3 Differentiation: Basic Concepts

Differentiation is a technique that enables us to find out how a function changes when its argument changes.

3.1 New Vocabulary

Differentiation (n)	A technique for finding out how a function changes when its argument changes.
Differentiate (v)	A process of applying differentiation technique to a function
Derivative (n)	An outcome of differentiating a function.
Differentiable (adj)	When a derivative of a function can be obtained, that function is said to be differentiable.

3.2 Definition of Derivative

3.2.1 Mathematical definition

The derivative is given by the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: $\lim\limits_{h \to 0}$ is for "the limit as h tends to zero".

According to the formula, $f(x)$ is **differentiable** at $x = c$ if $f'(c)$ exists; that is, if the limit that defines $f'(x)$ exists when $x = c$.

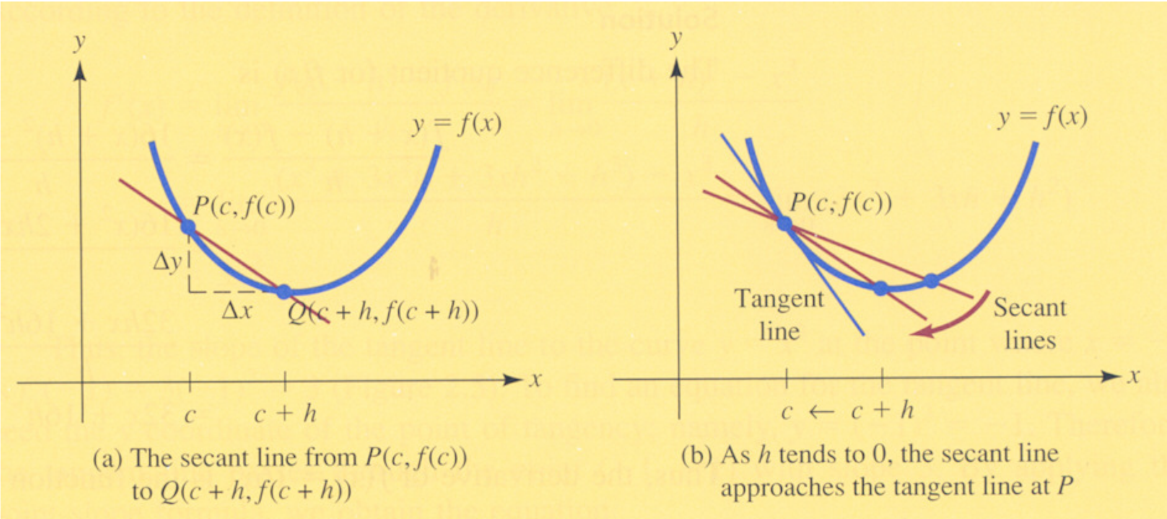


Figure 3.1: An overview of derivative formula

Hence, the derivative of a function $f(x)$ is the **SLOPE** of the **TANGENT** of the function $f(x)$ at point *x*.

- The mathematical symbols of a derivative of a function are normally $f'(x)$ or $\dfrac{dy}{dx}$ given $y = f(x)$.
- "Slope" measures the **STEEPNESS** of a line from left to right.
- "Tangent" of a curve at a particular point is a straight line which just touches the curve only at that point and only one point.

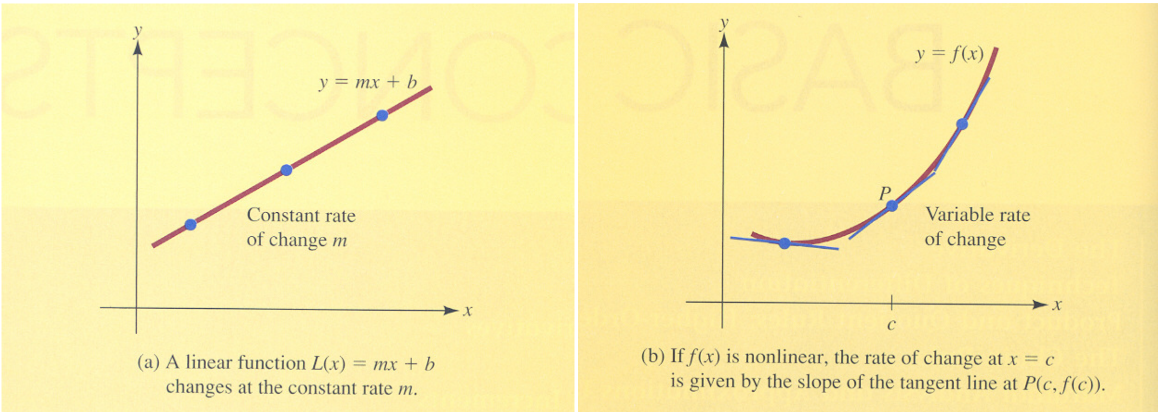


Figure 3.2: Definition of slope and tangent: (a) a linear function; (b) curve function.

3.2.2 Definition *vs*. Meaning

Slope as a Derivative: The slope of the tangent line to a curve $y = f(x)$ at the point $(c, f(c))$ is $m_{\tan} = f'(c)$.

Instantaneous Rate of Change as a Derivative: The rate of change of $f(x)$ with respect to *x* when $x = c$ is given by $f'(c)$.

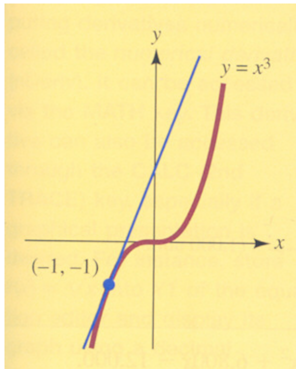
Mariginal as a derivative: In economics, the derivatives are often described by the adjective "marginal." For example, the derivative of a cost function is called the marginal cost function.

