

**Linear Independence**

**Definition**

A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in a vector space  $V$  is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution  $c_1 = 0, \dots, c_p = 0$ .

**Trivial combination, with all weights  $C_i=0$ , produces the zero vector.**

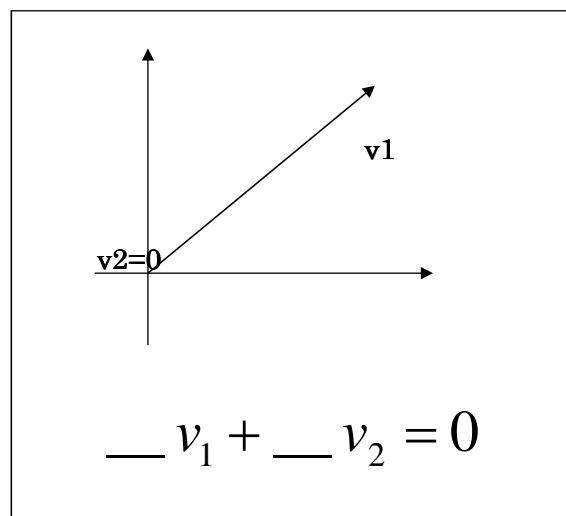
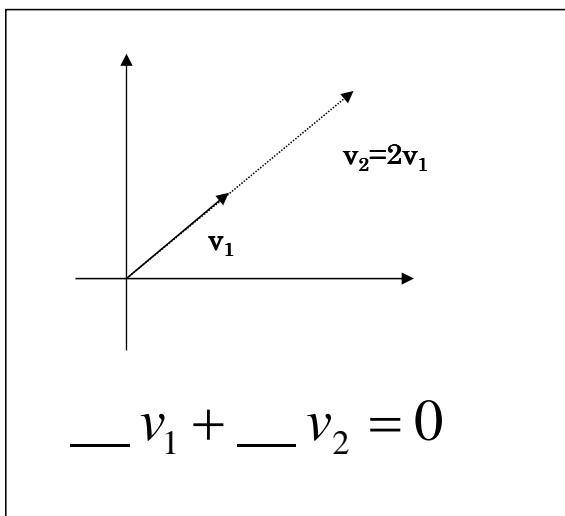
The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is said to be **linearly dependent** if there exists weights  $c_1, \dots, c_p$ , not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}.$$

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  are linear independent if no combination give zero vector (except the zero combination all  $c=0$ )

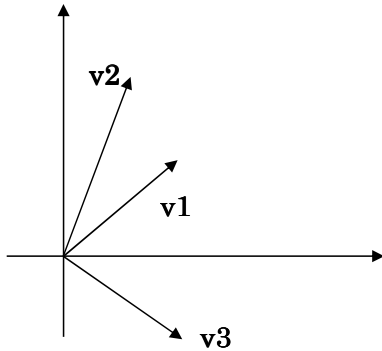
**Linear Independence**

**Linearly independence if no combination gives the zero vector except the zero combination.**



If one vector is zero dependence is said

**Linear Independence**



$$AC = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

- 2 vectors are dependent if they lie on the same line.
- 3 vectors are dependent if they lie in the same plane.
- A random choice of vectors, without any special accident, should produce linear independence.
- 4 vectors are always linearly dependent in  $\mathbb{R}^3$

**Linear Independence**

**Repeat the definition**

**When  $v_1, v_2, \dots, v_n$  are columns of A**

- They are independent if the null space of A is the zero vector only
- They are dependent if there are something in the null space.

$$AC = 0 \quad \text{for nonzero } C$$

**The columns of an  $m \times n$  matrix are independent**

- All columns are \_\_\_\_\_ columns
- Rank = \_\_\_\_\_  $\rightarrow$  the null space of A is only 0 (no free variable)
- Rank < \_\_\_\_\_  $\rightarrow$  at least one free variable

### Linear Independence

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

• What is the combination of column with weights  $-3, 1, 0, 0$ ?

•  $2 \times \text{row } 2 - 5 \times \text{row } 1 =$  \_\_\_\_\_

Dependent? or Independent?

The null space of  $A$  contains only the zero vector  $\rightarrow$  the columns of  $A$  are linearly independent

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Dependent? or Independent?

