Chapter 9
Equality Constrained Optimization: Applications

9.1 Cost Minimization and Conditional Input Demand

A firm in perfect competitive product and input markets minimizes its cost to produce a quantity q.

9.1.1 Sufficient Conditions The optimization with a single equality constraint is given by

$$\begin{aligned} &\boldsymbol{min}\, wL + rK \\ &st.\, f(L,K) = q \end{aligned}$$

The Lagrange function is written as

$$\mathcal{L}(L,K,\lambda) = wL + rK - \lambda\big(f(L,K) - q\big),$$

and the first-order condition is

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \lambda} = -\big(f(L^*,K^*) - q\big) = 0 \\ &\left.\begin{aligned} &\frac{\partial \mathcal{L}}{\partial L} = w - \lambda^* f_L(L^*,K^*) = 0 \\ &\frac{\partial \mathcal{L}}{\partial K} = r - \lambda^* f_K(L^*,K^*) = 0 \end{aligned}\right\} \Rightarrow \frac{w}{r} = \frac{f_L(L^*,K^*)}{f_K(L^*,K^*)} = MRTS_{LK} \end{aligned}$$

Test the second-order sufficient condition on the Bordered Hessian

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & -f_L & -f_K \\ -f_L & -\lambda^* f_{LL} & -\lambda^* f_{LK} \\ -f_K & -\lambda^* f_{KL} & -\lambda^* f_{KK} \end{bmatrix}.$$

That is the last $n-k=2-1=1$ leading principal minor of $\bar{\mathbf{H}}$, which is $\bar{\mathbf{H}}$ itself, has the sign $(-1)^k = (-1)^1 = -1$. That is, we need

$$\begin{aligned}
\left|\overline{\mathbf{H}}\right| &= f_L \begin{vmatrix} -f_L & -f_K \\ -\lambda^* f_{KL} & -\lambda^* f_{KK} \end{vmatrix} - f_K \begin{vmatrix} -f_L & -f_K \\ -\lambda^* f_{LL} & -\lambda^* f_{LK} \end{vmatrix} \\
&= f_L\left(\lambda^* f_L f_{KK} - \lambda^* f_K f_{KL}\right) - f_K\left(\lambda^* f_L f_{LK} - \lambda^* f_K f_{LL}\right) \\
&= \lambda^*\left(f_{LL}\left(f_K\right)^2 - 2 f_{LK} f_L f_K + f_{KK}\left(f_L\right)^2\right) < 0.
\end{aligned}$$

The optimal solution is $L^* = L(w,r,q)$ and $K^* = K(w,r,q)$, written as a function of the input prices and the desired quantity q of output level. We call $L(w,r,q)$ and $K(w,r,q)$ the ***conditional input demand functions*** because it is conditional on this desired output q.

9.1.2 Comparative Static Analysis

The first-order condition gives us the implicit functions

$$\nabla_{\begin{bmatrix}\lambda\\L\\K\end{bmatrix}} \mathcal{L}\left(L(w,r,q), K(w,r,q), \lambda(w,r,q); w,r,q\right) = \begin{bmatrix} -\left(f\left(L^*,K^*\right) - q\right) \\ w - \lambda^* f_L\left(L^*,K^*\right) \\ r - \lambda^* f_K\left(L^*,K^*\right) \end{bmatrix} = \mathbf{0}.$$

Since $\nabla^2_{\begin{bmatrix}\lambda\\L\\K\end{bmatrix}} \mathcal{L}\left(L^*, K^*, \lambda^*; w,r,q\right) = \overline{\mathbf{H}}$ is nonsingular, the Implicit Function Theorem gives

$$\begin{aligned}
\nabla_{\begin{bmatrix}w\\r\\q_0\end{bmatrix}} \begin{bmatrix}\lambda^*\\L^*\\K^*\end{bmatrix} &= -\overline{\mathbf{H}}^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & -f_L & -f_K \\ -f_L & -\lambda^* f_{LL} & -\lambda^* f_{LK} \\ -f_K & -\lambda^* f_{KL} & -\lambda^* f_{KK} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

