

Solution: Exercise 3 (Part 1)

1. (a) Prove the statement:

“There is a pair of real numbers x and y such that $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. ”

- (b) Disprove the statement: “For all real numbers x and y , $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. ”

Answer:

(a) To prove this *existential statement*, we can find a pair of x and y such that the given statement is true.

Let $x = 0$ and $y = 0$. Then $\lfloor x - y \rfloor = \lfloor 0 - 0 \rfloor = \lfloor 0 \rfloor = 0$, and $\lfloor x \rfloor = \lfloor 0 \rfloor = 0$ and $\lfloor y \rfloor = \lfloor 0 \rfloor = 0$. That is,

$$\lfloor 0 - 0 \rfloor = \lfloor 0 \rfloor - \lfloor 0 \rfloor$$

and hence the statement is true. ■

(b) To disprove the *universal statement*, we can find a pair of x and y such that the given statement is false.

Consider

$$\lfloor 1 \rfloor = 1, \quad \lfloor 0.5 \rfloor = 0, \quad \text{which imply } \lfloor 1 \rfloor - \lfloor 0 \rfloor = 1$$

but

$$\lfloor 1 - 0.5 \rfloor = \lfloor 0.5 \rfloor = 0.$$

That is, a counterexample is $x = 1, y = 0.5$. ■

2. Show that “for any integer n , if $n^3 + 5$ is odd, then n is even,” by using
 a) a proof by contraposition,
 b) a proof by contradiction.

Answer

a) **Proof by contraposition:** To prove by a contraposition, we consider the contrapositive of the given statement:

for any integer n , if n is odd, then $n^3 + 5$ is even.

We can do this by direct proof. Suppose n is odd. Then, we can write

$$n = 2k + 1,$$

where k is an integer and

$$n^3 + 5 = (2k + 1)^3 + 5 = (2k)^3 + 3(2k)^2 + 3(2k) + 1 + 5 = 2(4k^3 + 6k^2 + 3k + 3).$$

That is, we can write $n^3 + 5$ in terms of $n^3 + 5 = 2s$ where $s = 4k^3 + 6k^2 + 3k + 3$ is an integer (since it is the product and the sum of integers). Hence, $n^3 + 5$ is an even integer. ■

b) **Proof by contradiction:**

Suppose not. I.e., suppose the negation “ $n^3 + 5$ is odd, but n is not even” is true. Then, n is odd and we can write

$$n = 2k + 1$$

for some integer $k \in \mathbb{Z}$. So we have

$$n^3 + 5 = (2k + 1)^3 + 5 = (2k)^3 + 3(2k)^2 + 3(2k) + 1 + 5 = 2(4k^3 + 6k^2 + 3k + 3),$$

which implies that $n^3 + 5$ is an even integer (since we can write $n^3 + 5$ in terms of $n^3 + 5 = 2s$ where $s = 4k^3 + 6k^2 + 3k + 3$ is an integer).

This **contradicts** to the fact that $n^3 + 5$ is odd. Hence, the negation is false and cannot happen. That is, the given statement is true. ■

3. Prove by the **method of exhaustion** that “ $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$.”

Answer Given that an integer n such that $1 \leq n \leq 4$ implies that $n = 1, 2, 3$, or 4 .

For $n = 1$, $1^2 + 1 = 2^1$ and $n^2 + 1 \geq 2^n$ is true.

For $n = 2$, $2^2 + 1 = 5$ and $2^2 = 4$, so $n^2 + 1 \geq 2^n$ is true.

For $n = 3$, $3^2 + 1 = 10$ and $2^3 = 8$, so $n^2 + 1 \geq 2^n$ is true.

For $n = 4$, $4^2 + 1 = 17$ and $2^4 = 16$, so $n^2 + 1 \geq 2^n$ is true.

Therefore, $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$. ■

4. Use the **proof by cases** to show that “for any integer n , $n^2 \geq n$.”

[Hint: Consider 3 cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$]

Answer We will prove by cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$.

