# Matrices,
# Algebra of Matrices
# &
# Elementary Operations

## What is a Matrix ? (Not a movie trilogy staring Keanu Reaves)

A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called the entries (or elements) of the matrix.

For example,

$$\mathbf{A}=[a_{ij}]=\begin{array}{ccccl} \text{Column1} & \text{Column2} & & \text{Column n} & \\ a_{11} & a_{12} & \cdots & a_{1n} & \text{Row1} \\ a_{21} & a_{22} & \cdots & a_{2n} & \text{Row 2} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{m1} & . & \cdots & a_{mn} & \text{Row m} \end{array}$$

$\mathbf{A}$ is a matrix of dimension (size) $m$ x $n$ ($m$ rows and $n$ columns)

$a_{ij}$ is an entry or an element of the matrix

$a_{ii}$ is a main diagonal entry

$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}$ is a 2 x 3 matrix (Rectangular Matrix)

$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}$ is a 2 x 2 matrix (Square Matrix)

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3 x 3 matrix (Square Matrix)

$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is a diagonal matrix

$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is an identity matrix or a unit matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ is an upper triagular matrix

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ is a lower triagular matrix

$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ is a zero matrix

$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ is a 1 x 3 matrix (we usually call it a row vector)

$\begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$ is a 2 x 1 matrix (we usually call it a column vector)

## Vector

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector. We shall denote vectors by ***lowercase*** boldface letters **a, b,** $\cdots$ or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots, \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

## Basic operations of matrices

**Equality of Matrices**

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal,** written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different.** Thus, matrices of different sizes are always different.

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

## Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

$$\mathbf{A}+\mathbf{B}=\left[a_{jk}+b_{jk}\right]_{m\times n}$$

For example;

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{bmatrix}$$

## Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** $c$ (number $c$) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by $c$.

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

If $\mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}$, then $-\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$, $\frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$, $0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\mathbf{A}=\begin{bmatrix}4 & 0 & 5\\ -1 & 3 & 2\end{bmatrix} \quad \mathbf{B}=\begin{bmatrix}1 & 1 & 1\\ 3 & 5 & 7\end{bmatrix} \quad \mathbf{C}=\begin{bmatrix}2 & -3\\ 0 & 1\end{bmatrix}$$

$\mathbf{A}+\mathbf{B}=$

$\mathbf{A}+\mathbf{C}=$

$2\mathbf{B}=$

$\mathbf{A}-2\mathbf{B}$

---

Properties of Matrix Addition

Cummulative law (a) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

Associative law of addition (b) $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$ (written $\mathbf{A}+\mathbf{B}+\mathbf{C}$)

(c) $\mathbf{A}+\mathbf{0}=\mathbf{A}$

(d) $\mathbf{A}+(-\mathbf{A})=\mathbf{0}.$

Properties of Scalar Multiplication

Distributive laws (a) $c(\mathbf{A}+\mathbf{B})=c\mathbf{A}+c\mathbf{B}$

(b) $(c+k)\mathbf{A}=c\mathbf{A}+k\mathbf{A}$

(c) $c(k\mathbf{A})=(ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A}=\mathbf{A}.$

## Matrix multiplication

Three different ways with the same answer:

Method 1: Each entry of $AB$ is the product of a row and a column.

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix}\begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

$(AB)$ij=row i of $A$ times column j of $B$

This single entry is the inner product of the two vectors.

Example

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$$

**EXAMPLE** $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute $AB$, if it is defined.

*Solution:* Since $A$ is $2 \times 3$ and $B$ is $3 \times 2$, then $AB$ is defined and $AB$ is ______×______.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

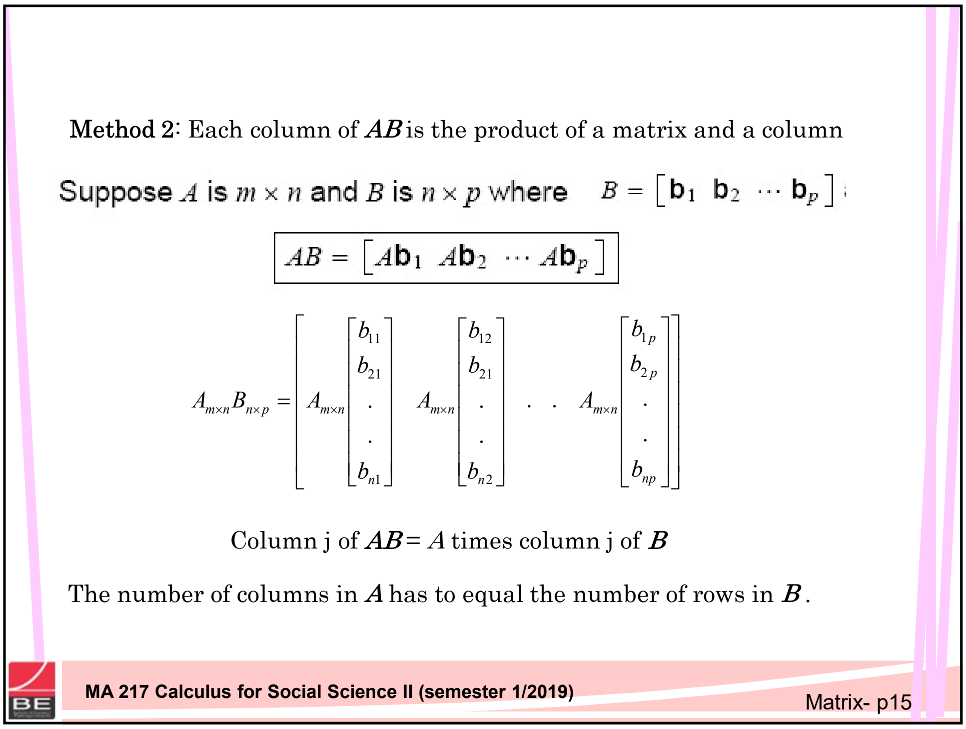
**EXAMPLE:** If $A$ is $4 \times 3$ and $B$ is $3 \times 2$, then what are the sizes of $AB$ and $BA$?

*Solution:*

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}$$

$BA$ would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$ Which is ________________

$\boxed{\text{If } A \text{ is } m \times n \text{ and } B \text{ is } n \times p, \text{ then } AB \text{ is } m \times p.}$



**Method 2**: Each column of $AB$ is the product of a matrix and a column

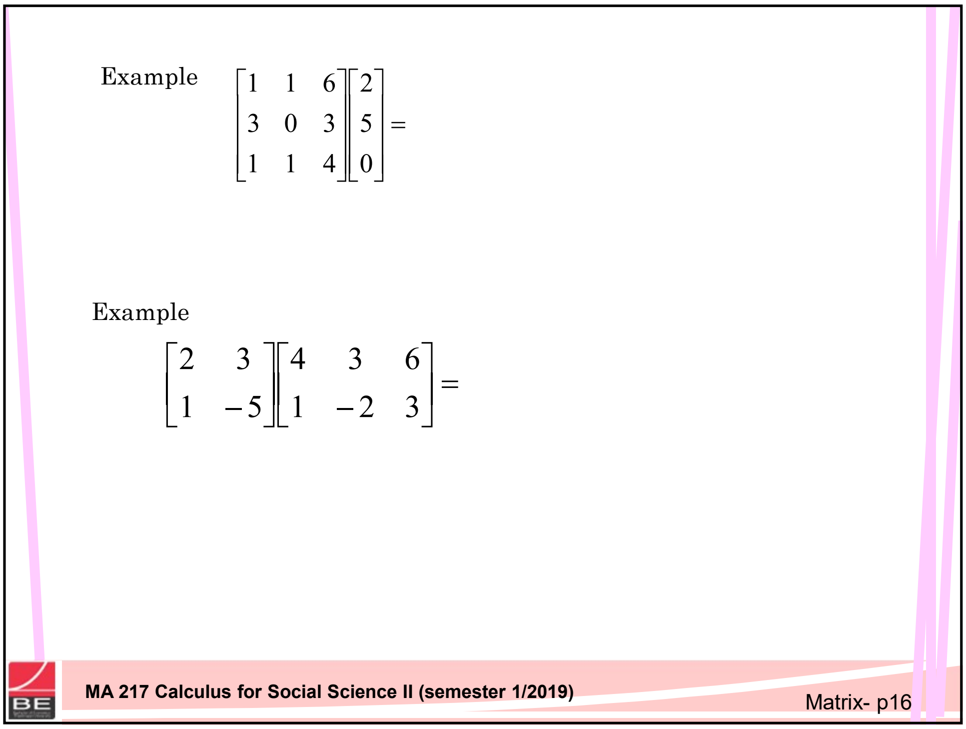
Suppose $A$ is $m \times n$ and $B$ is $n \times p$ where $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_p]$ :

$$\boxed{AB = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_p]}$$

$$A_{m\times n}B_{n\times p} = \begin{bmatrix} A_{m\times n}\begin{bmatrix} b_{11} \\ b_{21} \\ . \\ . \\ b_{n1} \end{bmatrix} & A_{m\times n}\begin{bmatrix} b_{12} \\ b_{21} \\ . \\ . \\ b_{n2} \end{bmatrix} & . \ . & A_{m\times n}\begin{bmatrix} b_{1p} \\ b_{2p} \\ . \\ . \\ b_{np} \end{bmatrix} \end{bmatrix}$$

Column j of $AB$ = $A$ times column j of $B$

The number of columns in $A$ has to equal the number of rows in $B$.



Example $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$

Example

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} =$$

**EXAMPLE:** Compute $AB$ where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

*Solution:*

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

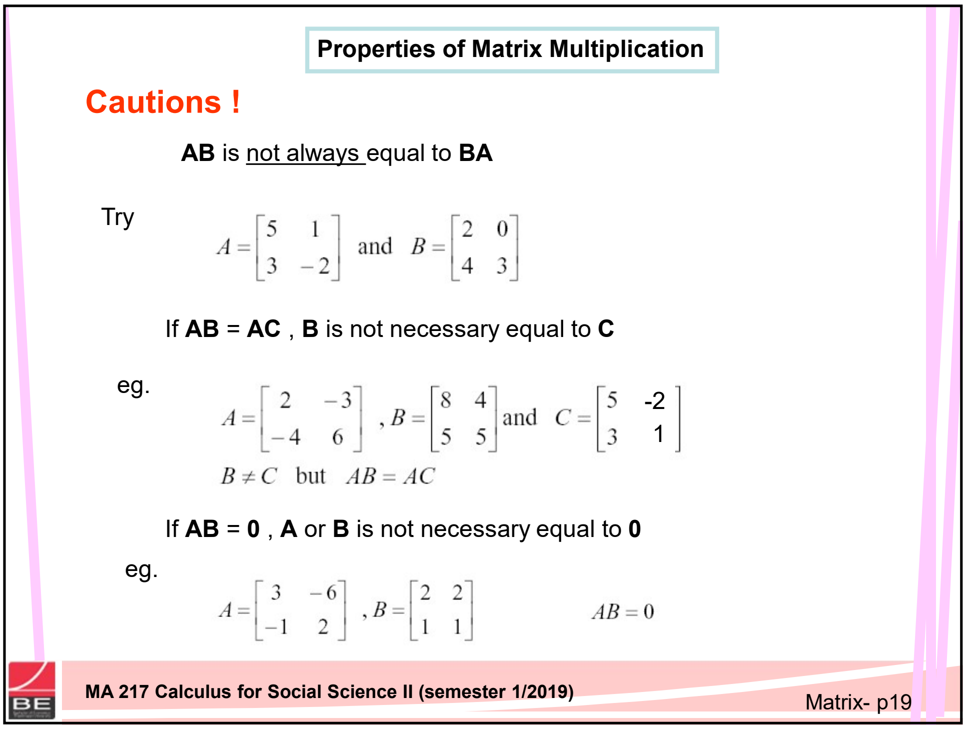
$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of $A$ and $A\mathbf{b}_2$ is a linear combination of the columns of $A$.