By Cramer's Rule,

$$\frac{\partial L^*}{\partial w}=-\frac{\begin{vmatrix}0 & 0 & -f_K\\ -f_L & 1 & -\lambda^* f_{LK}\\ -f_K & 0 & -\lambda^* f_{KK}\end{vmatrix}}{|\bar{\mathbf{H}}|}=\frac{\left(f_K\right)^2}{|\bar{\mathbf{H}}|}<0$$

$$\frac{\partial L^*}{\partial r}=-\frac{\begin{vmatrix}0 & 0 & -f_K\\ -f_L & 0 & -\lambda^* f_{LK}\\ -f_K & 1 & -\lambda^* f_{KK}\end{vmatrix}}{|\bar{\mathbf{H}}|}=-\frac{f_L f_K}{|\bar{\mathbf{H}}|}>0$$

$$\frac{\partial L^*}{\partial q}=-\frac{\begin{vmatrix}0 & 1 & -f_K\\ -f_L & 0 & -\lambda^* f_{LK}\\ -f_K & 0 & -\lambda^* f_{KK}\end{vmatrix}}{|\bar{\mathbf{H}}|}=-\frac{\lambda^*\left(f_K f_{LK}-f_L f_{KK}\right)}{|\bar{\mathbf{H}}|}><0?$$

The last term $\frac{\partial L^*}{\partial q}$ can be positive or negative depending on the expansion path.

HW Baldani, p. 277,
#10.1 (a,c) Conditional demand,
#10.2 (a,b,d,e,f) for 10.1 (a,c)

9.2 Utility Maximization: Log Utility Function

The consumer solves the problem

$$\max\ U(x,y)=\ln x+\ln y$$
$$st.\ p_x x+p_y y=I.$$

The Lagrange function is given by

$$\mathcal{L}(x,y,\lambda)=\ln x+\ln y-\lambda\left(p_x x+p_y y-I\right).$$

First-order sufficient condition:

$$\nabla_{\begin{bmatrix}\lambda\\x\\y\end{bmatrix}}\mathcal{L}\left(x^*,y^*,\lambda^*\right)=\begin{bmatrix}-p_x x^*-p_y y^*+I\\ \dfrac{1}{x^*}-\lambda^* p_x\\ \dfrac{1}{y^*}-\lambda^* p_y\end{bmatrix}=\mathbf{0},$$

which implies $\dfrac{1}{x^*}\Big/\dfrac{1}{y^*}=\dfrac{y^*}{x^*}=\dfrac{p_x}{p_y}$. We can solve for $\left(x^*,y^*,\lambda^*\right)$ explicitly. The bordered Hessian is

$$\overline{\mathbf{H}}=\begin{bmatrix}0 & -p_x & -p_y\\ -p_x & -\dfrac{1}{x^{*2}} & 0\\ -p_y & 0 & -\dfrac{1}{y^{*2}}\end{bmatrix},$$

which is satisfy the second-order sufficient condition because the last $n-k=2-1=1$ leading principal minor, which is the determinant of $\overline{\mathbf{H}}$ itself, is positive. That is,

$$\begin{aligned}\left|\overline{\mathbf{H}}\right|&=p_x\begin{vmatrix}-p_x & -p_y\\ 0 & -\dfrac{1}{y^{*2}}\end{vmatrix}-p_y\begin{vmatrix}-p_x & -p_y\\ -\dfrac{1}{x^{*2}} & 0\end{vmatrix}\\ &=p_x^2\frac{1}{y^{*2}}+p_y^2\frac{1}{x^{*2}}>0.\end{aligned}$$

We can also do the comparative static analysis to see the effects of changes in p_x, p_y and I on the optimal values of the endogeneous variables x, y and λ.

