MA216 Midterm exam **Thursday October 1, 2020; 09.00-11.00 hrs.**
40% (7 problems)

Chapter 1: Limits and Continuity

- Computing Limits: Conjugate/Using Factorization
- Limits involving Infinity → Asymptotes
- Limits of Trigonometric Functions
- Squeeze Theorem
- $\boxed{\text{Continuity,}}$ Intermediate Value Theorem

$x^2 \sin\left(\frac{1}{x}\right)$, $|x-1| \cos\left(\frac{1}{x-1}\right)$

Chapter 2: Differentiation

- The Derivative definition (using limits)
- Derivative formulas:
  - Power functions
  - Trigonometric functions: sin/cos/tan/sec/csc/cot (No inverse, no hyperbolic functions)
  - Logarithmic and Exponential Functions
- Product/Quotient Chain Rules
- Implicit Differentiation → x, y
- Logarithmic Differentiation → $f(x) = [g(x)]^{h(x)}$
- Higher Derivatives
- Linear Approximations and (Differentials)

$dy = f'(x)\,dx$

E.g. $x^y = \sin(y)$. Find $\frac{dy}{dx}$.

Chapter 3: Applications of Differentiation

- L'Hospital's Rule; Indeterminate Forms for finding limits
- Mean value theorem

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$

0^0, $\boxed{1^\infty}$, 0^∞

$f(x) = x^x + \sin(\ln(x)) + e^{\sin(x)}$

Find $f'(x)$

Log. Diff

$\boxed{y = x^x}$

I $\ln(y) = \ln(x^x)$

$\ln(y) = x \ln(x)$

II $\frac{d}{dx} \ln(y) = \frac{d}{dx}[x \ln(x)]$.

$\frac{1}{y} \boxed{\frac{dy}{dx}} = \left[\cancel{x} \cdot \frac{1}{\cancel{x}} + \ln(x)\right]$

III $\frac{dy}{dx} = y\,[1 + \ln(x)]$

$\frac{d}{dx} x^x = x^x [1 + \ln(x)]$

Ex) $x^y = \sin^2(y) + \log_3(\ln(x))$

Find $\frac{dy}{dx}$

$$\frac{d}{dx}\underbrace{x^y}_{A} = \frac{d}{dx}\left[\sin^2(y) + \log_3(\ln(x))\right]$$

$$\boxed{\frac{dA}{dx}} = 2\sin(y)\cos(y)\frac{dy}{dx} + \frac{1}{\ln(x)\ln(3)} \cdot \frac{1}{x}$$

$A = x^y$

$\ln(A) = \ln(x^y)$

$\ln(A) = y\ln(x)$

$\frac{d}{dx}\ln(A) = \frac{d}{dx}\left[y\ln(x)\right]$

$\frac{1}{A} \cdot \frac{dA}{dx} = y \cdot \frac{1}{x} + \ln(x)\frac{dy}{dx}$

$\frac{dA}{dx} = A\left[\frac{y}{x} + \ln(x)\frac{dy}{dx}\right]$

x^y

$f(x) = \log_x(\ln(x))$

$y = \log_x(\ln(x))$

$x^y = \ln(x)$

$\log_B(A) = L$

$A = B^L$

$y = \log_x(\ln(x))$

$\log_B(A) = \dfrac{\ln(A)}{\ln(B)}$

$$\boxed{y = \frac{\ln(\ln(x))}{\ln(x)}}$$

- $\lim_{x \to 0} |x|^4 \sin\left(\frac{1}{x}\right) = ?$

$\lim_{x \to \frac{1}{2}} (2x-1)^2 \sin\left(\frac{5}{1-2x}\right)$

- $$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$ ← Assume

$\Downarrow$

$\lim_{x \to 0} \dfrac{x}{\sin(x)} \Rightarrow \lim_{x \to 0} \dfrac{x}{\tan(x)}$

