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times k}{X'X})^{-1} \underset{k\times n}{X'} \underset{n\times 1}{y}$$

WLS Estimation

$$\underset{n\times n}{\Sigma} = \hat{\sigma}_i^2 \underset{n\times n}{I}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times 1}{y}}_{k\times 1}$$

Cochrane-Orcutt

$$\underset{n\times n}{\Sigma} = \sigma^2 \underset{n\times n}{\hat{\Omega}}$$

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Omega}}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Omega}}^{-1} \underset{n\times 1}{y}}_{k\times 1}$$

FGLS Estimation

$\underset{n\times n}{\Sigma}$ *is not known*

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\hat{\Sigma}}^{-1} \underset{n\times 1}{y}}_{k\times 1}$$

GLS Estimation

$\underset{n\times n}{\Sigma}$ *is known*

$$\underset{k\times 1}{\hat{\beta}} = (\underbrace{\underset{k\times n}{X'} \underset{n\times n}{\Sigma}^{-1} \underset{n\times k}{X}}_{k\times k})^{-1} \underbrace{\underset{k\times n}{X'} \underset{n\times n}{\Sigma}^{-1} \underset{n\times 1}{y}}_{k\times 1}$$