HW Baldani, p. 278,
#10.3 (b,c) Hicksian and Marshallian demands
#10.4 for 10.3 (b,c)

9.3 Utility Maximization Subject to Budget and Time Constraints

A consumer purchases n products at constant prices and consumes each at a constant time per unit to maximize his utility. The problem is given by

$$\begin{aligned} &\boldsymbol{max}\ U(\mathbf{x}) \\ &st.\ \mathbf{p}^{\mathrm{T}}\mathbf{x} = I \\ &\quad\ \mathbf{t}^{\mathrm{T}}\mathbf{x} = T. \end{aligned}$$

9.3.1 Sufficient Conditions The Lagrange function is written as

$$\begin{aligned} \mathcal{L}(\mathbf{x},\lambda_1,\lambda_2) &= U(\mathbf{x}) - \lambda_1(\mathbf{p}^{\mathrm{T}}\mathbf{x} - I) - \lambda_2(\mathbf{t}^{\mathrm{T}}\mathbf{x} - T) \\ &= U(\mathbf{x}) - \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{p}^{\mathrm{T}}\mathbf{x} - I \\ \mathbf{t}^{\mathrm{T}}\mathbf{x} - T \end{bmatrix} \\ &= U(\mathbf{x}) - \boldsymbol{\lambda}^{\mathrm{T}}\left(\begin{bmatrix} \mathbf{p}^{\mathrm{T}}\mathbf{x} \\ \mathbf{t}^{\mathrm{T}}\mathbf{x} \end{bmatrix} - \begin{bmatrix} I \\ T \end{bmatrix} \right). \end{aligned}$$

The first-order sufficient condition is

$$\begin{aligned} \nabla_{\left[\begin{smallmatrix}\lambda\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*) &= \begin{bmatrix} -(\mathbf{p}^{\mathrm{T}}\mathbf{x}^* - I) \\ -(\mathbf{t}^{\mathrm{T}}\mathbf{x}^* - T) \\ \nabla U(\mathbf{x}^*) - \nabla \mathbf{g}(\mathbf{x}^*)^{\mathrm{T}}\boldsymbol{\lambda}^* \end{bmatrix} \\ &= \begin{bmatrix} -(\mathbf{p}^{\mathrm{T}}\mathbf{x}^* - I) \\ -(\mathbf{t}^{\mathrm{T}}\mathbf{x}^* - T) \\ \nabla U(\mathbf{x}^*) - [\mathbf{p}\quad \mathbf{t}]\boldsymbol{\lambda}^* \end{bmatrix} \\ &= \begin{bmatrix} -(\mathbf{p}^{\mathrm{T}}\mathbf{x}^* - I) \\ -(\mathbf{t}^{\mathrm{T}}\mathbf{x}^* - T) \\ U_1(\mathbf{x}^*) - \lambda_1^* p_1 - \lambda_2^* t_1 \\ \vdots \\ U_n(\mathbf{x}^*) - \lambda_1^* p_n - \lambda_2^* t_n \end{bmatrix} = \mathbf{0}. \end{aligned}$$

This implies

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} = \frac{\lambda_1^* p_i + \lambda_2^* t_i}{\lambda_1^* p_j + \lambda_2^* t_j}$$

$MRS =$ ratio of values of i over j.

The last equality above is justified by noting that, as to be seen later in the chapter of Envelope Theorem, the Lagrange multipliers here is the rate of change of the utility when there is a change in the right-hand-side I or T.

The bordered Hessian has to be negative definite: the last $n-2$ leading principal minors have the sign $(-1)^n$.

$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{0} & -\nabla \mathbf{g}(\mathbf{x}^*) \\ -\nabla \mathbf{g}(\mathbf{x}^*)^{\mathrm{T}} & \nabla^2 U(\mathbf{x}^*) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\mathbf{p}^{\mathrm{T}} \\ 0 & 0 & -\mathbf{t}^{\mathrm{T}} \\ -\mathbf{p} & -\mathbf{t} & \nabla^2 U(\mathbf{x}^*) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -p_1 & -p_2 & \cdots & -p_n \\ 0 & 0 & -t_1 & -t_2 & \cdots & -t_n \\ -p_1 & -t_1 & U_{11} & U_{12} & \cdots & U_{1n} \\ -p_2 & -t_2 & U_{21} & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_n & -t_n & U_{n1} & U_{n2} & \cdots & U_{nn} \end{bmatrix}.$$

9.3.2 Sensitivity Analysis Given $(\mathbf{p}_0, \mathbf{t}_0)$, the optimal solution $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is found from the first-order sufficient condition being the following implicit function.