Method 3: Each row of $AB$ is the product of a row and a matrix

row i of $AB$ = row i of $A$ times $B$

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} =$$



## Properties of Matrix Multiplication

**Cautions !**

**AB** is <u>not always</u> equal to **BA**

Try

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

If **AB** = **AC** , **B** is not necessary equal to **C**

eg.

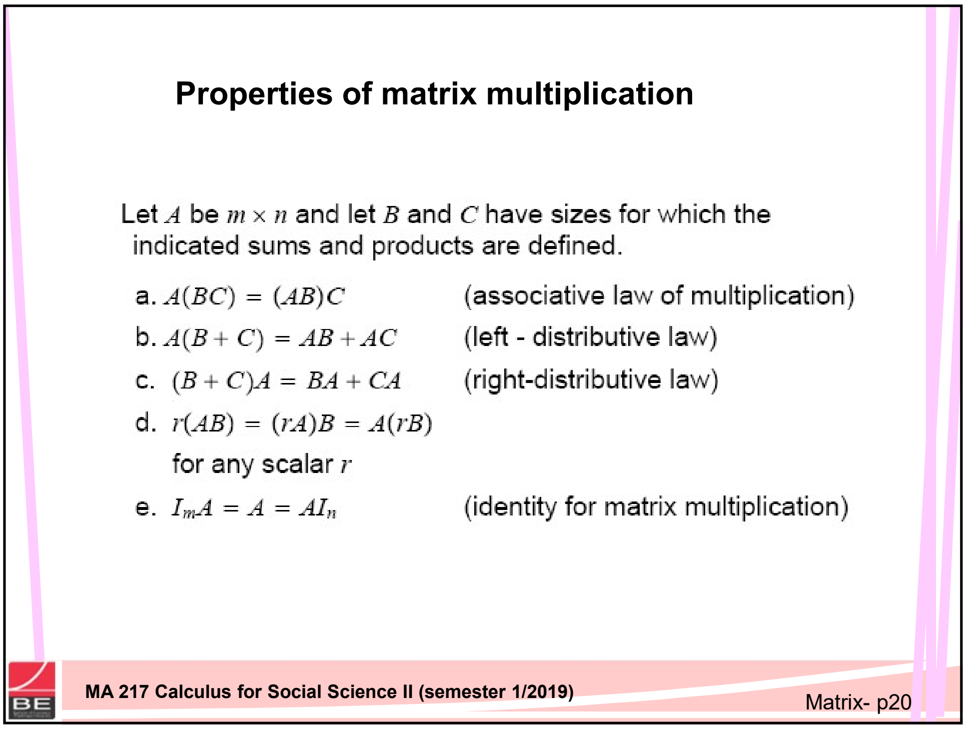
$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$$B \neq C \text{ but } AB = AC$$

If **AB** = **0** , **A** or **B** is not necessary equal to **0**

eg.

$$A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \qquad AB = 0$$



## Properties of matrix multiplication

Let $A$ be $m \times n$ and let $B$ and $C$ have sizes for which the indicated sums and products are defined.

| | |
|---|---|
| a. $A(BC) = (AB)C$ | (associative law of multiplication) |
| b. $A(B + C) = AB + AC$ | (left - distributive law) |
| c. $(B + C)A = BA + CA$ | (right-distributive law) |
| d. $r(AB) = (rA)B = A(rB)$<br>for any scalar $r$ | |
| e. $I_m A = A = A I_n$ | (identity for matrix multiplication) |

**Transposition of Matrices and Vectors**

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^{\mathsf{T}}$ (read *A transpose*) that has the first *row* of **A** as its first *column*, the second *row* of **A** as its second *column*, and so on.

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & . & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{A}^T = \left[a_{ji}\right] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

**Transposition of Matrices and Vectors**

If $\quad \mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \quad$ then $\quad \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}.$

**EXAMPLE:** Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute $AB$, $(AB)^T$, $A^TB^T$ and $B^TA^T$.

*Solution:*

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

$$\boxed{(AB)^T =}$$

## Properties of Matrix Transposition

Let $A$ and $B$ denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of $A^T$ is $A$)

b. $(A+B)^T = A^T + B^T$

c. For any scalar $r$, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order. )

**EXAMPLE:** Prove that $(ABC)^T =$ ________.

*Solution:* By Theorem 3d,

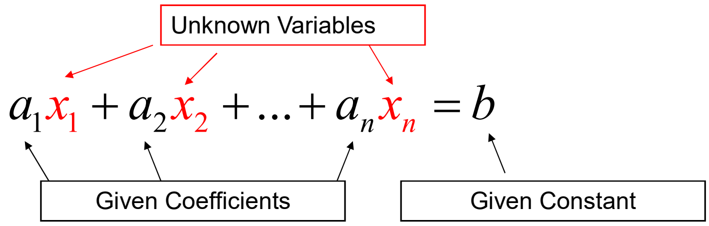
$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T$$

$$= C^T(\qquad) = \text{\_\_\_\_\_\_\_\_}.$$

# Solution of System of Linear Equations

## System of Linear Equations

A linear equation


Unknown Variables

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Given Coefficients

Given Constant

---

A linear equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad (*)$$

eg.

$$2x_1 - x_2 + 5x_3 = 2\sqrt{5}$$

Non-linear equation

Anything that is not in the form of a linear equation (*)

eg.

$$2x_1 - x^{2}{}_{2} + 5x_3 \sin(x_1) = 25$$

## System of Linear Equations

A collection of one or more linear equations involving the same set of variables.

A system of linear equations with 2 variables:

$$\begin{aligned} ax+by&=h \\ cx+dy&=k \end{aligned} \qquad \text{E.g.} \quad \begin{aligned} 2x+y&=8 \\ x+3y&=9 \end{aligned}$$

A system of linear equations with 3 variables:

$$\begin{aligned} 6x_1+x_2+x_3&=6 \\ 5x_1+x_2+2x_3&=4 \\ 4x_1+x_2-x_3&=-2 \end{aligned}$$

The whole idea of linear algebra is to solve

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}}=\underline{\mathbf{b}}$$

A system of linear equations can be written in matrix form

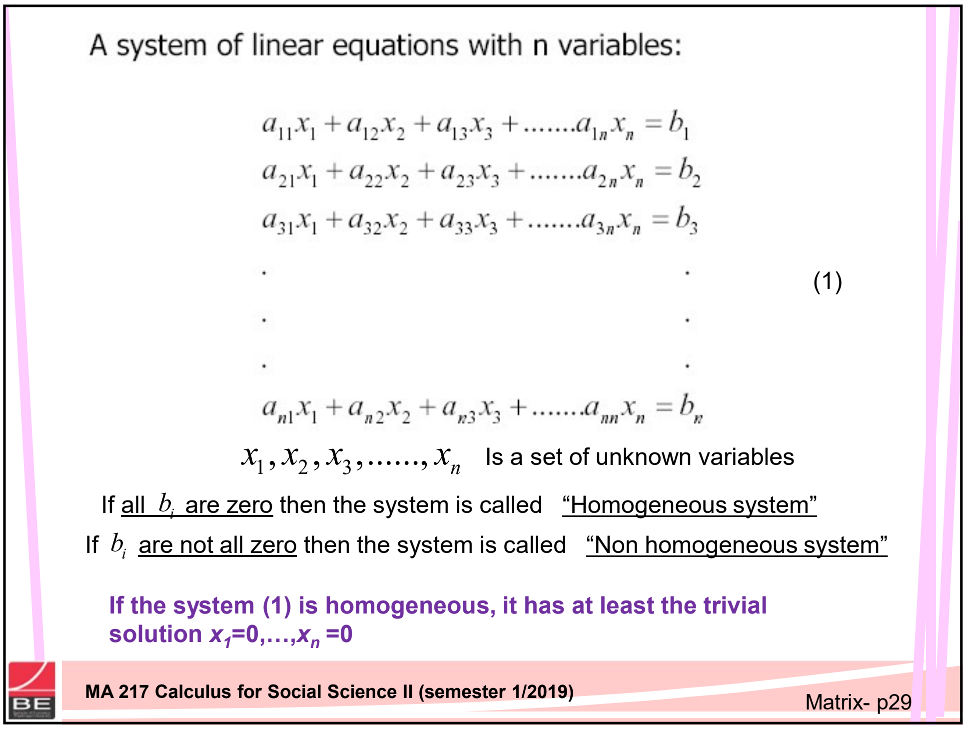
$$\begin{aligned} 6x_1+x_2+x_3&=6 \\ 5x_1+x_2+2x_3&=4 \\ 4x_1+x_2-x_3&=-2 \end{aligned}$$

$$\begin{bmatrix} 6 & 1 & 1 \\ 5 & 1 & 2 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 6 & 1 & 1 & 6 \\ 5 & 1 & 2 & 4 \\ 4 & 1 & -1 & -2 \end{array}\right]$$

An augmented matrix form

$$[\mathbf{A}|\mathbf{b}]$$

A matrix equation: $\underline{\mathbf{A}}\,\underline{\mathbf{x}}=\underline{\mathbf{b}}$



A system of linear equations with n variables:

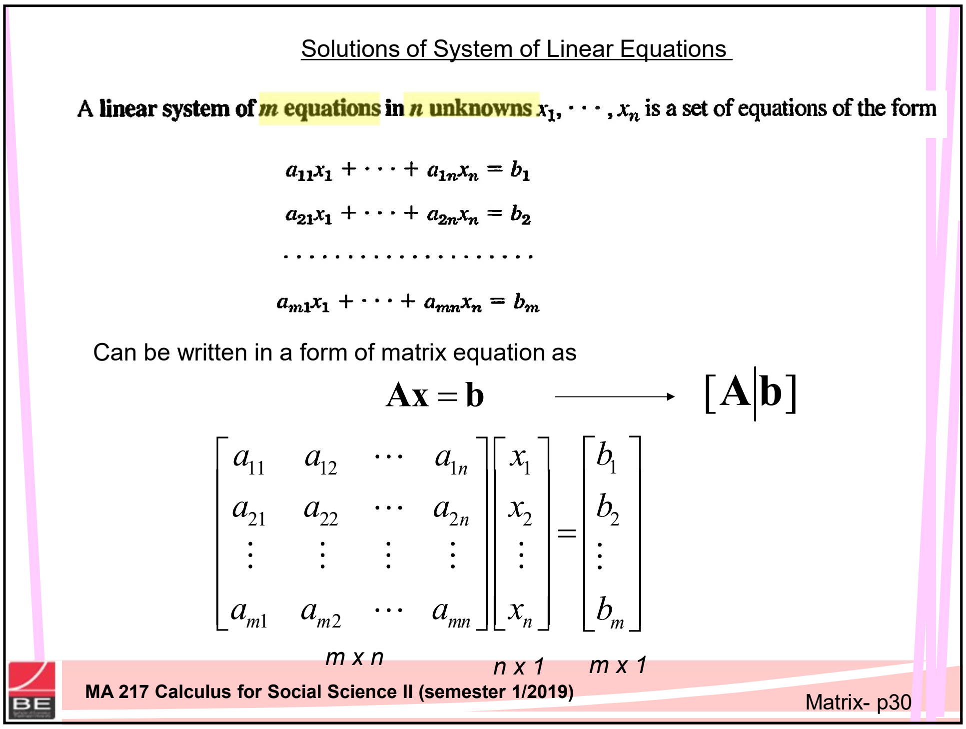
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots a_{3n}x_n &= b_3 \\ \cdot \qquad\qquad\qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad\qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad\qquad\qquad &\quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots a_{nn}x_n &= b_n \end{aligned} \qquad (1)$$

$x_1, x_2, x_3, \ldots\ldots, x_n$ Is a set of unknown variables

If all $b_i$ are zero then the system is called "Homogeneous system"

If $b_i$ are not all zero then the system is called "Non homogeneous system"