3.2.3 Notation

Differentiate $y = f(x)$ with respect to ("w.r.t.") x equals to find:

- $f'(x)$ is pronounced "*f* dash *x*" or "*f* prime *x*"
- $\dfrac{dy}{dx}$ "dee *y* by dee *x*" or "dee *y*, dee *x*"
- $\dfrac{d}{dx}f(x)$ "dee by dee *x* of *f* *x*"
- $\dot{f}$ "*f* dot" A special case when *f* is differentiated w.r.t. time.
- $f'(c)$, $\left.\dfrac{dy}{dx}\right|_{x=c}$, $\left.\dfrac{d}{dx}f(x)\right|_{x=c}$ The derivative at $x = c$.

Ex. 1: Use ***<u>the definition of differentiation</u>*** to find the derivatives of the following functions.



(a) For $f(x) = x$, find $f'(1)$ and $f'(2)$. Ans: 1, 2

(b) For $f(x) = x^2$, find $f'(1)$, $f'(3)$ and $f'(-2)$ and the equation of the tangent at point (3,9).

Ans: $f'(x) = 2x$, 2, 6,-4, $y = 6x - 9$

(c) For $f(x) = x^3$, find $f'(x)$, $f'(-1)$ and the equation of the tangent where $x = -1$. Ans: $3x^2$, 3, $y = 3x + 2$

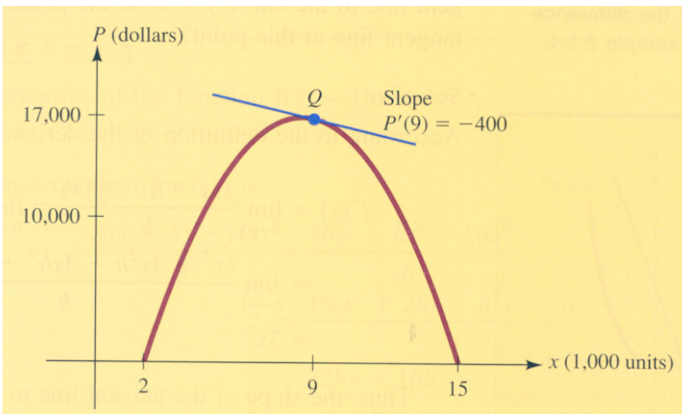
(d) For $f(x) = 9x^3 - 7x^2 + 15$, find $f'(x)$. Ans: $27x^2 - 14x$

Ex. 2: A manufacturer determines that when <u>*x* thousand units</u> of a particular commodity are produced, the profit generated will be

$$P(x) = -400x^2 + 6{,}800x - 12{,}000 \quad \text{dollars}$$

At what rate is profit changing with respect to the level of production *x* when 9,000 units are produced?

Ans: -400



3.2.4 Significance of the Sign of the Derivative $f'(x)$

If the function *f* is differentiable at *x* = *c*, then

f is **increasing** at $x = \text{c}$ if $f'(c) > 0$

and

f is **decreasing** at $x = \text{c}$ if $f'(c) < 0$

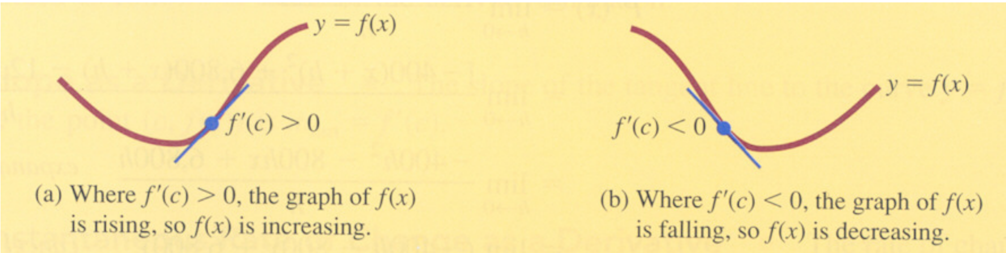


Figure 3.3: Increasing and decreasing functions

Ex. 3: Use the definition of differentiation to compute the derivative of $f(x) = \sqrt{x}$, then use the derivative to:

(a) Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point where $x = 4$.

(b) Find the rate at which $y = \sqrt{x}$ is changing with respect to *x* when $x = 1$.

Ans: $\dfrac{df}{dx} = \dfrac{1}{2\sqrt{x}}$, $f'(4) = \dfrac{1}{4}$, $y = \dfrac{1}{4}x + 1$, $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{1}{2}$

3.3 Continuity of a Differentiable Function

If the function *f* (*x*) is differentiable at *x* = *c*, then it is also continuous at point *x* = *c*.

However, a continuous function *f* (*x*) is not always differentiable. A continuous function *f*(*x*) will not be differentiable at *x* = *c* if $f'(x)$ becomes infinite at *x* = *c* or if the graph of *f*(*x*) has a "shape" point at $P(c, f(c))$; that is, a point where the curve makes an abrupt change in direction.