Case (i): $n \in \mathbb{Z}^-$

In this case, $n < 0$ and $n^2 > 0$. That is $n < 0 < n^2$ and $n^2 \geq n$ is true.

Case (ii): $n = 0$

In this case, $n^2 = 0$. So $n = n^2$ and $n^2 \geq n$ is true.

Case (iii): $n \in \mathbb{Z}^+$

In this case, consider $n^2 - n = n(n - 1)$. Since $n \in \mathbb{Z}^+$ implies that n is a positive integer and the smallest value of n is 1. That is $n \geq 1$, which implies $n - 1 \geq 0$. Since $n > 0$ and $n - 1 \geq 0$,

$$n^2 - n = n(n - 1) \geq 0, \text{ which implies } n^2 \geq n,$$

and the given statement is true.

Since any integer n can be in either case (i), (ii), or (iii), then the given statement is true. ■

5. Consider the statement: for $n \geq 1$,

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

Suppose we want to prove the above statement by **mathematical induction**.

- (a) What is $P(n)$?
- (b) Write $P(1)$: Is $P(1)$ true?
- (c) Write $P(k)$:
- (d) Write $P(k+1)$:
- (e) Prove the above statement: $\sum_{j=1}^n \frac{1}{2^j} = \frac{2^n-1}{2^n}$, by using **mathematical induction**.

Answer:

- (a) $P(n)$ is a statement

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n},$$

for $n \geq 1$

- (b) Write $P(1)$:

$$\frac{1}{2} = \frac{2^{(1-1)}}{2^1},$$

Yes, $P(1)$ is true because $\frac{2^{(1-1)}}{2^1} = \frac{1}{2}$.

- (c) $P(k)$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

- (d) $P(k+1)$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}.$$

- (e) Prove the above statement: $\sum_{j=1}^n \frac{1}{2^j} = \frac{2^n-1}{2^n}$, by using **mathematical induction**.

Let $P(n)$ be the statement $\sum_{j=1}^n \frac{1}{2^j} = \frac{2^n-1}{2^n}$.

- (I) Basis step: Show that $P(1)$ is true. $P(1)$:

$$\frac{1}{2} = \frac{2^{(1-1)}}{2^1},$$

$P(1)$ is true because $\frac{2^{(1-1)}}{2^1} = \frac{1}{2}$.

- (II) Inductive step: Show that if $P(k)$ is true, then $P(k+1)$ is true.

Assume that $P(k)$ is true, or

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

We want to show that $P(k+1)$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}.$$

$$\begin{aligned}
\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{k+1}} &= \underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^k}}_{=\frac{2^k-1}{2^k}} + \frac{1}{2^{k+1}} \quad \text{by inductive hypothesis } P(k) \\
&= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{2(2^k - 1) + 1}{2^{k+1}} \\
&= \frac{(2^{k+1} - 2) + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 1}{2^{k+1}}
\end{aligned}$$

and the statement $P(k+1)$ is true.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 1$ by mathematical induction proof. ■

6. Use mathematical induction proof to show that

$$n! < n^n,$$

for any integer n that is greater than 1.

Answer:

Proof by mathematical induction:

Let $P(n)$ be the statement $n! < n^n$. We want to prove that $P(n)$ is true for all integer $n > 1$. Note that for $n \in \mathbb{Z}$, $n > 1$ is equivalent to $n \geq 2$ and we therefore have to use $n = 2$ in the basis step.

(I) **Basis step:** Show that $P(2)$ is true.

$P(2)$: $2! < 2^2$.

Since $2! = 2$ and $2^2 = 4$. Hence $2! < 2^2$ and $P(2)$ is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 2$.

Assume that $P(k) : k! < k^k$ is true.