**If the system (1) is homogeneous, it has at least the trivial solution $x_1$=0,…,$x_n$ =0**



Solutions of System of Linear Equations

A linear system of **$m$ equations** in **$n$ unknowns** $x_1, \cdots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Can be written in a form of matrix equation as

$$\mathbf{Ax} = \mathbf{b} \longrightarrow [\mathbf{A}|\mathbf{b}]$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \text{ x } n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n \text{ x } 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m \text{ x } 1}$$

## Examples of a system of linear equations

A charity wishes to endow a fund that will provide \$50000 per year for cancer research. The charity has \$480000 and , to reduce risk, wants to invest in two banks paying 10% and 11 % respectively.
How much should be invested in each bank?

$$\begin{aligned} x + \quad\; y &= 480000 \\ 0.1\,x + 0.11\,y &= 50000 \end{aligned}$$

$$\begin{aligned} 10x+10y&=4800000 \\ 10x+11y&=5000000 \end{aligned}$$

$$\begin{aligned} y &= 200000 \\ x &= 280000 \end{aligned}$$

## Examples of a system of linear equations

Tax benefits of charitable contribution

A company earns before-tax profits of \$100000. It has agreed to contribute 10% of its after-tax profits to Red Cross Relief Fund. It must pay a state tax of 5 % of its profits (after the Red Cross donation) and a federal tax of 40 % of its profit (after the donation and state taxes are paid). How much does the company pay in state taxes, federal taxes and Red Cross donation?

$$\begin{aligned} C + 0.1S + 0.1F &= 10000 \\ 0.05C + \quad S \quad\quad &= 5000 \\ 0.4C + 0.4S + \quad F &= 40000 \end{aligned}$$

C = 5956, S = 4,702, F = 35,737

### Examples of a system of linear equations

Suppose that you are at a bench-press contest and have just witnessed the Lightweight Champion press a bar weighted with 2 large disks and 1 small disk on each side and the weight is announced as 275 pounds. Then the eventual middleweight champion pressed a bar with 3 large disk and 1 small disk on each side, the weight is announce 365 pounds. The heavyweight champion lifts a bar weighted with 4 large disks and 1 small disk, but you cannot hear announced weight. Assuming the bar weighted 45 pounds,how much did the heavyweight lift?

Ans:X=45 and y=25

A foolish dieter, following the instructions of his nutritionist to the letter, is insistent that fat constitute exactly 10% of his diet. On a certain day, he must Select from a menu consisting strictly of 92% fat-free turkey bologna and hog Lard, which is virtually 100% fat. How much of each should he consume if he wishes to eat 10 pounds of food.

Ans: X=9.8, y=0.2

A manufacturer produces three products: A,B, and C. The profits for each unit of A ,B and C sold are $1,$2 and $3 respectively. Fixed costs are $17000 per year, and the costs of producing each unit of A,B and C **are $4,$5, $7,** respectively. Next year, a total of 11,000 units of all three products is to be produced and sold, and the total profit of $25,000 is to be realised. If the total cost is to be $80,000, how many units of each of the products should be produced next year?

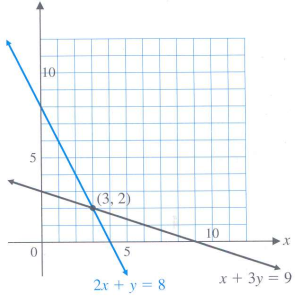
A=2000,B=4000,C=5000

$$2x+y=8 \quad (1)$$
$$x+3y=9 \quad (2)$$

To obtain solutions to this system of linear equations we can either use, for examples

1. Graph method
2. Substitution method
3. Elimination of unknowns method
4. Elementary row operations
5. Inverse of a matrix
6. Cramer's rule

1. Graph method



2. Substitution method

From (1): $2x+y=8 \Rightarrow y=\boxed{8-2x}$

From (2): $x+3y=9$

$x+3(8-2x)=9$

$x+24-6x=9$

$-5x=-15$

$x=3$ and

From (1): $y=8-2x=8-2(3) \Rightarrow y=2$

## 3. Elimination of unknowns

$$2x + y = 8 \quad (1)$$
$$x + 3y = 9 \quad (2)$$

To eliminate $y$,

$(1) \times -3$: $\quad -6x - 3y = -24 \quad (3)$

$(2) + (3)$: $\quad -5x = -15$

$x = 3$ and

From (1): $\quad y = 8 - 2x = 8 - 2(3) \Rightarrow y = 2$

Methods 4-6 require the use of matrix algebra.

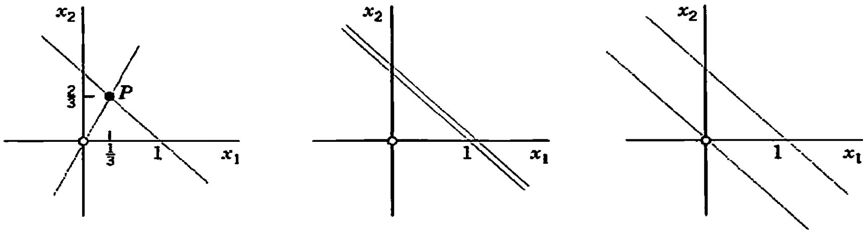
## Geometric Interpretation. Existence and Uniqueness of Solutions

If $m = n = 2$, we have two equations in two unknowns $x_1, x_2$

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

There are 3 possible cases

| | | |
|---|---|---|
| $x_1 + x_2 = 1$ | $x_1 + x_2 = 1$ | $x_1 + x_2 = 1$ |
| $2x_1 - x_2 = 0$ | $2x_1 + 2x_2 = 2$ | $x_1 + x_2 = 0$ |
| Case (*a*) | Case (*b*) | Case (*c*) |

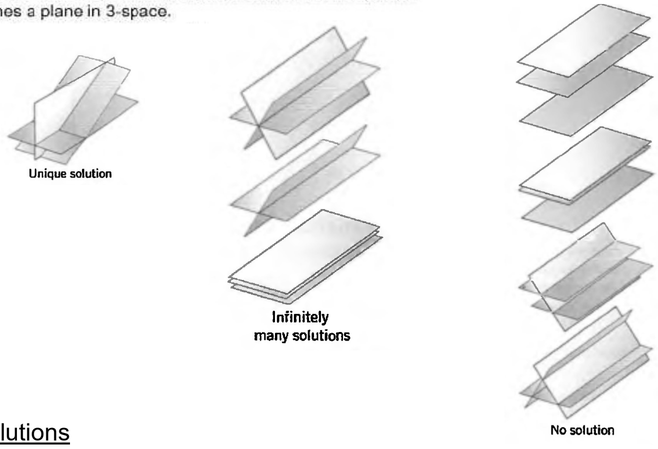


(*b*) Infinitely many solutions if the lines coincide.

(*a*) Precisely one solution if the lines intersect.

(*c*) No solution if the lines are parallel

**EXAMPLE:** Three equations in three variables. Each equation determines a plane in 3-space.



Solutions

Unique or Infinitely many solutions ⟶ Consistent system

No solution ⟶ Inconsistent system

**How can we know ?**

## Strategy for solving a linear system

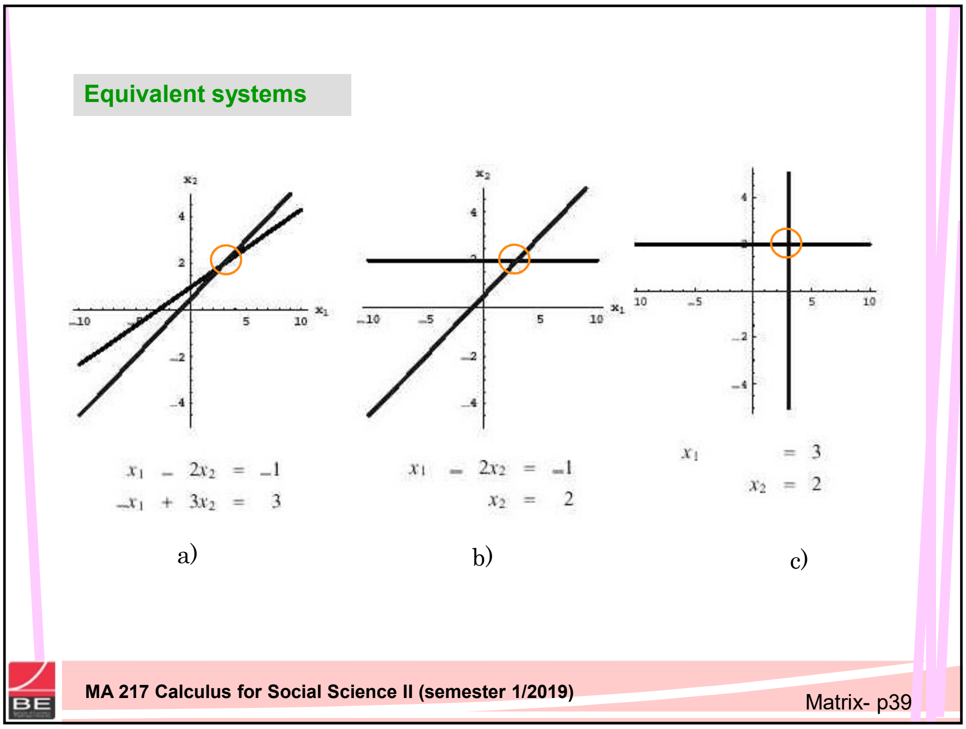
Replace one system with **an equivalent system** (one with the same solution set) that is easier to solve.

example

a)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

b)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

c)
$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$



## Equivalent systems

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

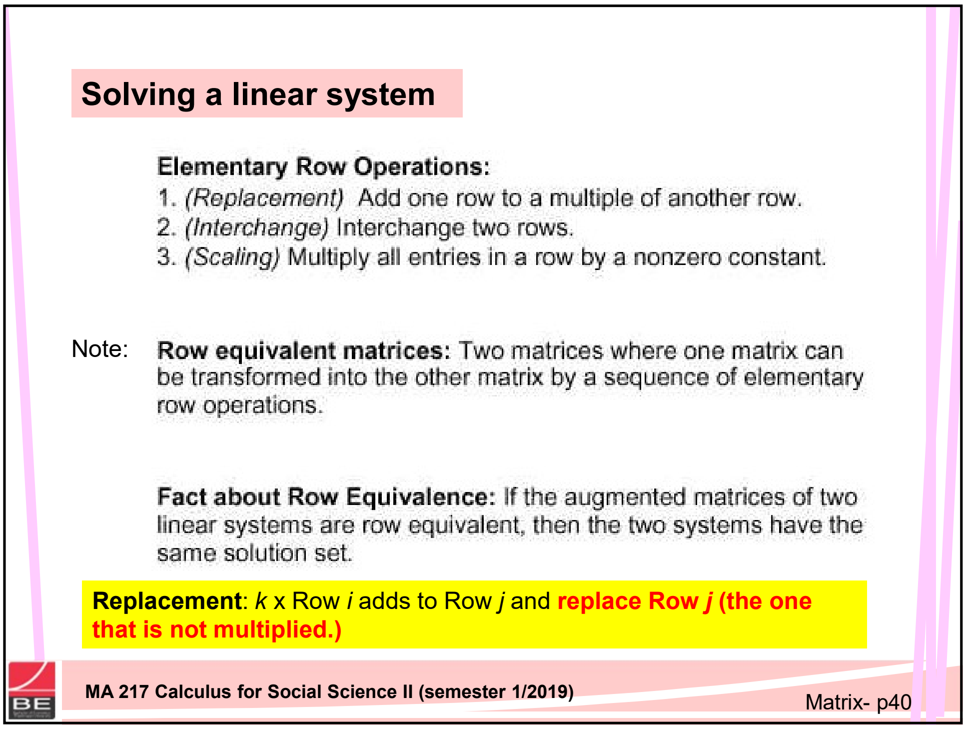
a)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

b)

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

c)



## Solving a linear system

**Elementary Row Operations:**

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Note: **Row equivalent matrices:** Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

**Fact about Row Equivalence:** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

**Replacement**: *k* x Row *i* adds to Row *j* and **replace Row *j* (the one that is not multiplied.)**

**Example** Solving a system of linear equations using augmented matrix methods.

$$3x_1 + 4x_2 = 1$$

$$x_1 - 2x_2 = 7$$

1. Augmented matrix corresponding to the system of linear equations.

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

2. $R_1 \leftrightarrow R_2$ (To get a 1 in the upper left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

3. $(-3)R_1 + R_2 \to R_2$ (To get a 0 in the lower left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

4. $\left(\frac{1}{10}\right)R_2 \to R_2$ (To get a 1 in the lower right corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

5. $(2)R_2 + R_1 \to R_1$ (To get a 0 in the upper right corner.)