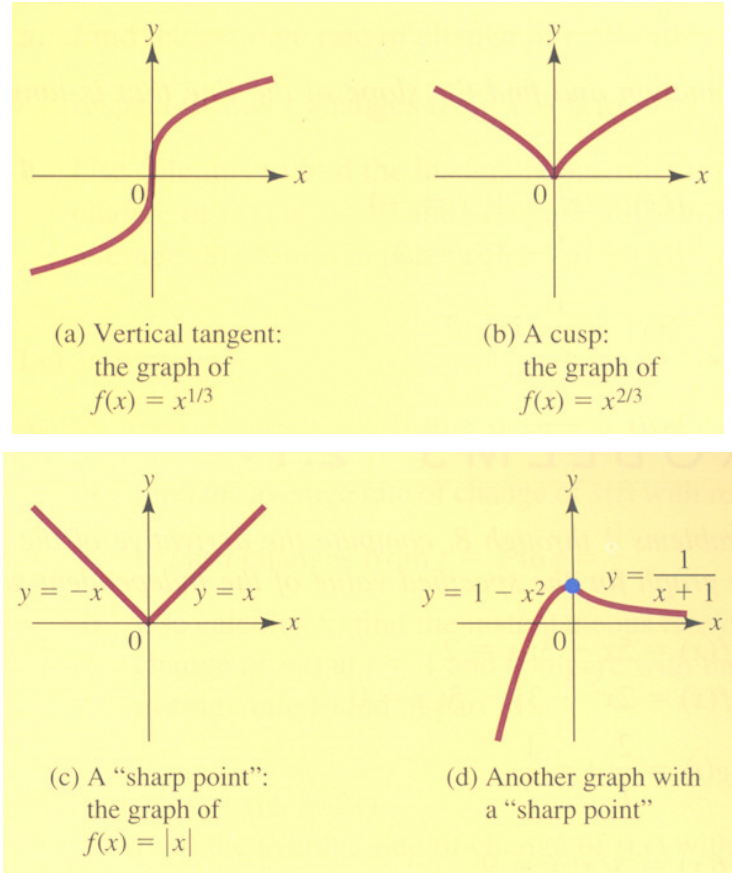
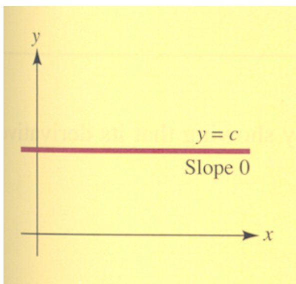


Figure 3.4: Examples of continuous functions that are not differentiable.

3.4 Differentiation Techniques

Finding derivatives using the definition could be very tedious. Instead there are simple formulas that can be used to help finding derivatives. These formulas can be derived using definition of derivative.

3.4.1 Constant Function Rule



For any constant c,

$$\frac{d}{dx}(c) = 0$$

The derivative of any constant function is zero.

(Prove using definition of derivative!!!)

3.4.2 Power Rule

For any real number *n*,

$$\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$$

The derivative of *x* to the power *n* is *n* times *x* to the power *n* minus 1.

(Prove using definition of derivative!!!)

Ex. 4: Find the derivatives of x, x^2 and x^3 [Ex. 1 (a)-(c)]

Ex. 5: Verify the power rule for the function $f(x) = \dfrac{1}{x^2} = x^{-2}$ by showing that its derivative is $f'(x) = -2x^{-3}$.

3.4.3 Constant Multiply Property

If *c* is a constant and $f(x)$ is differentiable, then

$$\frac{d}{dx}\left[c\,f(x)\right] = c\,f'(x)$$

The derivative of a constant times a differentiable function is the constant times the derivative of the function.

Ex. 6: Find the derivatives of the following functions.

(a) $f(x) = 9x^3$

Ans: $27x^2$

(b) $f(x) = 4\sqrt{x}$

Ans: $2/\sqrt{x}$

(c) $f(x) = -2x^{-1.3}$

Ans: $2.6x^{-2.3}$

(d) $f(x) = \dfrac{-7}{\sqrt{x}}$

Ans: $\dfrac{7}{2}x^{-\frac{3}{2}}$

3.4.4 Sum and Difference Property

If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

The derivative of the sum/difference of two differentiable functions is the sum/difference of the derivatives.

Ex. 7: Find the derivatives of the following functions.

(a) $f(x) = 9x^3 - 7x^2 + 15$ [Ex. 1(d)] Ans: $27x^2 - 14x$

(b) $f(x) = x^{-2} + 7$ Ans: $-2x^{-3}$

(c) $f(x) = 2x^5 - 3x^{-7}$ Ans: $10x^4 + 21x^{-8}$

(d) $f(x) = 5x^3 - 4x^2 + 12x - 3$ Ans: $15x^2 - 8x + 12$

3.4.5 Product Rule

If $f(x)$ and $g(x)$ are differentiable at x, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Ex. 8: Determine the derivative of the product of $f(x)$ and $g(x)$ by (i) expanding the expression into polynomials and (ii) using the product rule without expanding where

(a) $f(x) = x^2$ and $g(x) = x^3$ Ans: $5x^4$

(b) $f(x) = x - 1$ and $g(x) = 3x - 2$ Ans: $6x - 5$

(c) $f(x) = 5x + 3$ and $g(x) = x^2 - x + 1$ Ans: $15x^2 - 4x + 2$

Ex. 9: Find the derivative of $y(x) = 2x^{\frac{3}{2}}\left(x^3 - 2\right)$. Ans: $9x^{\frac{7}{2}} - 6x^{\frac{1}{2}}$

Ex. 10: For the curve $y = (2x+1)(2x^2 - x - 1)$,

(a) Find y'.

(b) Find an equation for the tangent line to the curve at the point where $x = 1$.

(c) Find all points on the curve where the tangent line is horizontal.

