Solution: Quiz 8

1. Consider a relation

$$f = \left\{ (x,y) \in \mathbb{R} \times \mathbb{R} \middle| \quad y = (x^2+x+1)^{1/5} \right\}.$$

Determine x and y intercepts, symmetry, relative extrema, concavity , and the inflection points of f. Sketch the curve of f.

Note: $f(x) = (x^2+x+1)^{1/5}$, $f'(x) = \dfrac{2x+1}{5(x^2+x+1)^{4/5}}$, $f''(x) = -\dfrac{6}{25}\dfrac{x^2+x-1}{(x^2+x+1)^{9/5}}$.

Solution:

- **x-intercepts** can be found by setting $y=0$. However, $x^2+x+1>0$ for all $x \in \mathbb{R}$ (using $b^2-4ac = 1^2-4(1)(1) = -3 < 0$ and $a=1>0$). That is $0 \neq (x^2+x+1)^{1/5}$ and there is no x-intercept.

  y-intercepts can be found by setting $x=0$, which gives $y = (0^2+0+1)^{1/5} = 1$. I.e. the y-intercept is $(0,1)$.
- **Symmetry:** From $f(-x) = ((-x)^2+(-x)+1)^{1/5} = (x^2-x+1)^{1/5}$, we have $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, which imply that $f(x)$ is not **symmetric about the y-axis** and $f(x)$ is not **symmetric about the origin**, respectively. Also, since, for $(x,y) \in f$, $(x,-y) \notin f$ (i.e. f is a function), it is **not** symmetric about the x-axis.
- **Relative extremum:** Notice that, since $(x^2+x+1)^{1/5} > 0$, the critical points only occurs when $f'(x) = 0$ or $2x+1=0$, which implies that $x = -\frac{1}{2}$. I.e., the only critical point is $x = -\frac{1}{2}$.

  $f'(x) < 0 \quad \Leftrightarrow \quad 2x+1<0 \quad \Leftrightarrow \quad x < -0.5 \Leftrightarrow f(x)$ is decreasing on $(-\infty, -0.5)$
  $f'(x) > 0 \quad \Leftrightarrow \quad 2x+1>0 \quad \Leftrightarrow \quad x > -0.5 \Leftrightarrow f(x)$ is increasing on $(-0.5, \infty)$
  By using the **First Derivative test**, $f(-0.5)$ is the relative minimum.
- **Concavity and inflection points:** From $f''(x) = -\dfrac{6}{25}\dfrac{x^2+x-1}{(x^2+x+1)^{9/5}}$,

  since $(x^2+x+1)^{9/5} > 0$ for all $x \in \mathbb{R}$, $f''(x) = 0 \Leftrightarrow x^2+x-1 = 0$ or $x = \frac{-1\pm\sqrt{1-4(1)(-1)}}{2} = \frac{-1\pm\sqrt{5}}{2}$. Let $x_1 = \frac{-1-\sqrt{5}}{2}$, $x_2 = \frac{-1+\sqrt{5}}{2}$. Then $x^2+x+1 = (x-x_1)(x-x_2)$ and the sign of $f''(x)$ is opposite to the sign of $x^2+x+1 = (x-x_1)(x-x_2)$.

	$x \in (-\infty, x_1)$	$x \in (x_1, x_2)$	$x \in (x_2, \infty)$
sign of $(x-x_1)(x-x_2)$	$(-)(-)$	$(-)(+)$	$(+)(+)$
sign of $f''(x)$	$(-)$	$(+)$	$(-)$

  From above, $f(x)$ is concave up for $x \in (x_1, x_2)$ and $f(x)$ is concave down for $x \in (-\infty, x_1) \cup (x_2, \infty)$. The inflection points are $(x_1, f(x_1))$ and $(x_2, f(x_2))$ since $f''(x)$ changes sign at these two points.
  The sketch of this curve is shown below.

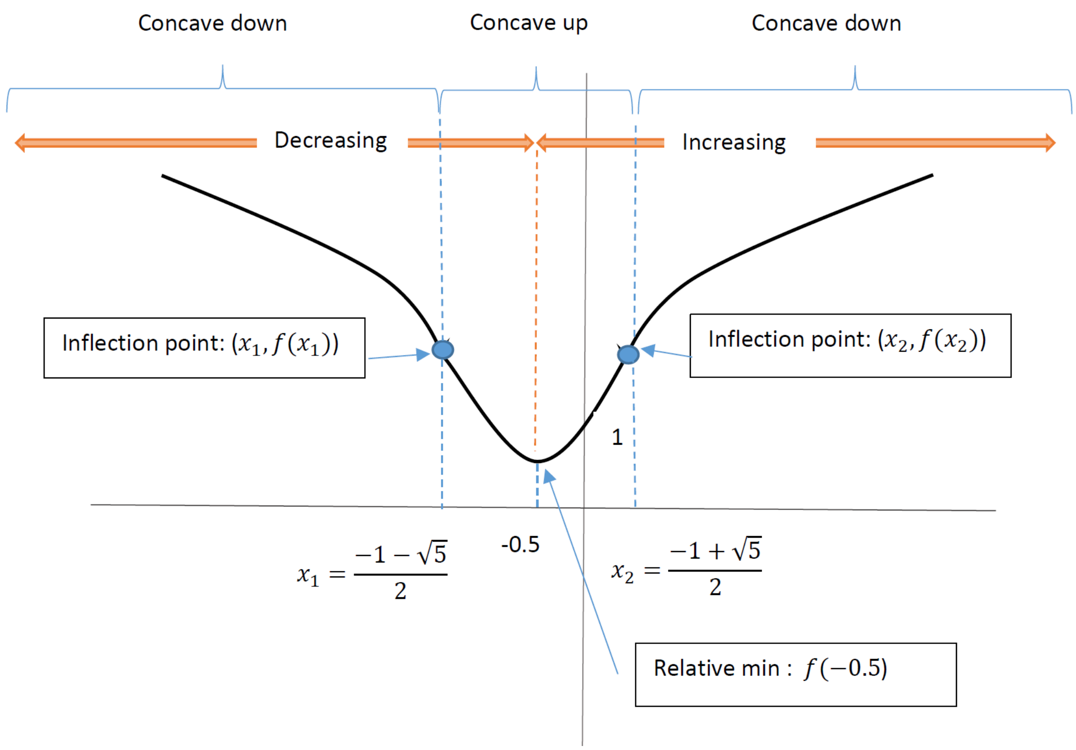

Figure 1: $f = \{(x, y) | y = (x^2 + x + 1)^{1/5}\}$

Note: To specify relative extremum from the critical point $x = -1/2$, we can also use the **Second Derivative test**:

since $f''(x) = -\dfrac{6}{25}\dfrac{x^2+x-1}{(x^2+x+1)^{9/5}}$ and $(x^2+x+1)^{9/5} > 0$ for all $x \in \mathbb{R}$, then

$$f''(-1/2) = -\frac{6}{25}\frac{\overbrace{(-1/2)^2 + (-1/2) - 1}^{<0}}{\underbrace{((-1/2)^2 + (-1/2) + 1)^{9/5}}_{>0}} > 0,$$ which implies that $f(-0.5)$ is the relative minimum of f.