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

Hence, $x_1 = 3$ and $x_2 = -2$.

We can stop the row operation process at step 4 and perform back substitution to obtain the solution set. This method is known as "**Gauss Elimination**" method.

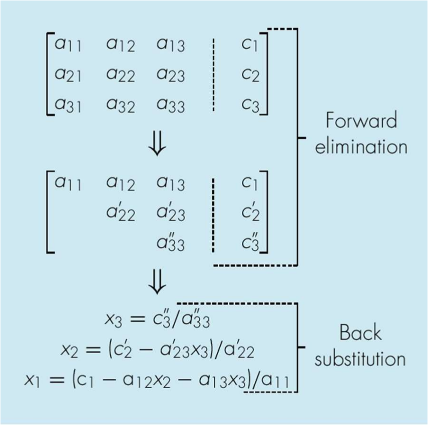
4. $\left(\frac{1}{10}\right)R_2 \to R_2$ (To get a 1 in the lower right corner.)

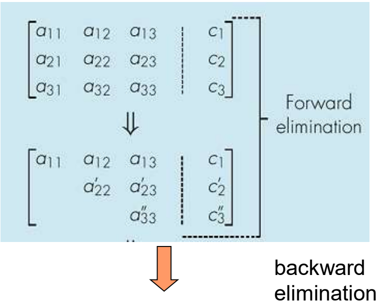
$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

$$x - 2y = 7 \qquad (1)$$

$$y = -2 \qquad (2)$$

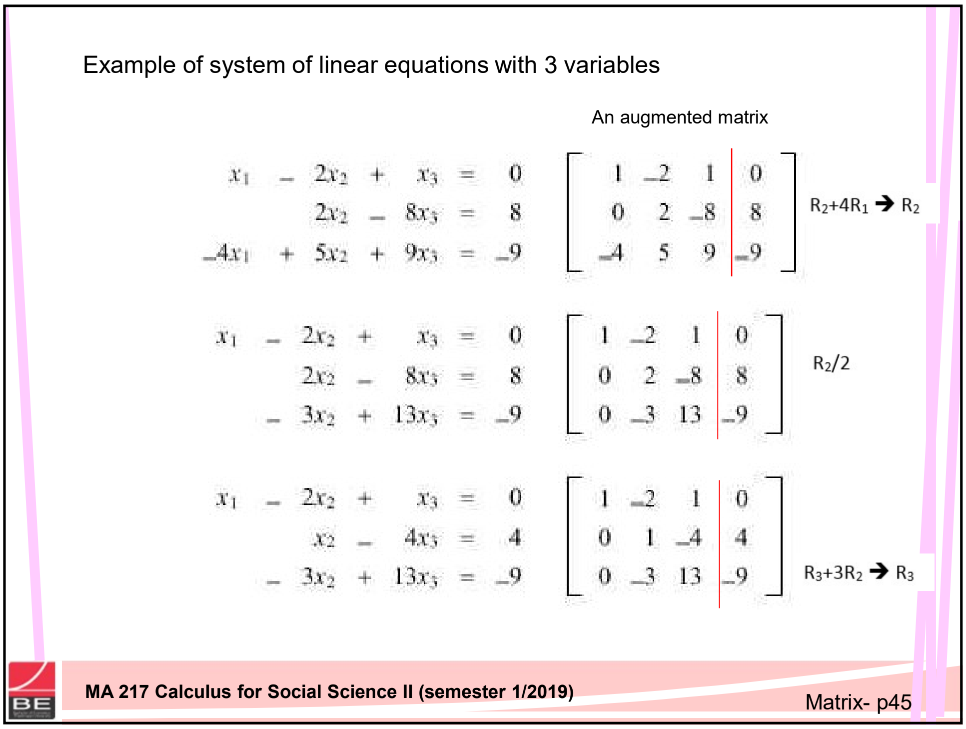
Solve for y first in eq. (2)
and then substitute y into eq.(1)
to solve for x





backward elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & C''_1 \\ 0 & 1 & 0 & C''_2 \\ 0 & 0 & 1 & C''_3 \end{array}\right]$$



Example of system of linear equations with 3 variables

An augmented matrix

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

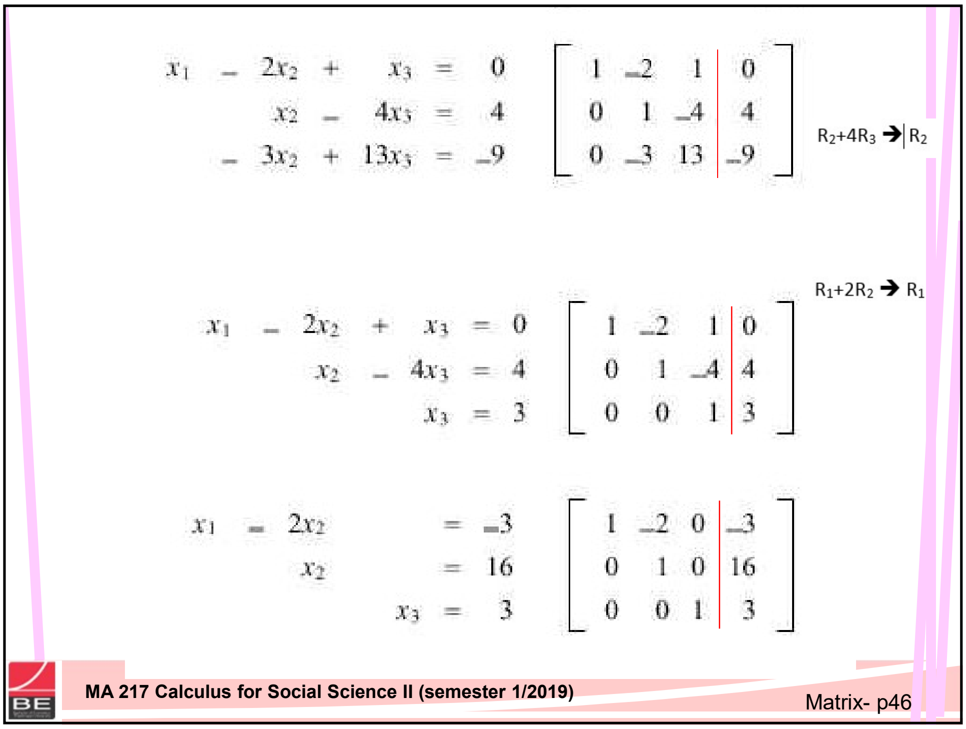
$R_2+4R_1 \rightarrow R_2$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$R_2/2$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$R_3+3R_2 \rightarrow R_3$



$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$R_2+4R_3 \rightarrow R_2$

$R_1+2R_2 \rightarrow R_1$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 \qquad\qquad &= 29 \\ x_2 \qquad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

**Check:** Is $(29, 16, 3)$ a solution of the *original* system?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$\begin{aligned} (29) - 2(16) + 3 &= 29 - 32 + 3 &&= 0 \\ 2(16) - 8(3) &= 32 - 24 &&= 8 \\ -4(29) + 5(16) + 9(3) &= -116 + 80 + 27 &&= -9 \end{aligned}$$

## Row operations are reversible.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution sets.

Example 1

$$6x_1 + x_2 + x_3 = 6$$
$$5x_1 + x_2 + 2x_3 = 4$$
$$4x_1 + x_2 - x_3 = -2$$

$x_1 = 3, x_2 = -13, x_3 = 1$

Example 2

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$x_1 = -1 + \frac{4}{3}x_3$$

$$x_2 = 2$$

Example 3

$$3x_2 - 6x_3 = 8$$

$$x_1 - 2x_2 + 3x_3 = -1$$

$$5x_1 - 7x_2 + 9x_3 = 0$$

Inconsistent

Example 4

$$2x_1 - 4x_2 + x_3 = -4$$
$$4x_1 - 8x_2 + 7x_3 = 2$$
$$-2x_1 - 4x_2 - 3x_3 = 5$$

Inconsistent

---

**Echelon form** (or **row echelon form**):

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

**EXAMPLE:** Echelon forms

(a) $$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

The whole purpose of doing the elimination is to get an upper triangular matrix (U) (echelon form)

**Reduced echelon form:** Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

**EXAMPLE** (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

## Important terms

**Pivot position**: a position of a leading entry in an echelon form of the matrix.

**Pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's

**Pivot column**: a column that contains a pivot position.

**Pivot row**: a row that contains a pivot position.

Number of pivots = **Rank of a matrix**

Example : 3 equations 3 unknowns

$$\text{(coefficient matrix)} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{2} & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

The first pivot

Pivot row

Pivot column

Replacement $\begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 8 & 3 \end{bmatrix}$ Replacement $\Rightarrow$ $\begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 0 & \boxed{1} \end{bmatrix}$

U

By definition, pivots cannot be zero. ( need to divide them!)

Example Row reduce to echelon form and locate the pivot columns.

pivot

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \underset{R_1 \leftrightarrow R_4}{\sim} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \underset{\substack{R_2+R_1 \rightarrow R_2 \\ R_3+2R_1 \rightarrow R_3}}{\sim} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot column

Possible Pivots:

$$\begin{array}{l} R_3-(5/2)R_2 \rightarrow R_3 \\ R_4+(3/2)R_2 \rightarrow R_4 \end{array} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Original Matrix: $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

$\uparrow \ \uparrow \quad \uparrow$

pivot columns: 1 2 4

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

**EXAMPLE:** Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$R_2 - R_1 \rightarrow R_2 \quad \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \overset{R_2/2}{\sim} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\underset{R_3 - 3R_2 \rightarrow R_3}{\sim} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{matrix} R_1 - 6R_3 \rightarrow R_1 \\ \\ R_2 - R_3 \rightarrow R_2 \end{matrix} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Echelon form

$$\overset{R_1 + 9R_2 \rightarrow R_1}{\sim} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \overset{R_1/3}{\sim} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Reduced echelon form

Example

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 2 & 6 & 8\\ 2 & 8 & 10\end{bmatrix}$$

How many pivot variables?

---

## Under what circumstances could the elimination stop?

If the algorithm produces ***n*** pivots (there are pivots in every column i.e. a full set of pivots), then there is only one solution to the equations (a unique solution)

**example** $$\begin{aligned}x_1+x_2&=10\\ -x_1+x_2&=0\end{aligned} \quad\Longleftrightarrow\quad \underline{\mathbf{Ax=b}}$$

If a **zero** appears in a pivot position, elimination has to stop!

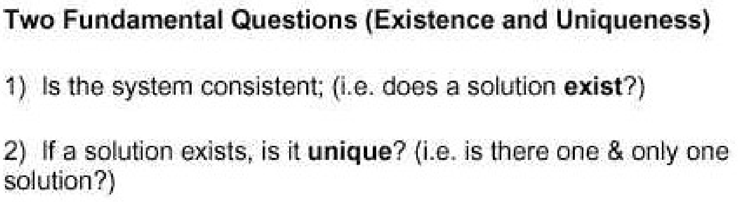
Stop temporarily➔ there is possibility to exchange with a lower row for a proper pivot.

Stop permanently➔ there is no exchange of row that can avoid zero.

**examples**

$$\begin{aligned}x_1-2x_2&=-3\\ 2x_1-4x_2&=8\end{aligned} \quad\Longleftrightarrow\quad \underline{\mathbf{Ax=b}} \qquad\qquad \begin{aligned}x_1+x_2&=3\\ -2x_1-2x_2&=-6\end{aligned} \quad\Longleftrightarrow\quad \underline{\mathbf{Ax=b}}$$

However, we do not know whether a zero will appear until we try.