$\lim_{x \to 1} \boxed{\dfrac{x^2 - 2x + 1}{\sin^2(|x-1|)}} = \dfrac{(x-1)^2}{(\sin(|x-1|))^2}$

Find $\lim\limits_{x \to 1} \dfrac{(x-1)^2}{\sin^2(|x-1|)}$

$|x-1| = \begin{cases} -(x-1), & x<1 \\ (x-1), & x \geq 1 \end{cases}$

$\sin(-\theta) = -\sin(\theta)$

$x<1$

$\lim\limits_{x \to 1^-} \dfrac{(x-1)^2}{[\sin(-(x-1))]^2}$

$= \lim\limits_{x \to 1^-} \dfrac{(x-1)^2}{[-\sin(x-1)]^2}$

$= \lim\limits_{x \to 1^-} \left[\dfrac{(x-1)}{\sin(x-1)}\right]^2 = \left[\lim\limits_{x \to 1^-} \dfrac{x-1}{\sin(x-1)}\right]^2$

$= \lim\limits_{x \to 1^-} \left(\dfrac{x-1}{\sin(x-1)}\right)\left(\dfrac{x-1}{\sin(x-1)}\right) = 1$ #

$= \lim\limits_{\theta \to 0^-} \left[\dfrac{\theta}{\sin(\theta)}\right]\left[\dfrac{\theta}{\sin\theta}\right] = 1$

$\theta = x-1$

$x \to 1^- \Rightarrow \theta \to 0^-$

$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ *

$\Downarrow$

$\lim\limits_{x \to 0} \dfrac{x}{\sin(x)} = 1$

$= \lim\limits_{x \to 0} \dfrac{1}{\frac{\sin(x)}{x}}$

$\dfrac{d}{dx}\left(\sqrt{\dfrac{2}{x}} + \sqrt{\dfrac{x}{2}}\right) = \dfrac{d}{dx} \sqrt{2}\, x^{-1/2} + \dfrac{x^{1/2}}{\sqrt{2}}$

$= \sqrt{2}\left(-\dfrac{1}{2} \cdot x^{-3/2}\right) + \dfrac{1}{2\sqrt{2}} x^{-1/2}$

Find $\lim_{x \to 0} \frac{1-\cos(x)}{x}$

$\cos^2(x) + \sin^2(x) = 1$ *

$(A-B)(A+B) = A^2 - B^2$

$1-\cos^2(x) = \sin^2(x)$

$$= \lim_{x \to 0} \frac{(1-\cos(x))}{x} \cdot \frac{(1+\cos(x))}{1+\cos(x)}$$

$$= \lim_{x \to 0} \frac{\boxed{1-\cos^2(x)}}{x\,(1+\cos(x))}$$

$$= \lim_{x \to 0} \frac{\sin^2(x)}{x\,(1+\cos(x))}$$

$$= \lim_{x \to 0} \boxed{\frac{\sin(x)}{x}} \cdot \boxed{\frac{\sin(x)}{1+\cos(x)}}$$

$$= \boxed{\lim_{x \to 0} \frac{\sin(x)}{x}} \cdot \boxed{\lim_{x \to 0} \frac{\sin(x)}{1+\cos(x)}}$$

$\frac{\sin(0) = 0}{1+\cos(0)} = 0$

$$= 1 \cdot (0) = \boxed{0}$$

Ex

Approx. $\cos(1)$ by **linearization**.

$f(x) = \cos(x)$
$a = 0$

$\boxed{L(x) = f'(0)(x-0) + f(0)}$ *

$=$

$\rightarrow f'(x) = -\sin(x) \Rightarrow f'(0) = 0$

$f(0) = \cos(0) = 1$

$L(x) = 0(x-0) + 1$

$L(x) = 1$ *

$\cos(1) = f(1) \approx L(1) = 1$

$\boxed{A = \pi R^2}$

t:

Approx dA

$\Delta A \approx \boxed{dA}$

$\frac{dA}{dR} \cdot \Delta R$