Ans: $y' = 12x^2 - 3$, $y = 9x - 9$, $\left(\frac{1}{2}, -2\right), \left(-\frac{1}{2}, 0\right)$

3.4.6 Quotient Rule

If $f(x)$ and $g(x)$ are differentiable at x, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

The derivative of the quotient of two functions is the bottom function times the derivative of the top function minus the top function times the derivative of the bottom function, all over the bottom function squared.

Ex. 11: Differentiate the quotient $Q(x) = \dfrac{x^2 - 5x + 7}{2x}$ by

(a) Dividing through first.

(b) Using the quotient rule. Ans: $\dfrac{1}{2} - \dfrac{7}{2x^2}$

Ex. 12: Differentiate the following rational functions

(a) $y(x) = \dfrac{3x}{x^2 + 2}$ Ans: $\dfrac{6 - 3x^2}{\left(x^2 + 2\right)^2}$

(b) $y(x) = \dfrac{3x - 2}{2x + 1}$ Ans: $\dfrac{7}{\left(2x + 1\right)^2}$

3.4.7 The Chain Rule

If $y = f(u)$ is differentiable function of *u* and *u* = *g*(*x*) is in turn a differentiable function of *x*, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

A change in *x* causes a change in *u* and leads to a change in *y*. This is a composite function $y = f(g(x))$.

Ex. 13: Differentiate $y = (3x+1)^2$ by

(a) Expanding into polynomials.

(b) Chain rule where $y = f(u) = u^2$ and $u = g(x) = 3x+1$.

Ans: $18x+6$

Ex. 14: Differentiate the following functions with respect to *x*.

(a) $y(x) = 4(2x^3-1)^7$ Ans: $168x^2(2x^3-1)^6$

(b) $y(u) = u^n$ and *u* is a function of *x* Ans: $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

3.4.8 General Power Rule

(This is a special case of the **chain rule**.)

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Ex. 15: Differentiate the following functions.

(a) $y(x) = \frac{1}{1+x^2}$ Ans: $\frac{-2x}{(1+x^2)^2}$

(b) $y(x) = \sqrt{2x-3}$ Ans: $\frac{1}{\sqrt{2x-3}}$

(c) $y(x) = (2x^4 - x)^3$ Ans: $3(2x^4-x)^2(8x^3-1)$

(d) $y(x) = \sqrt{x^2+3x+2}$ Ans: $\frac{2x+3}{2\sqrt{x^2+3x+2}}$

(e) $y(x) = \frac{1}{(2x+3)^5}$ Ans: $-\frac{10}{(2x+3)^6}$

Ex. 16: Differentiate the following functions. (Combined Rules)

(a) $y(x) = x^3\left(x^2+1\right)^4$ Hint: Use Product Rule then Chain Rule

Ans: $x^2\left(x^2+1\right)^3\left(11x^2+3\right)$

(b) $y(x) = \left(\dfrac{x}{x+\alpha}\right)^{\beta}$ Hint: Use Chain Rule then Quotient Rule

Ans: $\dfrac{\alpha\beta x^{\beta-1}}{\left(x+\alpha\right)^{\beta+1}}$

(c) $y(x) = 5\left(1+\sqrt{x^3+1}\right)^{25}$ Hint: Use Chain Rule twice.

Ans: $\dfrac{375}{2}\left(1+\sqrt{x^3+1}\right)^{24}\left(x^3+1\right)^{-\frac{1}{2}}x^2$

Ex. 17: Differentiate the function $y(x) = (3x+1)^4(2x-1)^5$ and simplify your answer. Then find all values of $x = c$ for which the tangent line to the graph of $f(x)$ at $(c, f(c))$ is horizontal.

Ans: $f'(x) = 2(3x+1)^3(2x-1)^4(27x-1)$, $c = -\dfrac{1}{3}, \dfrac{1}{2}$ or $\dfrac{1}{27}$.

Ex. 18: The manager of an appliance manufacturing firm determines that when blenders are priced at p dollars apiece, the number sold each month can be modeled by

$$D(p) = \frac{8{,}000}{p}.$$

The manager estimates that t months from now, the unit price of the blenders will be $p(t) = 0.06t^{\frac{3}{2}} + 22.5$ dollars. At what rate will the monthly demand for blenders $D(p)$ be changing 25 months from now? Will it be increasing or decreasing at this time?

Ans: $\left.\dfrac{dD}{dt}\right|_{\substack{t=25\\p=30}} = -4$

3.5 Relative and Percentage Rate of Change:

The relative rate of change of a quantity $Q(x)$ with respect to *x* is given by the ratio

$$\text{Relative rate of change of } Q(x) = \frac{Q'(x)}{Q(x)} = \frac{\frac{dQ}{dx}}{Q}$$

The corresponding **percentage rate of change** of $Q(x)$ with respect to *x* is

$$\text{Percentage rate of change of } Q(x) = \frac{100 \cdot Q'(x)}{Q(x)}$$

Ex. 19: The gross domestic product (GDP) of a certain county was $N(t) = t^2 + 5t + 106$ billion dollars *t* years from 1995.

(a) At what rate was the GDP changing with respect to time in 2005?

(b) At what percentage rate was the GDP changing with respect to time in 2005?

Ans: $N'(t) = 2t + 5$, 25 billion dollars per year, 9.77% per year

3.6 The Higher-Order Derivative

It is possible that the rate of change of a function is itself a rate of change. For example, the acceleration of a car is the rate of change with respect to time of its velocity, which in turn is the rate of change with respect to time of its position.

$y = f(x)$

$y' = \frac{dy}{dx}$ is the derivative of *y*.