**Two Fundamental Questions (Existence and Uniqueness)**

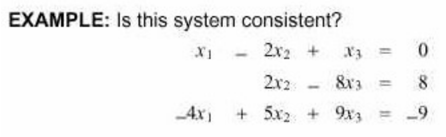
1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

If a solution exists <u>either a unique solution or infinitely many solutions</u>, the system is said to be consistent. Otherwise the system is inconsistent.

How an echelon form (or rref) of a matrix (obtained by performing row operations on a matrix ) tell us about the nature of the solution (e.g. Unique solution, infinitely many solutions or no solution)?

---

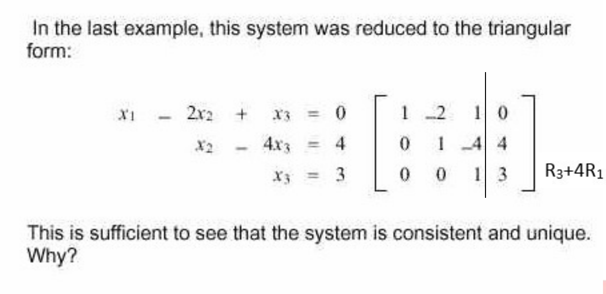


**EXAMPLE:** Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix} \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

**Perform row operation to obtain echelon form**



In the last example, this system was reduced to the triangular form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_3+4R_1$ ➔ $R_3$

This is sufficient to see that the system is consistent and unique. Why?

**EXAMPLE:** Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

**Solution:** Row operations produce:

$$\left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right] \xrightarrow[R_3-5R_1 \rightarrow R_3]{R_2 \longleftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \quad \leftarrow \textit{Never true} \end{aligned}$$

The original system is inconsistent!

**EXAMPLE:** For what values of $h$ will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

**Solution:** Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

## Homogeneous linear systems

**$\underline{Ax=0}$**

E.g. $$\begin{bmatrix}2&4\\1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}2&4&0\\1&3&0\end{array}\right]\overset{R_2-\frac{1}{2}R_1}{\sim}\left[\begin{array}{cc|c}2&4&0\\0&1&0\end{array}\right]$$

$\therefore x_2=0$ and $x_1=0$ or $\underline{x}=\begin{bmatrix}0\\0\end{bmatrix}$

E.g. $$\begin{bmatrix}2&4\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}2&4&0\\1&2&0\end{array}\right]\overset{R_2-\frac{1}{2}R_1}{\sim}\left[\begin{array}{cc|c}2&4&0\\0&0&0\end{array}\right]$$

$\therefore x_2\in R$
and
$0=2x_1+4x_2$

or $\underline{x}=C\begin{bmatrix}-2\\1\end{bmatrix};C\in R$

## Nonhomogeneous linear systems

**$\underline{Ax=b}$**

e.g.

(a) $$\begin{bmatrix}2&4\\1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}2&4&4\\1&3&1\end{array}\right]\overset{R_2-\frac{1}{2}R_1}{\sim}\left[\begin{array}{cc|c}2&4&4\\0&1&-1\end{array}\right]$$

$\therefore x_2=-1$ and $x_1=4$

(b) $$\left[\begin{array}{cc|c}2&4&4\\1&2&1\end{array}\right]\overset{R_2-\frac{1}{2}R_1}{\sim}\left[\begin{array}{cc|c}2&4&4\\0&0&-1\end{array}\right]$$

No solution

(c) $$\left[\begin{array}{cc|c}2&4&4\\1&2&2\end{array}\right]\overset{R_2-\frac{1}{2}R_1}{\sim}\left[\begin{array}{cc|c}2&4&4\\0&0&0\end{array}\right]$$

Infinitely many solutions

$\therefore x_2\in R$ and
$x_1=2-2x_2$

or $\underline{x}=\begin{bmatrix}2\\0\end{bmatrix}+C\begin{bmatrix}-2\\1\end{bmatrix};C\in R$

## Example

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \ldots\ldots \qquad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General solutions

$$x_1=-1+\frac{4}{3}x_3$$

$$x_2=2$$

---

As a vector the general solutions of **Ax=b** has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1+\frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}+\begin{bmatrix} -\frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} \quad ; \quad x\in\Re$$

$x_3$ can be any real number.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}+C\begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \quad ; \quad C\in R$$

## Rectangular (coefficient) matrices $\underline{\mathbf{A}}_{m\times n}$

$$\underline{\mathbf{A}}_{m\times n}\,\underline{\mathbf{x}}=\underline{\mathbf{b}}$$

This matrix equation is equivalent to a system of linear equations of *m* equations *n* unknowns.

$$\begin{aligned} x_1+2x_2&=2\\ x_1+5x_2&=6\\ 2x_1+7x_2&=8\\ 4x_1+14x_2&=16 \end{aligned} \iff \begin{bmatrix}1&2\\1&5\\2&7\\4&14\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\6\\8\\16\end{bmatrix} \iff \left[\begin{array}{cc|c}1&2&2\\1&5&6\\2&7&8\\4&14&16\end{array}\right]$$

The solution of *m* equations *n* unknowns $(m\neq n)$

## Rectangular (coefficient) matrices

**3 equations 4 unknowns➔** $\underline{\mathbf{Ax}}=\mathbf{0}$

$$A=\begin{bmatrix}\boxed{1}&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\xrightarrow[\text{row operation}]{\text{Perform}}\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$$

Candidate for the 2nd pivot has become zero and
The entry below is also zero. **" go onto the next column"**

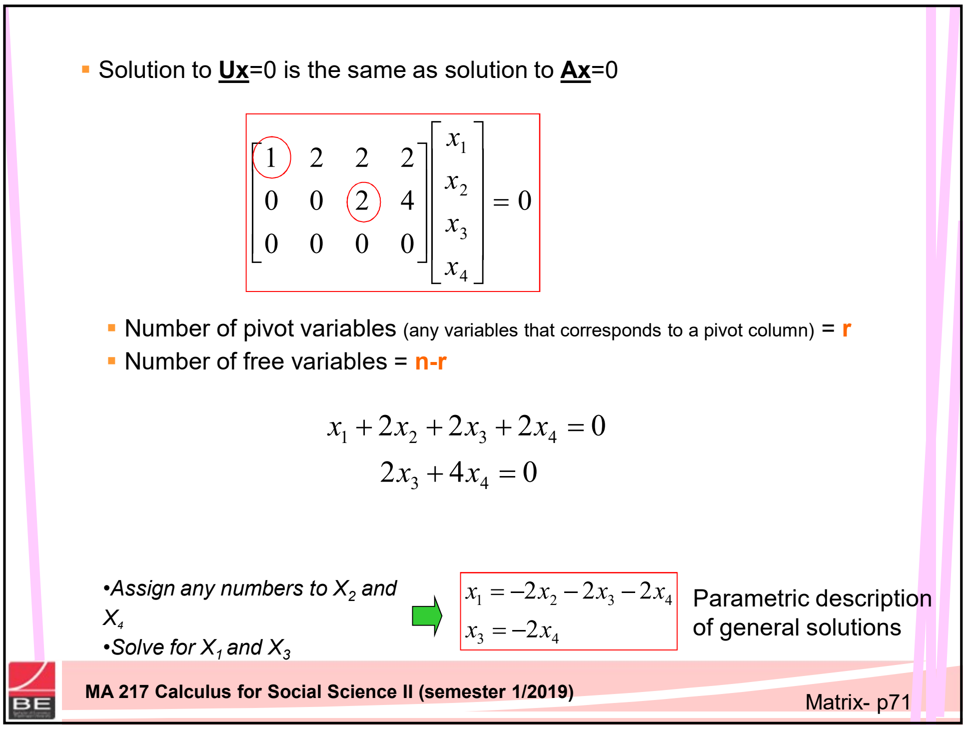
$$\begin{bmatrix}1&2&2&2\\0&0&\boxed{2}&4\\0&0&0&0\end{bmatrix}\longleftarrow$$

**U** ( Echelon form)

**Ux** = 0

Row of zero occurs because of row 3 was a combination of row 1 and 2

Number of pivots = 2 (rank of the matrix)
Solution is not unique



- Solution to **Ux**=0 is the same as solution to **Ax**=0

$$\boxed{\begin{bmatrix} \enclose{circle}{1} & 2 & 2 & 2 \\ 0 & 0 & \enclose{circle}{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0}$$

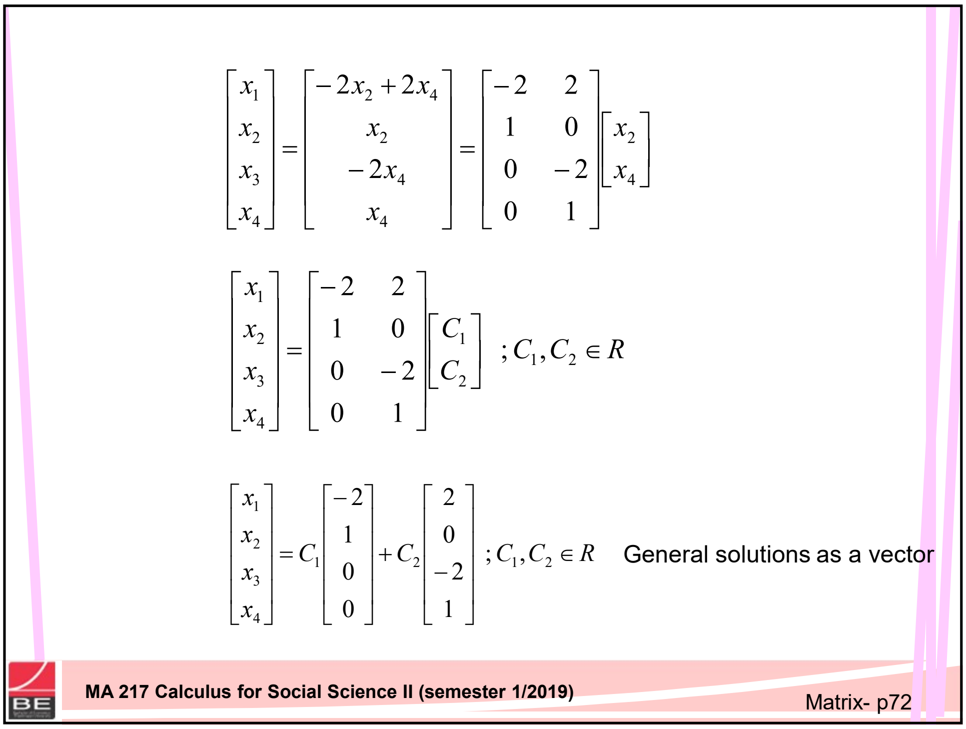
- Number of pivot variables (any variables that corresponds to a pivot column) = r
- Number of free variables = n-r

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_3 + 4x_4 &= 0 \end{aligned}$$

*•Assign any numbers to $X_2$ and $X_4$*
*•Solve for $X_1$ and $X_3$*

$$\Rightarrow \boxed{\begin{aligned} x_1 &= -2x_2 - 2x_3 - 2x_4 \\ x_3 &= -2x_4 \end{aligned}}$$

Parametric description of general solutions



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad ; C_1, C_2 \in R$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = C_1\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad ; C_1, C_2 \in R$$

General solutions as a vector

## 4 equations 2 unknowns➔ $\underline{\mathbf{Ax}}=\mathbf{0}$

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 2 & 7 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## Rectangular (coefficient) matrices $\underline{\mathbf{A}}_{m\times n}$

