Note Title 9/7/2016

Last time

2.2 Higher-Order Differential Equations

2.2.1 2^{nd} order linear DE w constant coefficients and constant term

$$y''(t) + a_1 y'(t) + a_2 y(t) = b \qquad \text{--- (1)}$$

$$a_1, a_2, b \Rightarrow \text{constant}$$

General Sol^n: $\quad y(t) = y_c + y_p$

⊙ Find Particular Integral $(y_p) \Rightarrow 3$ cases.

Today

⊙ Find the complementary function (y_c)

we set $w(t) = 0$ in this case $b = 0$

$$\left[y''(t) + u_1(t) y'(t) + u_2(t) y(t) = w(t) \right]$$

Solving a homogeneous version of (1)

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0 \qquad \text{--- (5)}$$

Note $\quad y'(t) + a y(t) = 0$

$$y_c = A e^{at} \quad ; \quad A \neq 0$$

So we guess y_c in this case $= Ae^{rt}$; $A \neq 0$

$$\left.\begin{aligned} y(t) &= Ae^{rt} \\ y'(t) &= Are^{rt} \\ y''(t) &= Ar^2e^{rt} \end{aligned}\right\} \Rightarrow \text{into } (5)$$

$$Ar^2e^{rt} + a_1Are^{rt} + a_2Ae^{rt} = 0 \quad -(6)$$

$$\underbrace{Ae^{rt}}_{\neq 0}(r^2 + a_1r + a_2) = 0$$

$$\left[\; r^2 + a_1r + a_2 = 0 \quad -(6)' \;\right]$$

we find the values of 'r' that satisfy $(6)'$

$(6)'$ is the characteristic eqn of (5)

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \quad -(7)$$

$$r_1 = -\frac{a_1}{2} + \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

$$r_2 = -\frac{a_1}{2} - \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

Note

$$\boxed{r_1 + r_2 = -a_1}$$

$$r_1 \cdot r_2 = \left[\frac{-a_1}{2} + \frac{\sqrt{a_1^2 - 4a_2}}{2}\right]\left[-\frac{a_1}{2} - \frac{\sqrt{a_1^2-4a_2}}{2}\right]$$

$$= \left(\frac{-a_1}{2}\right)^2 - \left(\frac{\sqrt{a_1^2-4a_2}}{2}\right)^2 \qquad [x-y][x+y] = [x^2 - y^2]$$

$$= \frac{(-a_1)^2 - (a_1^2 - 4a_2)}{4}$$

$$= \ldots$$

$$= a_2$$

There are two sol$^{n's}$ which will work

$$y_1(t) = A_1 e^{r_1 t} \quad \text{and} \quad y_2(t) = A_2 e^{r_2 t}$$

<u>Claim</u> if $y_1(t)$ and $y_2(t)$ satisfy (5),

so does $y_1(t) + y_2(t)$

<u>Proof</u>

$$y_1''(t) + a_1 y_1'(t) + a_2 y_1(t) = 0 \qquad - (*)$$

$$y_2''(t) + a_1 y_2'(t) + a_2 y_2(t) = 0 \qquad - (**)$$

Let

$$\left.\begin{aligned} y(t) &= y_1(t) + y_2(t) \\ y'(t) &= y_1'(t) + y_2'(t) \\ y''(t) &= y_1''(t) + y_2''(t) \end{aligned}\right\} \Rightarrow (5)$$

$$[y_1''(t) + y_2''(t)] + a_1[y_1'(t) + y_2'(t)] + a_2[y_1(t) + y_2(t)] = 0$$

$$\underbrace{[y_1''(t) + a_1 y_1'(t) + a_2 y_1(t)]}_{=0 \text{ from } (*)} + \underbrace{[y_2''(t) + a_1 y_2'(t) + a_2 y_2(t)]}_{=0 \text{ from } (**)} = 0$$

$y_c = y_1(t) + y_2(t)$ satisfies (5)

(a) The general sol^{n} of the homogeneous DE (5)

$$y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

(b) The general sol^{n} of the nonhomogeneous DE (1)

$$y(t) = \underbrace{A_1 e^{r_1 t} + A_2 e^{r_2 t}}_{y_c} + y_p$$

Case 1: (Two) Distinct Real Roots $a_1^2 > 4a_2$

Case 2: (Two) Repeated Roots $a_1^2 = 4a_2$

Case 3: (Two) Complex Roots $a_1^2 < 4a_2$

<u>Case 1</u> (Two) Distinct Real Roots

$$a_1^2 > 4a_2$$

r_1 and r_2 ⇒ distinct real roots

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad -(8)$$

Ex

$$y''(t) + y'(t) - 2y(t) = -10$$

$y(0) = 12$ and $y'(0) = 2$

Find the general solⁿ ; $y(t) = ?$

⊙ Particular integral

$$y_p = \frac{b}{a_2} = \frac{-10}{-2} = 5$$

$$y_p = k$$

$$\left.\begin{aligned} y'(t) &= 0 \\ y''(t) &= 0 \end{aligned}\right\} \text{sub into the original DE}$$