$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$ is the second derivative of *y*.

$y''' = \frac{dy''}{dx} = \frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = \frac{d^3y}{dx^3} \neq \left(\frac{dy}{dx}\right)^3$ is the third derivative of *y*.

Ex. 20: Find the second derivative of the function $f(x) = \dfrac{3x-2}{(x-1)^2}$.

Ans: $f'(x) = \dfrac{1-3x}{(x-1)^3}, f''(x) = \dfrac{6x}{(x-1)^4}$

Ex. 21: Determine $y', y'', y''', y^{(4)}$ and $y^{(5)}$, for

(a) $y = f(x) = x^4 + x^3 + x^2 + x + 1$

(b) $y = f(x) = \dfrac{1}{x}$

3.7 Marginal Analysis and Approximations Using Increments

Economists often use the word **marginal** to signify a rate of change, that is, a derivative, for example ***marginal cost, marginal revenue, and marginal profit.***

3.7.1 Marginal cost

If x is the number of product produced and sold and total cost is a function $C(x)$ then marginal cost can be defined as:

$$\textbf{\textit{Marginal cost}} = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

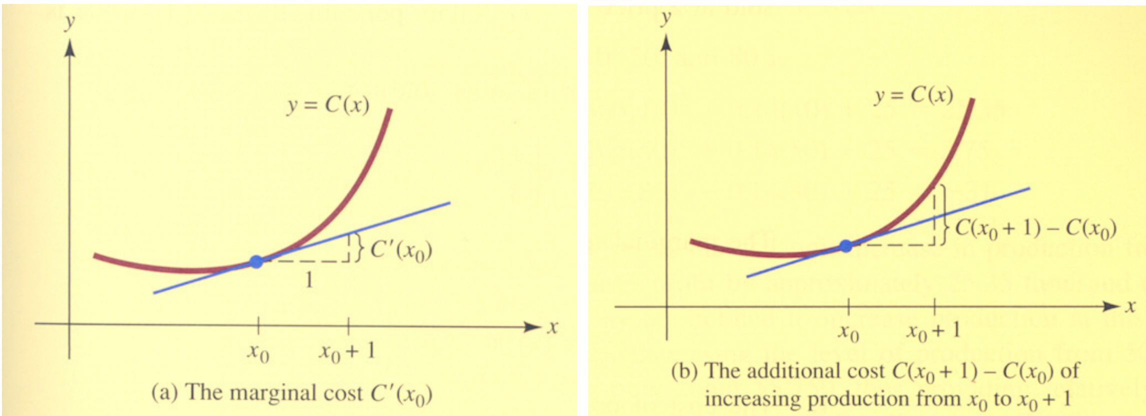


Figure 3.5 Graphical presentations of marginal cost: (a) definition and (b) meaning.

According to Figure 3.5(a), the marginal cost when the number of products produced/sold is x_0, can be found as a slope of a tangent to the curve $C(x)$ at (x_0, $C(x_0)$). The slop in turn approximates $\Delta C / \Delta x$ or ΔC when Δx equals to 1.

Hence, the marginal cost approximates the additional cost $C(x_0 + 1) - C(x_0)$ of increasing the level of production by 1 unit from x_0 to $x_0 + 1$ (See Figure 3.5(b)).

$$C'(x_0) \approx \frac{C(x_0 + 1) - C(x_0)}{1} = C(x_0 + 1) - C(x_0)$$

3.7.2 Marginal revenue

As aforementioned in Chapter 2, total revenue, $R(x) = p(x) \bullet x$ then marginal revenue is defined as:

$$\textit{\textbf{Marginal revenue}} = \quad R'(x_0) = \lim_{h \to 0} \frac{R(x_0 + h) - R(x_0)}{h}$$

Similarly to marginal cost,

$$R'(x_0) \approx \frac{R(x_0 + 1) - R(x_0)}{1} = R(x_0 + 1) - R(x_0)$$

Hence, the marginal revenue approximates the additional revenue $R(x_0 + 1) - R(x_0)$ of increasing the level of selling by 1 unit from x_0 to $x_0 + 1$.

3.7.3 Marginal profit

As Total profit $= P(x) = R(x) - C(x)$, marginal profit is then:

$$\textit{\textbf{Marginal profit}} = \quad P'(x) = R'(x) - C'(x)$$

$$= \textit{\textbf{Marginal revenue}} - \textit{\textbf{Marginal cost}}$$

Similarly to marginal cost and marginal revenue, the marginal profit approximates the additional profit obtained by selling one more unit.

Note: Marginal ***Something*** is the instantaneous rate of change of that ***Something*** relative to number of product produced and sold at a given production level.

Ex. 22: A manufacturer estimates that when x units of a particular commodity are produced, the total cost will be $C(x) = \frac{1}{8}x^2 + 3x + 98$ dollars. Furthermore, all x units will be sold when the price is $P(x) = \frac{1}{3}(75 - x)$ dollars per unit.

(a) Find the marginal cost and marginal revenue.

(b) Use marginal cost to estimate the cost of producing the ninth unit.

(c) What is the actual cost of producing the ninth unit?

(d) Use marginal revenue to estimate the revenue derived from the sale of the ninth unit.

(e) What is the actual revenue derived from the sale of the ninth unit?