**m<n** → <u>No unique solution exists.</u> There may be

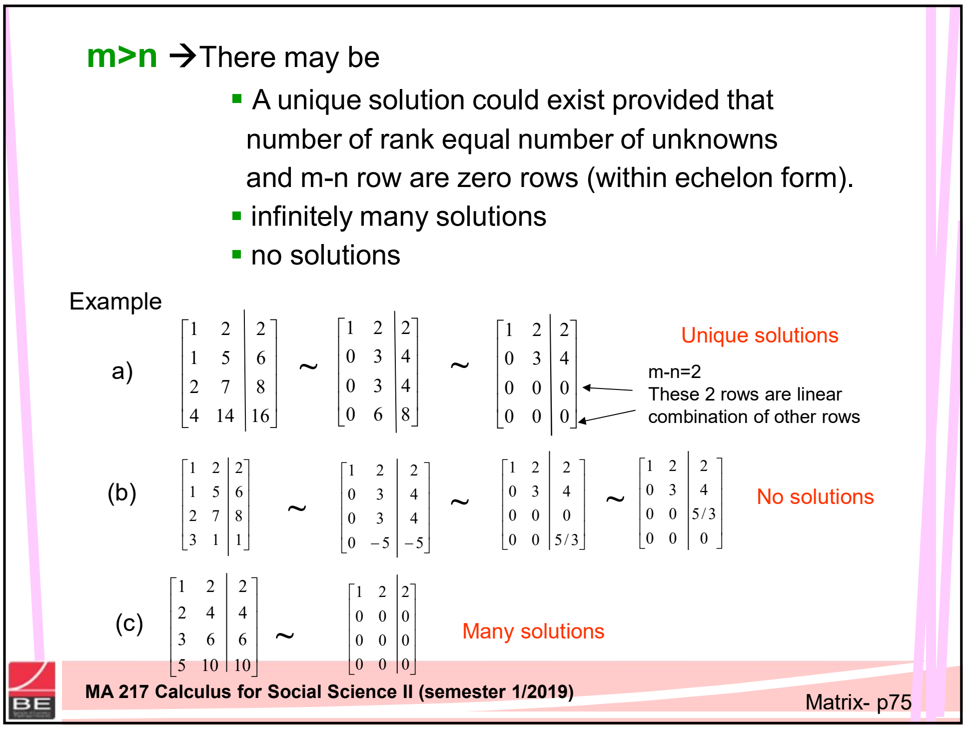
- infinitely many solutions
- no solution

(if the system in echelon form contains equations of the form 0=b with b nonzero.)

e.g.

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & -b_2+2b_1+b_3-3b_1 \end{array}\right]$$



**m>n** →There may be

- A unique solution could exist provided that number of rank equal number of unknowns and m-n row are zero rows (within echelon form).
- infinitely many solutions
- no solutions

Example

a) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 4 & 14 & 16 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Unique solutions

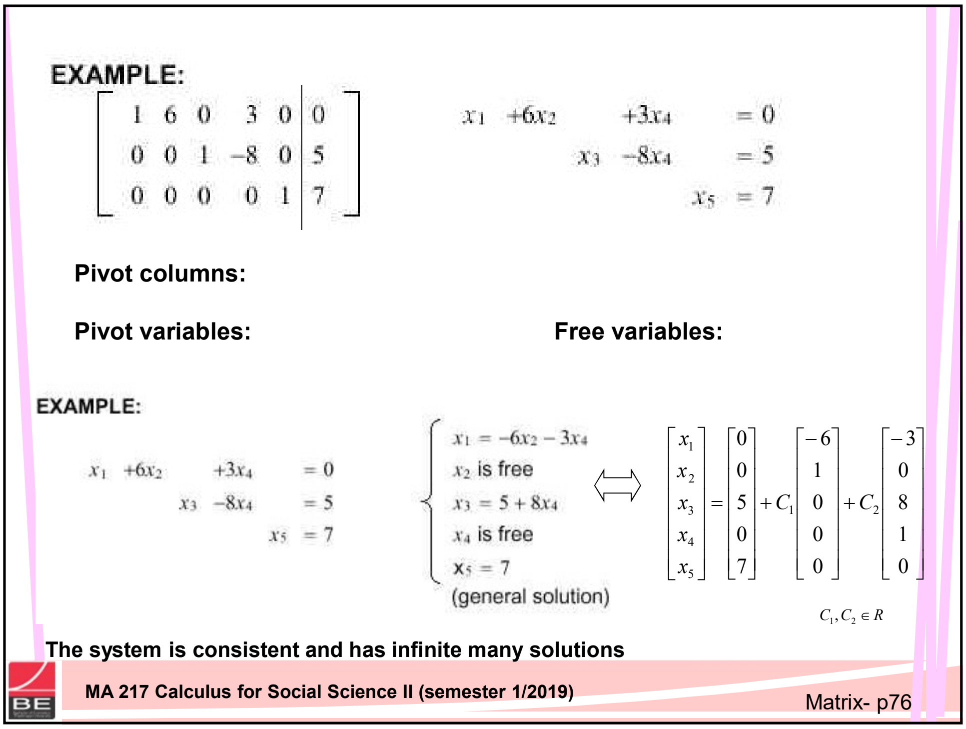
m-n=2
These 2 rows are linear combination of other rows

(b) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & -5 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 5/3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5/3 \\ 0 & 0 & 0 \end{array}\right]$

No solutions

(c) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 6 \\ 5 & 10 & 10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Many solutions



**EXAMPLE:**

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \qquad \begin{array}{l} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 8x_4 = 5 \\ x_5 = 7 \end{array}$$

**Pivot columns:**

**Pivot variables:**

**Free variables:**

**EXAMPLE:**

$$\begin{array}{l} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 8x_4 = 5 \\ x_5 = 7 \end{array} \qquad \left\{ \begin{array}{l} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{array} \right. \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} + C_1 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}$$

(general solution)

$C_1, C_2 \in R$

**The system is consistent and has infinite many solutions**

**EXAMPLE:**

$$\begin{array}{rrrrrl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{array}$$

Perform sequences of row operation

$$\left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

**Free variables:** $x_3$ and $x_4$

$\boxed{\text{Consistent system with free variables}}$ ⇒ **infinitely many solutions.**

**What is general solutions in a vector form?**

**EXAMPLE:**

$$\begin{array}{rrl} 3x_1 & +4x_2 & = -3 \\ 2x_1 & +5x_2 & = 5 \\ -2x_1 & -3x_2 & = 1 \end{array} \rightarrow \left[\begin{array}{rr|r} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rr|r} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

$\boxed{\text{Consistent system, no free variables}}$ ⇒ **unique solution.**

$$\begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Example

$$\mathbf{A}=\begin{bmatrix}1 & 3\\ 2 & 1\\ 6 & 1\\ 5 & 1\end{bmatrix} \quad \& \quad \mathbf{b}=\begin{bmatrix}4\\ 3\\ 7\\ 6\end{bmatrix}$$

RREF

$$\left[\begin{array}{cc|c}1 & 3 & 4\\ 2 & 1 & 3\\ 6 & 1 & 7\\ 5 & 1 & 6\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & 4\\ 0 & -5 & -5\\ 0 & -17 & -17\\ 0 & -14 & -14\end{array}\right] \sim \left[\begin{array}{cc|c}\boxed{1} & 0 & 1\\ 0 & \boxed{1} & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$$

r=n <m
(full column rank)

## Summary

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}}=\underline{\mathbf{b}} \qquad \underline{\mathbf{b}}\neq\underline{0}$$

r = m = n ➔ one solution exists (full rank)

r = n < m ➔ 0 or 1 solution (full column rank)

r = m < n ➔ infinitely many solutions
(no zero row) (full row rank)

r < m and r < n ➔ no solution
or infinite many solutions

# Inverse of Matrices

## Matrix Inverses

The inverse of a real number $a$ is denoted by $a^{-1}$. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix $A$ is said to be **invertible** if there is an $n \times n$ matrix $C$ satisfying

$$CA = AC = I_n$$

where $I_n$ is the $n \times n$ identity matrix. We call $C$ the **inverse** of $A$ .

example $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ ; $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ $AC =$ $CA =$

The inverse of $A$ is usually denoted by $A^{-1}$.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

***Not all*** $n \times n$ ***matrices are invertible.*** A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

**Fact 1** If $A$ is invertible, then the inverse is unique.

*Proof:* Assume $B$ and $C$ are both inverses of $A$. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\_\_\_\_\_\_ = I\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

**Fact 2** The inverse of $A^{-1}$ is A itself.

**Fact 3** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then $A$ is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $A$ is not invertible.

Assume $A$ is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$\_\_\_\_\_A\mathbf{x} = \_\_\_\_\_\mathbf{b}$ and so

$I\mathbf{x} = \_\_\_\_\_\_\_\_$ or $\mathbf{x} = \_\_\_\_\_\_\_\_$.

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$\_\_\_\_\_A\mathbf{w} = \_\_\_\_\_\mathbf{b}$ which means $\mathbf{w} = A^{-1}\mathbf{b}$.

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

**Fact 4** If $A$ is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

**EXAMPLE:** Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{array}{rcrcl} -7x_1 & + & 3x_2 & = & 2 \\ 5x_1 & - & 2x_2 & = & 1 \end{array}.$$

*Solution:* Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

## Properties of Inverses

Suppose A and B are invertible. Then the following results hold:

a. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$ (i.e. $A$ is the inverse of $A^{-1}$).

b. $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$

*Partial proof of part b:*

$$(AB)(B^{-1}A^{-1}) = A(\_\_\_\_\_\_\_\_)A^{-1}$$

$$= A(\_\_\_\_\_\_\_\_)A^{-1} = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

*Proof part c*

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

## Matrix inversion algorithm

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

➔Take a column at a time, that equation determines the columns of $A^{-1}$

$A$ times column j of $A^{-1}$ = column j of $I$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Carry out elimination on
*all systems simultaneously.*

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$[A \mid I]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$[I \mid A^{-1}]$$

The Gauss-Jordan method

## Matrix inversion algorithm

Place $A$ and $I$ side-by-side to form an augmented matrix $[A \mid I]$. Then perform row operations on this matrix (which will produce identical operations on $A$ and $I$).

$$[A \mid I] \text{ will row reduce to } [I \mid A^{-1}]$$

or $A$ is not invertible.

**EXAMPLE:** Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

*Solution:*

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ , if it exists

# Determinant of Matrices

**Determinants**

Determinant of a square matrix **A** is denoted by $|\mathbf{A}|$ or det **A** is a uniquely defined <u>SCALAR</u> associated with that matrix. Determinants are defined only for <u>square matrices</u>

*A second-order determinant*

For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$ is a scalar

Example : Given

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 5 & 9 \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (6)(9) - (-3)(5) = 69$$

*A Third-order determinant*

For a 3x3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}) - (a_{13}a_{31}a_{22} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

or

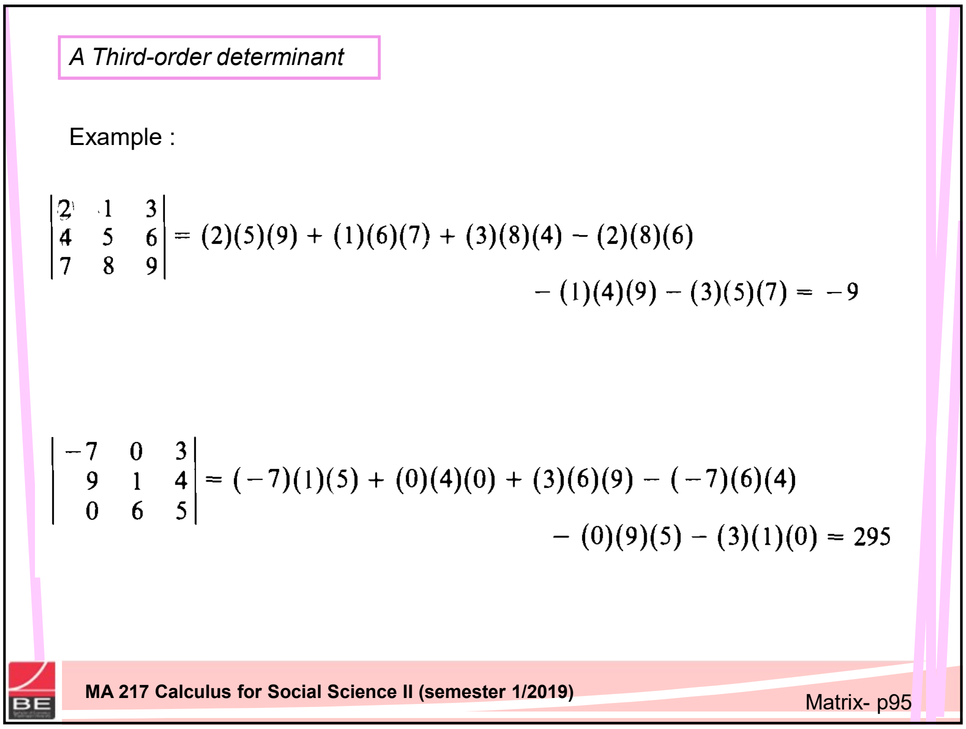
$$\det \mathbf{A} = |\mathbf{A}| = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