$$\nabla_{\begin{bmatrix}\lambda \\ \mathbf{x}\end{bmatrix}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*; \mathbf{p}_0, \mathbf{t}_0) = \begin{bmatrix} -(\mathbf{p}_0^{\mathrm{T}}\mathbf{x}^* - I) \\ -(\mathbf{t}_0^{\mathrm{T}}\mathbf{x}^* - T) \\ \nabla U(\mathbf{x}^*) - [\mathbf{p}_0 \quad \mathbf{t}_0]\boldsymbol{\lambda}^* \end{bmatrix} = \mathbf{0}.$$

If $\bar{\mathbf{H}} = \nabla^2_{\begin{bmatrix}\lambda \\ \mathbf{x}\end{bmatrix}} \mathcal{L}(\mathbf{x}^*, \lambda^*; \mathbf{p}_0, \mathbf{t}_0)$ satisfies the second-order sufficient condition, it is nonsingular and the Implicit Function Theorem applies and yields

$$\nabla_{\begin{bmatrix}\mathbf{p}\\\mathbf{t}\end{bmatrix}}\begin{bmatrix}\boldsymbol{\lambda}^*\\\mathbf{x}^*\end{bmatrix} = -\bar{\mathbf{H}}^{-1}\nabla_{\begin{bmatrix}\mathbf{p}\\\mathbf{t}\end{bmatrix}}\left[\nabla_{\begin{bmatrix}\lambda\\\mathbf{x}\end{bmatrix}}\mathcal{L}\left(\mathbf{x}^*,\boldsymbol{\lambda}^*;\mathbf{p}_0,\mathbf{t}_0\right)\right]$$

$$= -\begin{bmatrix} 0 & 0 & -\mathbf{p}_0^{\mathrm{T}} \\ 0 & 0 & -\mathbf{t}_0^{\mathrm{T}} \\ -\mathbf{p}_0 & -\mathbf{t}_0 & \nabla^2 U\left(\mathbf{x}^*\right)\end{bmatrix}_{(n+2)\times(n+2)}^{-1}\begin{bmatrix} -\mathbf{x}^{*\mathrm{T}} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{1\times n} & -\mathbf{x}^{*\mathrm{T}} \\ -\lambda_1^*\mathbf{I}_n & -\lambda_2^*\mathbf{I}_n\end{bmatrix}_{(n+2)\times 2n}.$$

HW. For the case of $n=3$, using the Cramer's Rule to find the sign of the partial derivative of the optimal consumption x_1^* with respect to p_1 and t_2 respectively.

9.4 Intertemporal Consumption

9.4.1 2-Period Case For a 2-period model, the problem is given by

$$\boldsymbol{max}\, U\left(c_1,c_2\right) = c_1c_2$$

$$st.\; c_1 + \frac{c_2}{1+r} = I_1,$$

where r is the prevailing interest rate for both lending and borrowing. The consumer has I_1 current income and zero future income.

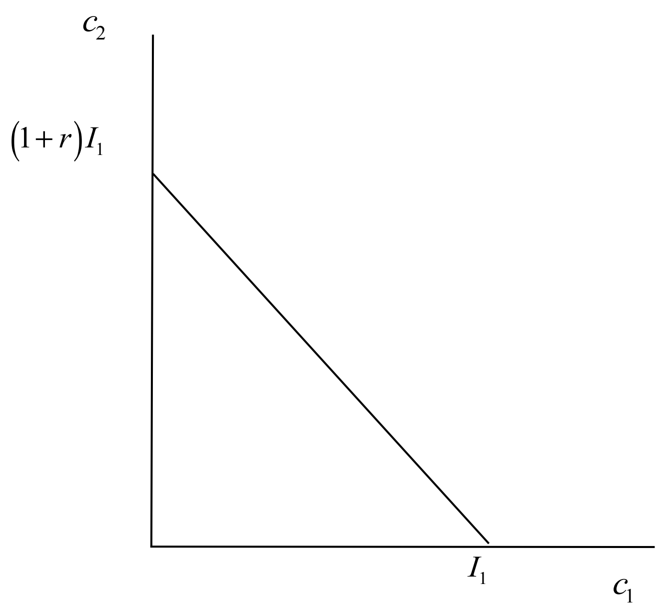

Figure 9.1 Intertemporal constraint for I_1 current period income and zero future income.