Ans:

$$C'(x) = \frac{1}{4}x + 3, R'(x) = 25 - \frac{2}{3}x, C'(8) = 5, C(9) - C(8) = 5.13, R'(8) = 19.67, R(9) - R(8) = 19.33$$

Ex. 23: A manufacture of digital cameras estimates that when x hundred cameras are produced, the total profit will be

$$P(x) = -0.0035x^3 + 0.07x^2 + 25x - 200 \qquad \text{thousand dollars.}$$

(a) Find the marginal profit.

(b) What is the marginal profit when the level of production is $x = 10$, $x = 50$, and $x = 80$?

(c) Interpret these results.

Ans: $P'(x) = -0.0105x^2 + 0.14x + 25, P'(10) = 25.35, P'(50) = 5.75, P'(80) = -31$

3.7.4 Approximation by Increments:

If $f(x)$ is differentiable at $x = x_0$ and Δx is a small change in *x*, then

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x$$

Or, equivalently, if $\Delta f = f(x_0 + \Delta x) - f(x_0)$, then

$$\Delta f \approx f'(x_0) \cdot \Delta x$$

Ex. 24: Suppose the total cost in dollars of manufacturing *q* units of a certain commodity is $C(q) = 3q^2 + 5q + 10$. If the current level of production is 40 units, estimate how the total cost will change if 42 units are produced.

Ans: $\Delta C \approx C'(40) \cdot 2 = 490$

For practice, compute the actual change in cost caused by the increase in the level of production from 40 to 42 and compare your answer with the approximate. Is the approximate a good one?

3.7.5 Approximation of Percentage Change:

The percentage rate of change of a quantity is defined as the change in the quantity as a percentage of its size prior to the change. If Δx is a (small) change in *x*, the corresponding percentage change in the function $f(x)$ is

$$\text{Percentage change in } f = 100\frac{\Delta f}{f(x)} \approx 100\frac{f'(x)\Delta x}{f(x)}$$

Ex. 25: The GDP of a certain country was $N(t) = t^2 + 5t + 200$ billion dollars *t* years after 1997. Use calculus to estimate the percentage change in the GDP during the first **quarter** of 2005.

Ans: Percentage change in $N \approx 100\frac{N'(t)\Delta t}{N(t)}$, $t = 8, \Delta t = 0.25, N'(t) = 2t + 5$, 1.73%

3.8 Differentials

If the **differential of *x*** is dx, and if $y = f(x)$ is a differentiable function of *x*, then $dy = f'(x)dx$ is the **differential of *y*.**

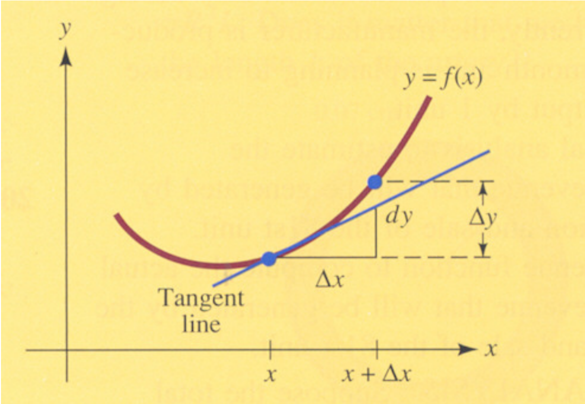


Figure 3.6 Difference of differential of *y* and Δ*y*.

For example, according to Figure 3.6, when $dx = \Delta x$, then the differential of *y*, $dy = f'(x)\Delta x$. It should be noted that it is not necessary for dy to be equal to Δy.

Ex. 26: In each case, find the differential of $y = f(x)$.

(a) $f(x) = x^3 - 7x^2 + 2$

(b) $f(x) = (x^2 + 5)(3 - x - 2x^2)$

Ans: $dy = (3x^2 - 14x)dx, dy = [(x^2 + 5)(-1 - 4x) + (2x)(3 - x - 2x^2)]dx$

3.9 Newton's Method

3.9.1 Introduction

In order to find real roots of $f(x) = 0$, $f(x) = 0$ may be solved mathematically, particularly when $f(x)$ is a polynomial function that can be factorized. However, for complex functions, the functions may be sketched as a graph $y = f(x)$ and the root is found by locating the point where $y = 0$.

Sometimes to accurately locate the roots from a sketch can be a problem. The good news is that calculus can be used to approximate real roots of an equation and the method is called Newton's method.