$$0 + 0 - 2k = -10$$

$$k = 5$$

$$\boxed{y_p = k = 5}$$

⊙ Complementary function

Solve for characteristic roots of

$$y''(t) + y'(t) - 2y(t) = 0$$

$a_1 = 1$ and $a_2 = -2$

So we solve $r^2 + r - 2 = 0$

$$r_1, r_2 = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-2)}}{2}$$

$$= 1, -2$$

$r_1 + r_2 = -1 = -a_1$

$r_1 \cdot r_2 = -2 = a_2$

$$y_c = A_1 e^{t} + A_2 e^{-2t}$$

The general sol^{n} $\quad y(t) = y_c + y_p$

$$= A_1 e^{t} + A_2 e^{-2t} + 5$$

$$y'(t) + a y(t) = b \Rightarrow y(t) = A e^{at} + \frac{b}{a}$$

Case 2 Repeated Real Roots $\quad a_1^2 = 4a_2$

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1, r_2 = \frac{-a_1}{2} = r$$

$$y_c = y_1 + y_2 = A_1 e^{rt} + A_2 e^{rt}$$

$$= (A_1 + A_2) e^{rt}$$

$$= A_3 e^{rt}$$

We claim that

$$y(t) = A_4 . t . e^{rt} \quad \text{— } (\star)$$

also satisfies (5) when $r = r_1 = r_2$

Proof

$$y'(t) = r A_4 t e^{rt} + A_4 e^{rt}$$

$$= (rt + 1) A_4 e^{rt} \quad \text{— } (\star\star)$$

$$y''(t) = r^2 A_4 t e^{rt} + r A_4 e^{rt} + r A_4 e^{rt}$$

$$= (r^2 t + 2r) A_4 e^{rt} \quad \text{— } (\star\star\star)$$

recall

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0 \quad \text{— } (5)$$

$$\underbrace{(r^2 t + 2r) A_4 e^{rt}}_{(\star\star\star)} + a_1 \underbrace{(rt+1) A_4 e^{rt}}_{(\star\star)} + a_2 \underbrace{A_4 t e^{rt}}_{(\star)} = 0$$

$$\underbrace{\left[(r^2 t + 2r) + a_1 (rt + 1) + a_2 t\right]} \underbrace{A_4 e^{rt}}_{\neq 0} = 0$$

$$r^2 t + 2r + a_1 r t + a_1 + a_2 t = 0$$

$$\underbrace{(r^2 + a_1 r + a_2)}_{= 0} t + (a_1 + 2r) = 0$$

from (6)

$$+ \underbrace{\left(a_1 + 2\left(-\frac{a_1}{2}\right)\right)}_{=0} = 0$$

$$0 + 0 = 0$$

$$\therefore \quad y = A_4 t e^{rt} \quad \text{also satisfies (5)}$$

$$\therefore \quad y_c = A_3 e^{rt} + A_4 t e^{rt} \qquad \text{— (9)}$$

Ex $\quad y''(t) + 6y'(t) + 9y(t) = 27 \qquad y(0) = 5$

$$y'(0) = -5$$

Find the general sol^n

② Find Particular Integral y_p

we set $\quad y_p = k$

$$y'(t) = 0$$

$$y''(t) = 0$$

sub into the original DE

$$0 + 6.0 + 9.k = 27$$

$$9k = 27$$

$$k = 3$$

The particular integral $y_p = 3$

(c) Find complementary function y_c

Solve for characteristic roots of

$$y''(t) + 6y'(t) + 9y(t) = 0$$

so we solve

$$r^2 + 6r + 9 = 0$$

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r = -3$$

We have two repeated real roots.

The complementary function take the form

$$y_c = A_3 e^{rt} + A_4 t e^{rt}$$

$$= A_3 e^{-3t} + A_4 t e^{-3t}$$

The general solⁿ

$$y(t) = y_c + y_p$$

$$= A_3 e^{-3t} + A_4 t e^{-3t} + 3$$

*** Find the definite solⁿ

$$y'(t) = (-3)A_3 e^{-3t} + (-3)A_4 t e^{-3t} + A_4 e^{-3t} \quad (*)$$

$$y(t) = A_3 e^{-3t} + A_4 t e^{-3t} + 3 \quad (**)$$

@ $t = 0$

$(*)$ $\quad -5 = -3A_3 + A_4$

$(**)$ $\quad 5 = A_3 + 3 \quad \Rightarrow A_3 = 2$

$$\therefore \quad -5 = -3(2) + A_4$$

$$A_4 = 1$$

$$\boxed{\text{The Definite Sol}^{\text{n}} \quad y(t) = 2e^{-3t} + te^{-3t} + 3}$$