The Lagrange function is

$$\mathcal{L}(c_1, c_2, \lambda) = c_1 c_2 - \lambda\left(c_1 + \frac{c_2}{1+r} - I_1\right).$$

FOSC:

$$\nabla_{\begin{bmatrix}\lambda \\ c_1 \\ c_2\end{bmatrix}} \mathcal{L}(c_1^*, c_2^*, \lambda^*) = \begin{bmatrix} -\left(c_1^* + \dfrac{c_2^*}{1+r} - I_1\right) \\ c_2^* - \lambda^* \\ c_1^* - \dfrac{\lambda^*}{1+r} \end{bmatrix} = \mathbf{0}.$$

We have $\dfrac{c_2^*}{c_1^*} = 1+r$, and $c_1^* = \dfrac{I_1}{2}$ and $c_2^* = \dfrac{(1+r)I_1}{2}$.

SOSC: The bordered Hessian is

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & -1 & -\dfrac{1}{1+r} \\ -1 & 0 & 1 \\ -\dfrac{1}{1+r} & 1 & 0 \end{bmatrix},$$

which is negative definite.

HW Baldani, p. 280, # 10.21, 10.22, 10.23 (Income is earned only in the seond period and sensitivity analysis), 10.24.

9.4.2 *n*-period Case The consumer has a stream of income $I_1, I_2, \ldots, I_n$ and seeks to maximize utility by finding the optimal consumption $c_1, c_2, \ldots, c_n$. The lending and borrowing interest rates are equal. Therefore, the consumer maximizes the utility such that the net present value of the consumption equals the net present value of the income, i.e.,

$$\boldsymbol{max} U(\mathbf{c}) = \prod_{t=1}^{n} c_t$$
$$st. \sum_{t=1}^{n} \frac{c_t}{(1+r)^{t-1}} = \sum_{t=1}^{n} \frac{I_t}{(1+r)^{t-1}}.$$

The Lagrange function is

$$\mathcal{L}(\mathbf{c},\lambda) = \prod_{t=1}^{n} c_t - \lambda\left(\sum_{t=1}^{n} \frac{c_t}{(1+r)^{t-1}} - \sum_{t=1}^{n} \frac{I_t}{(1+r)^{t-1}}\right).$$

FOSC:

$$\nabla_{\begin{bmatrix}\lambda\\ \mathbf{c}\end{bmatrix}} \mathcal{L}(\mathbf{c}^*,\lambda^*) = \begin{bmatrix} -\left(\sum_{t=1}^{n} \frac{c_t^*}{(1+r)^{t-1}} - \sum_{t=1}^{n} \frac{I_t}{(1+r)^{t-1}}\right) \\ \prod_{t=2}^{n} c_t^* - \lambda^* \\ \prod_{\substack{t=1\\ t\neq 2}}^{n} c_t^* - \frac{\lambda^*}{1+r} \\ \vdots \\ \prod_{\substack{t=1\\ t\neq i}}^{n} c_t^* - \frac{\lambda^*}{(1+r)^{i-1}} \\ \vdots \\ \prod_{t=1}^{n-1} c_t^* - \frac{\lambda^*}{(1+r)^{n-1}} \end{bmatrix} = \mathbf{0}.$$

This implies $\frac{c_t^*}{(1+r)^{t-1}} = \frac{c_{t+1}^*}{(1+r)^{t}}$ and $\frac{c_{t+1}^*}{c_t^*} = 1+r$. If $n = 2$, we have the critical point

$$c_1^* = \frac{I_1}{2} + \frac{I_2}{2(1+r)}$$
$$c_2^* = \frac{I_1(1+r) + I_2}{2}$$
$$\lambda^* = \frac{I_1(1+r) + I_2}{2}.$$

The bordered Hessian is identical as before and thus the critical point is a local maximum point.

HW Find the bordered Hessian of the above example for the case of n periods.

HW Baldani, p. 281, #10.26.

HW For the case of $n = 2$, do the sensitivity analysis by the Implicit Function Theorem to find the partial derivative of the optimal consumption in each of the two periods with respect to the income in each period. How would the interest rate affect the consumption in each period?