—————(★) “inductive hypothesis”

We want to show that $P(k+1) : (k+1)! < (k+1)^{(k+1)}$ is true. Consider

$$\begin{aligned}
(k+1)! &= k!(k+1) \\
&< k^k(k+1) && \text{by (★) “inductive hypothesis”} \\
&< (k+1)^k \cdot (k+1) && k^k < (k+1)^k \text{ for } k \geq 2 \\
&= (k+1)^{(k+1)}
\end{aligned}$$

Note: we have used the fact that since $k < k+1$ implies $k^k < (k+1)^k$ (using the same exponent). Therefore $P(k+1)$ is true.

Note also that we have used $k!(k+1) = \underbrace{1 \cdot 2 \cdot 3 \cdots k}_{k!} \cdot (k+1) = (k+1)!$.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 2$ by mathematical induction proof. ■

7. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
8. (Optional) Use the method of constructive proof to show that:
if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

Answer: Constructive proof

Let $r, s \in \mathbb{R}$ such that $r < s$. Let

$$x = \frac{r + s}{2}.$$

We will show that for this particular x has the value between the r and s .

$$\begin{array}{rcl} r & < & s \\ r + r & < & s + r \\ \underbrace{\frac{r + r}{2}}_{=r} & < & \underbrace{\frac{s + r}{2}}_{=x} \Rightarrow r < x \end{array}$$

$$\begin{array}{rcl} r & < & s \\ r + s & < & s + s \\ \underbrace{\frac{r + s}{2}}_{=x} & < & \underbrace{\frac{s + s}{2}}_{=s} \Rightarrow x < s \end{array}$$

That is, for any given r and s , we can always find $x = \frac{r+s}{2}$ such that $r < x < s$.

Note that it is also possible to use a different value of x . ■

9. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

Answer: Let r be any rational number and s be any irrational number. We want to show that $r - s$ is irrational.

To prove this by contradiction, we will suppose that $r - s$ is rational. Then we can write $r = \frac{a}{b}$ and $r - s = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$. That is,

$$\frac{a}{b} - s = \frac{c}{d}$$

and so

$$s = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

which implies that s is rational. This is a contradiction to the assumption that s is irrational. Therefore, the the given statement is true by contradiction proof. ■

10. (Optional) A sequence $a_1, a_2, \dots$ is defined recursively by

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for } i \geq 2.$$

Show that

$$a_n = 3 \cdot 7^{n-1} \quad \text{for } n > 1.$$

Answer:

Proof by mathematical induction:

From the given definition :

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for } i \geq 2. \quad \text{--- --- --- --- ---} \textcircled{*}$$

Let $P(n)$ be the statement $a_n = 3 \cdot 7^{n-1}$.

We want to prove that $P(n)$ is true for all integer $n \geq 1$.

(I) **Basis step:** Show that $P(1)$ is true.

$$P(1): a_1 = 3 \cdot 7^{1-1}.$$

Since $a_1 = 3 \cdot 7^{1-1} = 3 \cdot 7^0 = 3 \cdot 1 = 3$, which is the same as $a_1 = 3$ from the definition $\circledast$.

Hence $P(1)$ is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.

Assume that $P(k) : a_k = 3 \cdot 7^{k-1}$ is true.

Assume that $P(k) : a_k = 3 \cdot 7^{k-1}$ is true. ————— (★) “inductive hypothesis”
 We want to show that $P(k+1) : a_{k+1} = 3 \cdot 7^{(k+1)-1}$, or $a_{k+1} = 3 \cdot 7^k$ is true. Consider from the definition ⊗

$$\begin{aligned} a_{k+1} &= 7a_{[(k+1)-1]} \\ &= 7 a_k \\ &= 7 [3 \cdot 7^{k-1}] && \text{by } (\star) \text{ “inductive hypothesis : } a_k = 3 \cdot 7^{k-1} \text{ ”} \\ &= 3 \cdot 7^k \end{aligned}$$

and therefore $P(k + 1)$ is true.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 1$ by mathematical induction proof. $\blacksquare$