OR

Subdeterminant or Minor

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \quad [= \text{a scalar}]$$

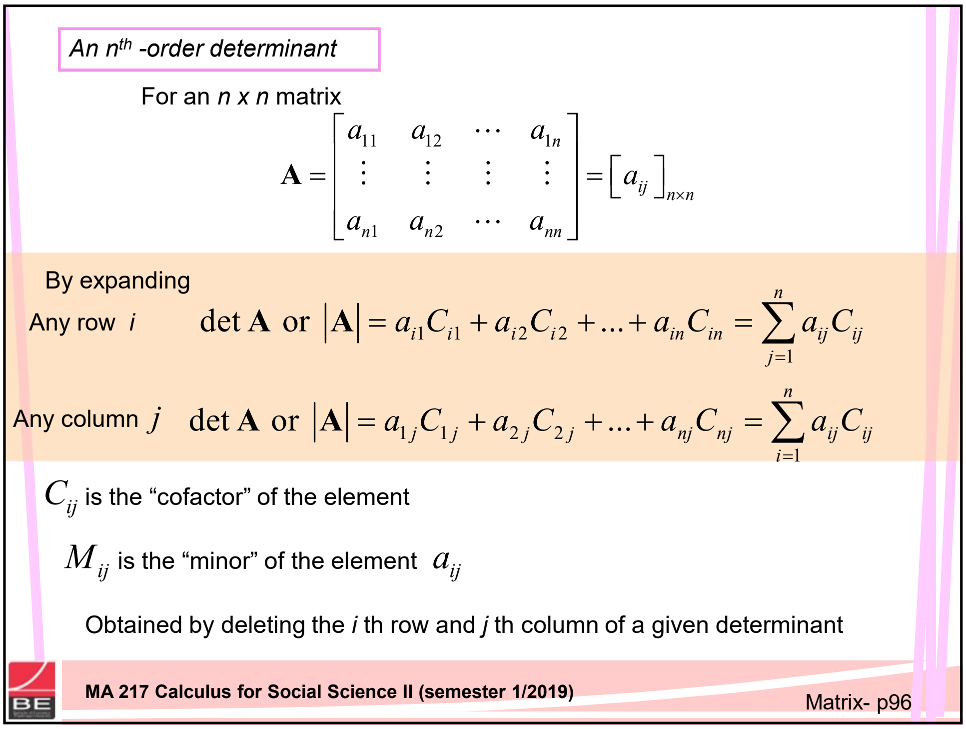


## A Third-order determinant

Example :

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (2)(5)(9) + (1)(6)(7) + (3)(8)(4) - (2)(8)(6) - (1)(4)(9) - (3)(5)(7) = -9$$

$$\begin{vmatrix} -7 & 0 & 3 \\ 9 & 1 & 4 \\ 0 & 6 & 5 \end{vmatrix} = (-7)(1)(5) + (0)(4)(0) + (3)(6)(9) - (-7)(6)(4) - (0)(9)(5) - (3)(1)(0) = 295$$



## An $n^{th}$ -order determinant

For an *n x n* matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \left[ a_{ij} \right]_{n \times n}$$

By expanding

Any row *i* $\det \mathbf{A} \text{ or } |\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$

Any column $j$ $\det \mathbf{A} \text{ or } |\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$

$C_{ij}$ is the “cofactor” of the element

$M_{ij}$ is the “minor” of the element $a_{ij}$

Obtained by deleting the *i* th row and *j* th column of a given determinant

The (i, j)-**cofactor** of $A$ is the number $C_{ij}$ where

$$C_{ij} = (-1)^{i+j}\det A_{ij}.$$

$$\det A = a_{11}.c_{11} + a_{12}c_{12} + ......a_{1n}c_{1n}$$

$$\boxed{\text{A cofactor expansion across the first row of } A}$$

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

Evaluate determinant of **A**, given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

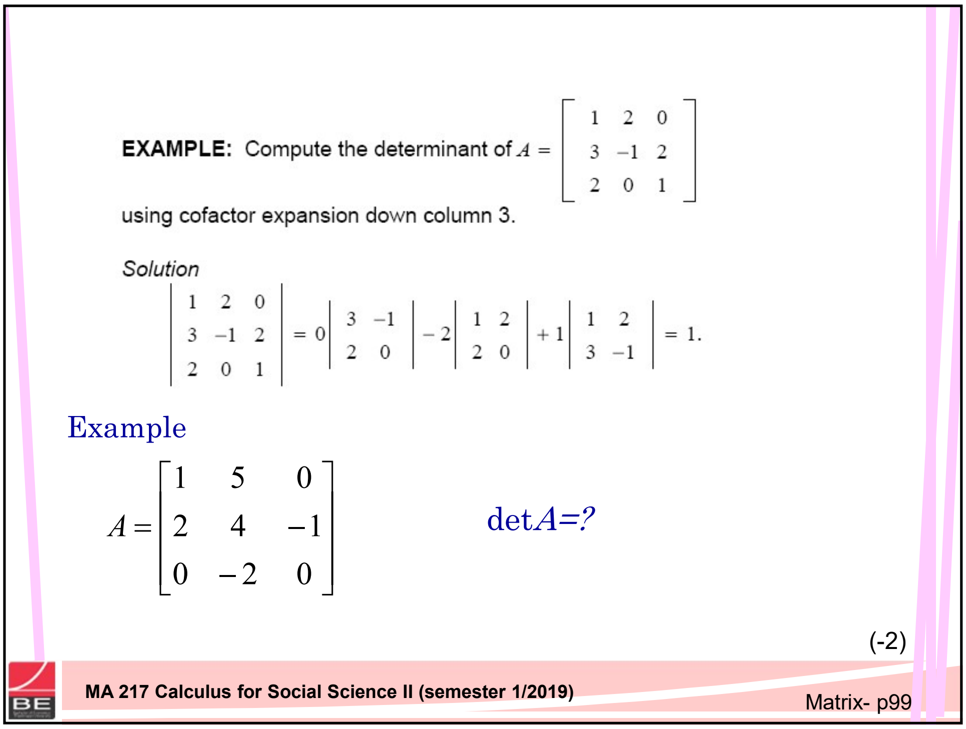
*Choose row 1 for expansion, since there is 0 in row 1*

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$C_{12} = (-1)^{1+2}(M_{12}) \rightarrow M_{12} = \begin{vmatrix} 1 & 4 \\ 5 & 7 \end{vmatrix} = (1)(7) - (4)(5) = -13$$

$$C_{13} = (-1)^{1+3}(M_{13}) \rightarrow M_{13} = \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} = (1)(6) - (3)(5) = -9$$

$$\det \mathbf{A} = (1)(3) + (2)(-9) = -5$$



**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

using cofactor expansion down column 3.

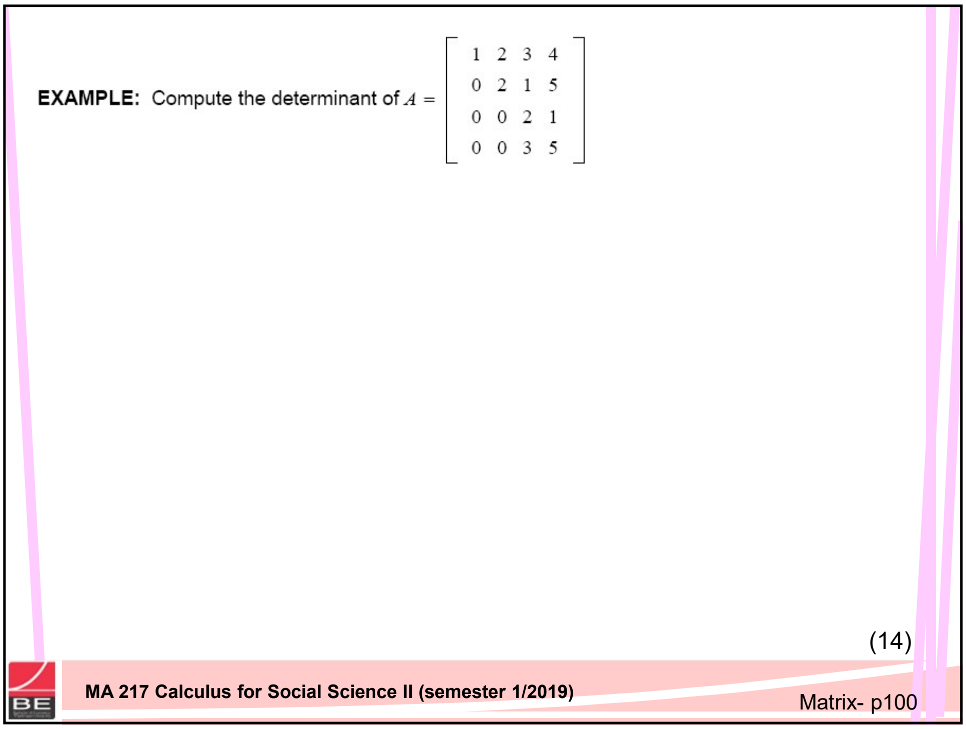
*Solution*

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

Example

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \qquad \det A = ?$$

(-2)



**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

(14)

## Basic properties of determinants

(1) $\det \mathbf{A}^T = \det \mathbf{A}$

***Example 1*** $\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 3 & 6 \end{vmatrix} = 9$

***Example 2*** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

**Hence, column operations = row operations in determinant. (2)-(4)**

(2) The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

*Example* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$ $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc)$

$\begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{vmatrix} = -26.$ $\begin{vmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 26.$

---

(3) The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold.

*Example*

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} 15a & 7b \\ 12c & 2d \end{vmatrix} = 3\begin{vmatrix} 5a & 7b \\ 4c & 2d \end{vmatrix} = 3(2)\begin{vmatrix} 5a & 7b \\ 2c & d \end{vmatrix} = 6(5ad - 14bc)$$

(4) The addition (subtraction) of a multiple of any row (or column) to (from) another row will leave the value of the determinant unaltered.