3.9.2 Newton's Method

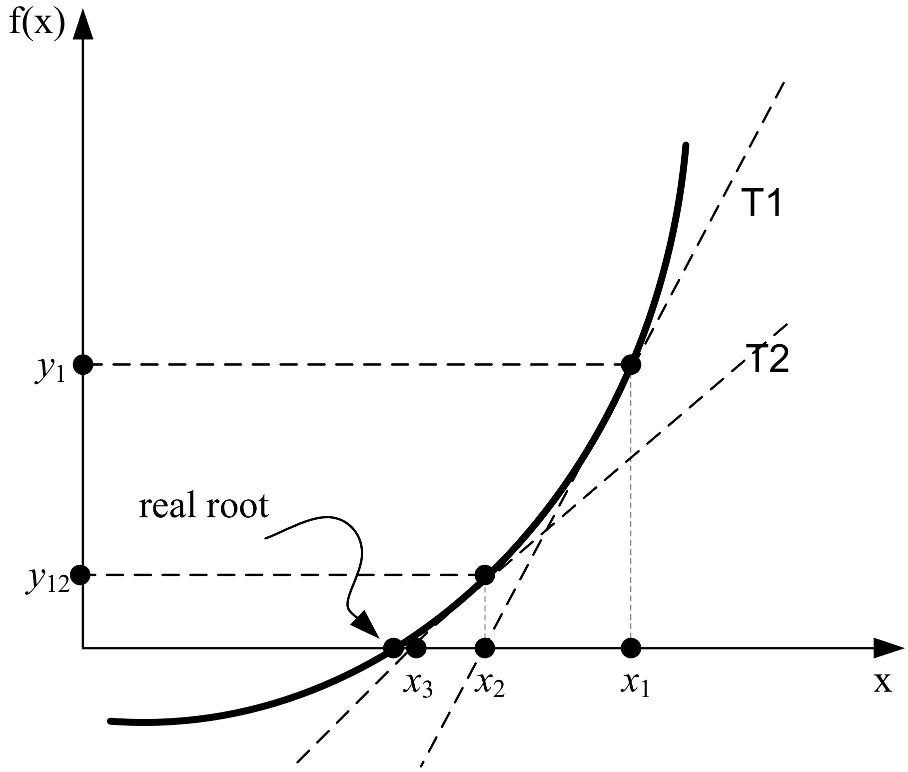


Figure 3.7 Finding real root of function using Newton's method

According to Figure 3.7, first the root is guessed to be close to x_1. Tangent line (dotted line T1) to the curve at $(x_1, f(x_1))$ intersects the x-axis at point $(x_2, 0)$ which is closer to the root (a better approximate than x_1). This process can continue ……

For T_1, $$f'(x_1) = \frac{f(x_1) - 0}{x_1 - x_2}$$

Rearrange, $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

For T_2, $$f'(x_2) = \frac{f(x_2) - 0}{x_2 - x_3}$$

Rearrange, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

Hence, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ $n = 1, 2, \ldots.$ **[Given!]**

3.9.3 Initial estimate

An initial estimate for a root of $f(x)$ or x_1 in Section 3.9.2 is required for Newton's method. There are **2** ways to do this:

1) Sketch a graph $y = f(x)$ and estimate the root which is a point where $y = 0$ (x-intercept) (See Figure 3.8(a)).

2) If $f(x)$ is continuous on the interval $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs (See Figure 3.8(b)), then the equation $f(x) = 0$ has at least one real root between a and b. Either a or b can be taken as the initial estimate. However, $f(x)$ that is closer to zero is recommended.

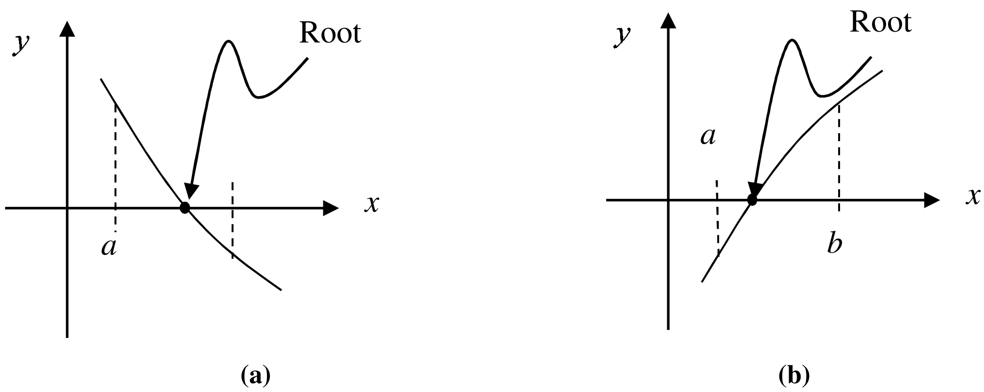


Figure 3.8 Initial estimates for Newton's method

Ex. 27: Use Newton's method to approximate the indicated root of the given equation. Continue the approximation procedure until the difference of two successive approximation is less than 0.0001.

(a) $x^3 - 71 = 0$, use $x_1 = 4$. [Ans.: $x_4 = 4.14081$]

(b) $x^3 - 2x - 5 = 0$, find a root between 2 and 3. [Ans.: $x_4 = 2.09455$]

(c) $x^3 = x + 1$, find a root between 1 and 2. [Ans.: $x_{n+1} = \dfrac{2x_n^3 + 1}{3x_n^2 - 1}$; $x_6 = 1.32472$]

(d) $x^4 = 1 - 4x$, find a root between -2 and -1.[Ans.: $x_{n+1} = \dfrac{3x_n^4 + 1}{4x_n^3 + 4}$; $x_5 = -1.66325$]

Ex. 28(a) $f(x) = x^3 - 71$

$f'(x) = 3x^2$

Newton's Approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 71}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + 71}{3x_n^2}$$

$x_1 = 4$

$n = 1$, $\quad x_2 = \dfrac{2(x_1)^3 + 71}{3(x_1)^2} = \dfrac{2(4)^3 + 71}{3(4)^2} = 4.14583$

$n = 2$, $\quad x_3 = \dfrac{2(x_2)^3 + 71}{3(x_2)^2} = \dfrac{2(4.14583)^3 + 71}{3(4.14583)^2} = 4.14082$

$n = 3$, $\quad x_4 = \dfrac{2(x_3)^3 + 71}{3(x_3)^2} = \dfrac{2(4.14082)^3 + 71}{3(4.14082)^2} = 4.14081$

$|x_4 - x_3| = 0.00001 < 0.0001$

Hence, the approximate root at $x^3 - 71 = 0$ is 4.14081 (x_4).

Ex. 28(b) $f(x) = x^3 - 2x - 5 \quad \& \quad x = [2,3]$

$$f(2) = (2)^3 - 2(2) - 5 = -1$$

$$f(3) = (3)^3 - 2(3) - 5 = 16$$

$f(2)$ and $f(3)$ have different signs so there is a root of $x^3 - 2x - 5 = 0$ between 2 and 3.