*Example*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

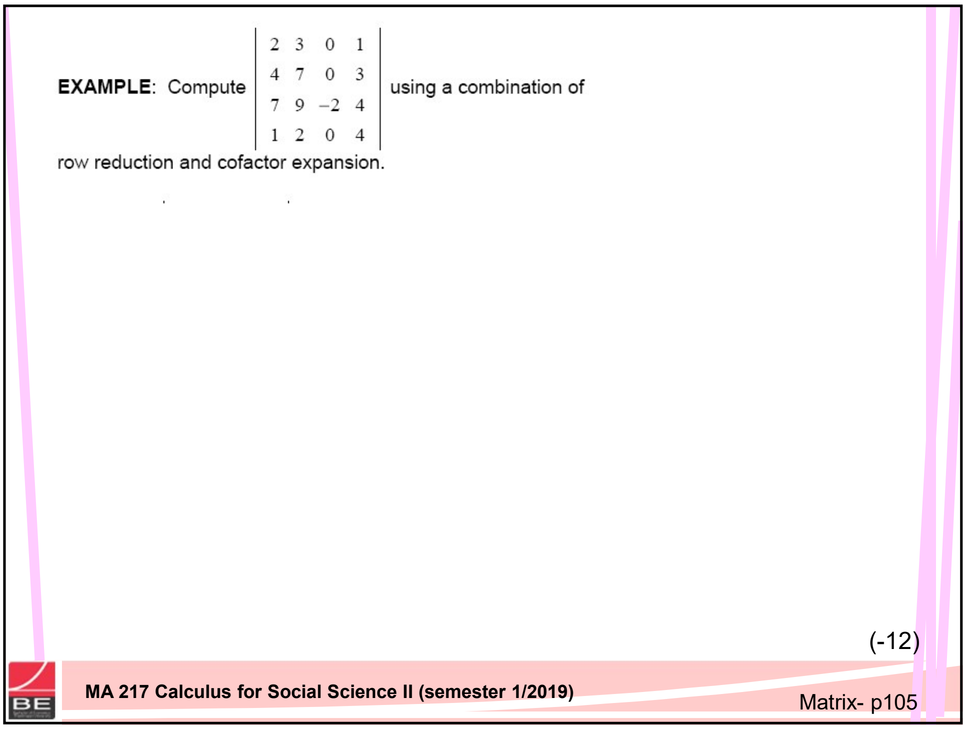
**EXAMPLE:** Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

*Solution*

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

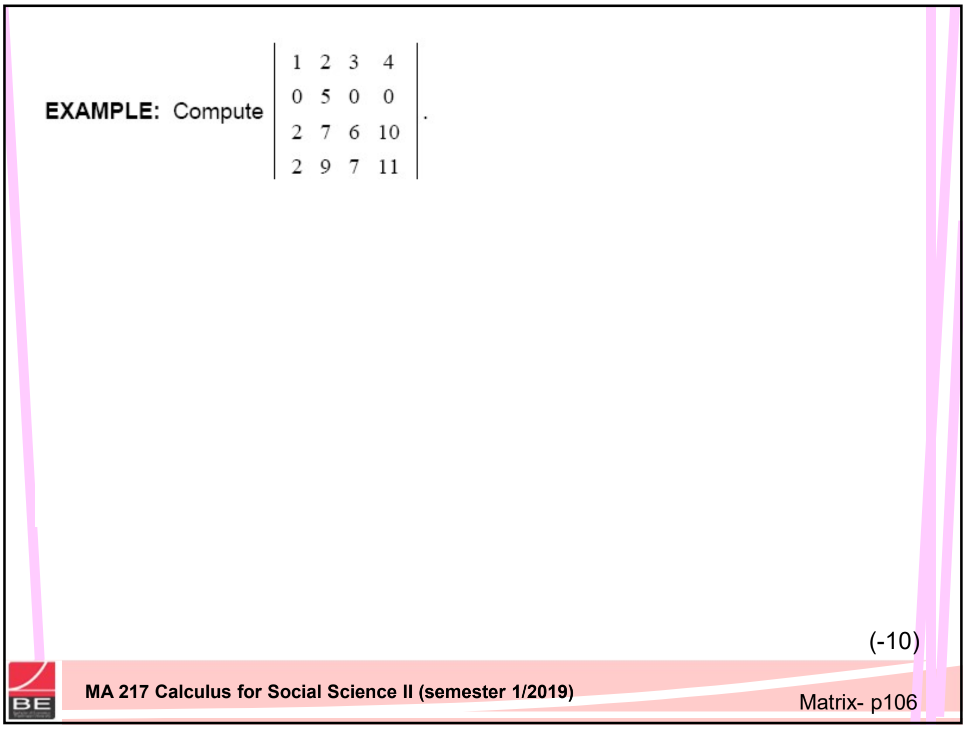
$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$



**EXAMPLE**: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

(-12)



**EXAMPLE**: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

(-10)

Compute Det(**A**)

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

Det(**A**) = 0 when **A** is not invertible (singular).

(0)

(5) If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

*Example*

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0 \qquad \begin{vmatrix} c & c \\ d & d \end{vmatrix} = cd - cd = 0$$

(6) A zero row or column renders the value of a determinant zero.

*Example*

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 = \begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 8 \\ 0 & 3 & 9 \end{vmatrix}$$

Further Properties

(7) $$\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$$

**EXAMPLE:** Compute $\det A^3$ if $\det A = 5$.

*Solution:* $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$

(8) If **A** is an *n* x *n* upper or lower triangular matrix

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

$$\det \mathbf{A} = a_{11}a_{22}a_{33}\ldots a_{nn}$$

Example

$$[A] = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 10 & 0 \\ 4 & 8 & 9 \end{bmatrix} \rightarrow |A| = (2)(10)(9) \ = \mathbf{180}$$

## Applications of determinants

### Cramer's Rule

Cramer's rule can be used to study how the solution of **Ax=b** affected by the changes in the entries of b.

Let $A$ be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $R^n$ , the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1,2,3 \ldots n$$

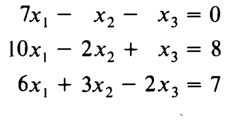
where

$$A_i(\mathbf{b}) = \left[\mathbf{a}_1 \quad . \quad . \quad \underset{\substack{\uparrow \\ Col\ i}}{\mathbf{b}} \quad . \quad . \quad \mathbf{a}_n\right]$$

Example

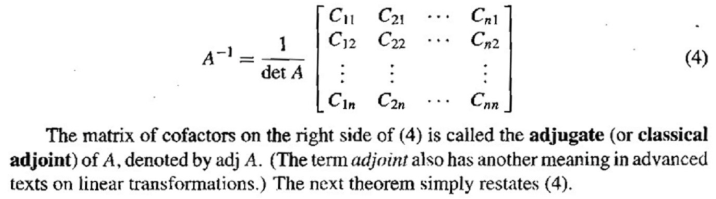
$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

Find the solution of the equation system



$$\begin{aligned} 7x_1 - x_2 - x_3 &= 0 \\ 10x_1 - 2x_2 + x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$

## The computation of A$^{-1}$



$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \quad (4)$$

The matrix of cofactors on the right side of (4) is called the **adjugate** (or **classical adjoint**) of $A$, denoted by adj $A$. (The term *adjoint* also has another meaning in advanced texts on linear transformations.) The next theorem simply restates (4).

The adjoint of $A_{nxn}$ is defined to be the transpose of the matrix of cofactors:

$$\boxed{adjA = \left[C_{ij}(A)\right]^T}$$

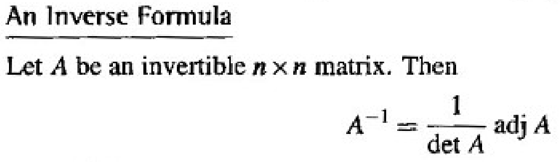
$$adjA ==> n \times n$$

Example

$$A=\begin{bmatrix}3 & 0 & -1\\ 4 & 7 & 3\\ -2 & 8 & 5\end{bmatrix}$$

$$adjA=\begin{bmatrix}11 & -26 & 46\\ -8 & 13 & -24\\ 7 & -13 & 21\end{bmatrix}^T=\begin{bmatrix}11 & -8 & 7\\ -26 & 13 & -13\\ 46 & -24 & 21\end{bmatrix}$$

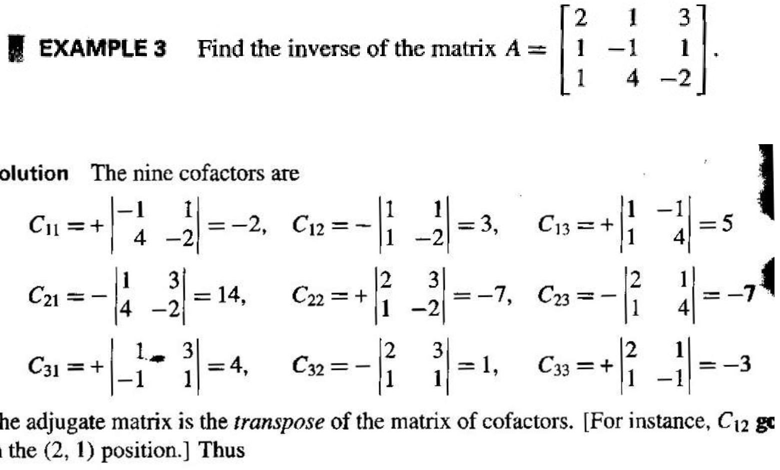
Theorem



**An Inverse Formula**

Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1}=\frac{1}{\det A}\operatorname{adj} A$$



**EXAMPLE 3** Find the inverse of the matrix $A=\begin{bmatrix}2 & 1 & 3\\ 1 & -1 & 1\\ 1 & 4 & -2\end{bmatrix}$.

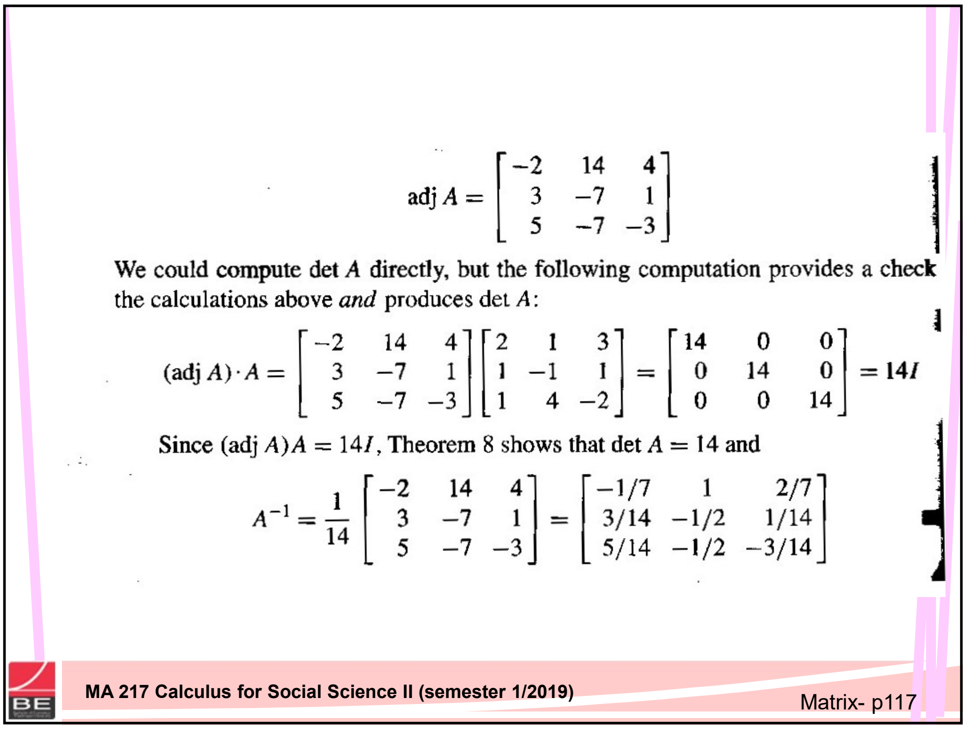
**Solution** The nine cofactors are

$$C_{11}=+\begin{vmatrix}-1 & 1\\ 4 & -2\end{vmatrix}=-2,\quad C_{12}=-\begin{vmatrix}1 & 1\\ 1 & -2\end{vmatrix}=3,\quad C_{13}=+\begin{vmatrix}1 & -1\\ 1 & 4\end{vmatrix}=5$$

$$C_{21}=-\begin{vmatrix}1 & 3\\ 4 & -2\end{vmatrix}=14,\quad C_{22}=+\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=-7,\quad C_{23}=-\begin{vmatrix}2 & 1\\ 1 & 4\end{vmatrix}=-7$$

$$C_{31}=+\begin{vmatrix}1 & 3\\ -1 & 1\end{vmatrix}=4,\quad C_{32}=-\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}=1,\quad C_{33}=+\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}=-3$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance, $C_{12}$ gc in the (2, 1) position.] Thus



$$\text{adj}\,A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

We could compute det $A$ directly, but the following computation provides a check the calculations above *and* produces det $A$:

$$(\text{adj}\,A)\cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since $(\text{adj}\,A)A = 14I$, Theorem 8 shows that det $A = 14$ and

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$