Ans: Since $f(2)$ is closer to zero than $f(3)$. Use $x_1 = 2$.

$$f'(x) = 3x^2 - 2$$

Newton's Approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$$x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

$n = 1$, $$x_2 = \frac{2(x_1)^3 + 5}{3(x_1)^2 - 2} = \frac{2(2)^3 + 5}{3(2)^2 - 2} = 2.10000$$

$n = 2$, $$x_3 = \frac{2(x_2)^3 + 5}{3(x_2)^2 - 2} = \frac{2(2.1)^3 + 5}{3(2.1)^2 - 2} = 2.09457$$

$n = 3$, $$x_4 = \frac{2(x_3)^3 + 5}{3(x_3)^2 - 2} = \frac{2(2.09457)^3 + 5}{3(2.09457)^2 - 2} = 2.09455$$

$|x_4 - x_3| = 0.00002 < 0.0001$

Hence, the approximate root at $x^3 - 2x - 5 = 0$ is 2.09455 (x_4).

Ex. 29: For $\ln x + x = 5$, find a root to three-desimal-place accuracy. Use $x_1 = 3$.

[Ans: $x_{n+1} = x_n - \dfrac{\ln(x_n) + x_n - 5}{\dfrac{1}{x_n} + 1}$; $x_4 = 3.693$]

3.10 Implicit Differentiation and Related Rates

3.10.1 Implicit differentiation

Implicit differentiation is a technique for differentiating functions that are not given in the usual form $y = f(x)$ or it is difficult to rearrange the function in this form. y is defined implicitly as a differentiable function of x.

Suppose an equation defined y implicitly as a differentiable function of x. To find $\frac{dy}{dx}$;

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x and use the chain rule when differentiating terms containing y.
2. Solve the differential equation algebraically for $\frac{dy}{dx}$.

Ex. 30: For $xy = 10$, find $\frac{dy}{dx}$ using

(a) Implicit differentiation

(b) Rearranging $y = f(x)$

Ans: $y' = -\frac{10}{x^2}$

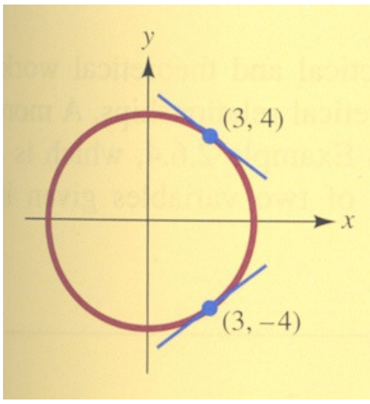
Ex. 31: Find $\frac{dy}{dx}$ for the following expressions

(a) $x^2 + y^2 = 25$

(b) $x^2 y + y^2 = x^3$

Ans: $\frac{dy}{dx} = -\frac{x}{y}, \frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}$

Ex. 32: Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3, 4). What is the slope at the point (3, -4)?



Ans: $\left.\dfrac{dy}{dx}\right|_{(3,4)} = -\dfrac{3}{4}, \left.\dfrac{dy}{dx}\right|_{(3,-4)} = \dfrac{3}{4}$

Ex. 33: Suppose the output at a certain factory is $Q = 2x^3 + x^2 y + y^3$ units, when x is the number of hours of skilled labour force consists of 30 hours of skilled labour and 20 hours of unskilled labour. Use calculus to estimate the change in unskilled labour y that should be made to offset a 1-hour increase in skilled labour x so that output will be maintained at its current level.

Ans: $\dfrac{dy}{dx} = -\dfrac{6x^2 + 2xy}{x^2 + 3y^2}, \left.\dfrac{dy}{dx}\right|_{\substack{x=30 \\ y=20}} = -3.14$

3.10.2 Related Rate

In certain practical problems, x and y are related by an equation and can be regarded as a functions of a third variable, t, which often represents time. Then implicit differentiation can be used to relate $\dfrac{dx}{dt}$ to $\dfrac{dy}{dt}$. This is known as **related rate.**

Ex. 34: The manager of a company determines that when q hundred units of a particular commodity are produced, the total cost of production is C thousand dollars, where $C^2 - 3q^3 = 4{,}275$. When 1,500 units are being produced, the level of production is increasing at the rate of 20 units per week. What is the total cost at this time and at what rate it changing?

Ans: $\frac{dC}{dt} = \frac{9q^2}{2C}\frac{dq}{dt}, q = 15, C = 120, \frac{dq}{dt} = 0.2, \frac{dC}{dt} = 1.6875$

The cost of producing 1,500 units is 120,000 dollars and at this level of production, the total cost is increasing at the rate of 1,687.50 dollars per week.

Ex. 35: When the price of a certain commodity is p dollars per unit, the manufacturer is willing to supply x thousand units, where

$$x^2 - 2x\sqrt{p} - p^2 = 31$$

How fast is the supply changing when the price is \$9 per unit and is increasing at the rate of 20 cents per week?

Ans: $p = 9, \frac{dp}{dt} = 0.2, x = 14, 2x\frac{dx}{dt} - 2\left[\left(\frac{dx}{dt}\right)\sqrt{p} + x\left(\frac{1}{2}\frac{1}{\sqrt{p}}\frac{dp}{dt}\right)\right] - 2p\frac{dp}{dt} = 0$, $\frac{dx}{dt} \approx 0.206$

This supply is increasing at the rate of 206 units per week.