### Linear Independence

For any echelon matrix  $U$  ; the nonzero rows must be \_\_\_\_\_

If we pick out the columns that contain the pivots  
 $\rightarrow$  linearly independent

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Column 1 and 3 are \_\_\_\_\_
- No set of 3 columns is \_\_\_\_\_
- Column 1 and column 4 are \_\_\_\_\_

The  $r$  nonzero rows of an echelon matrix  $U$  are linearly independent, and so are the  $r$  columns that contain pivots

MA332 LINEAR ALGEBRA  
**Linear Independence**

The columns of  $n$  by  $n$  identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Dependent? or Independent?

In  $\mathbb{R}^4$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

*Coordinate vectors*

MA332 LINEAR ALGEBRA  
**Linear Independence**

To check any set of vectors  $v_1, v_2, \dots, v_n$  for linear independence

- Form the matrix  $A$  whose  $n$  columns are the given vectors

$$AC = 0$$

- Solve for

**Dependent**  $\rightarrow$  a solution other than  $C=0$   
(zero vector)

**Independent**  $\rightarrow$  no free variables (rank is  $n$ )  $\rightarrow$  none of vector except  $C=0$  is in the null space  
 possible for the columns to be linearly independent  
 rank  $< n$  (we cannot have  $n$  pivots)

**A set of  $n$  vectors in  $\mathbb{R}^m$  must be linearly dependent if  $n > m$**

**Linear Independence****Example**

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

**Impossible to have 3 independent vectors in  $\mathbb{R}^2$**

$$A \rightarrow U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

**Spanning a subspace**

**A set of vectors  $w_1, \dots, w_l$  spans a space means the space consists of all linear combinations of those vectors.**

**General definition**

**If a vectors space  $V$  consists of all linear combinations of the particular vectors  $w_1, \dots, w_l$  then these vectors span the space. In other words, every vector  $v$  in  $V$  can be expressed as some combination of the  $w$ 's:**

$$v = c_1 w_1 + \dots + c_l w_l \quad \text{for some coefficients } c_i$$

**The coefficients need not be unique because the spanning set might be excessively large**

### Spanning a subspace

**Example** The column space of a matrix is exactly the space that is spanned by the columns.

~ Columns of a matrix span a column s

Let  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  be a set in  $V$  and let  $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ .

If one of the vectors in  $S$  - say  $\mathbf{v}_k$  - is a linear combination of the remaining vectors in  $S$ , then the set formed from  $S$  by removing  $\mathbf{v}_k$  still spans  $H$ .

<p>To decide if <math>b</math> is a combination of the columns <math>\rightarrow</math> solve <math>Ax=b</math></p> <p><i>Spanning involves the column space.</i></p>	<p>To decide if the columns are independent <math>\rightarrow</math> solve <math>Ax=0</math></p> <p><i>Independence involves the null space</i></p>
---	---

### Spanning a subspace

**EXAMPLE:** Suppose  $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$  and  $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ .

*Solution:* If  $\mathbf{x}$  is in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , then

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(\text{_____}\mathbf{v}_1 + \text{_____}\mathbf{v}_2) \\ &= \text{_____}\mathbf{v}_1 + \text{_____}\mathbf{v}_2 \end{aligned}$$

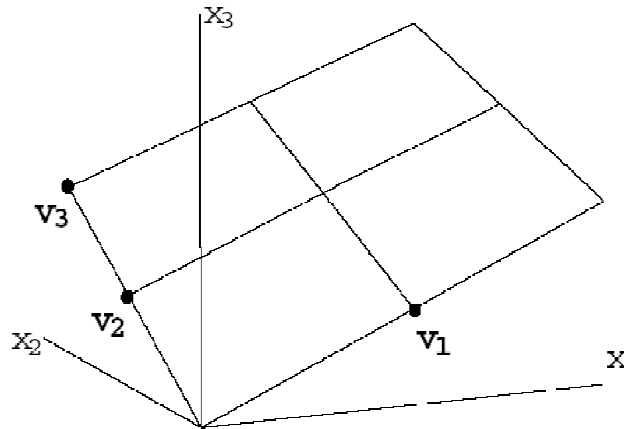
Therefore,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

## A Basis Set

Let  $H$  be the plane illustrated below. Which of the following are valid descriptions of  $H$ ?

- (a)  $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$       (b)  $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$   
 (c)  $H = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$       (d)  $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$



Dr. Julaluk Carmai

## A Basis Set

A *basis set* is an “efficient” spanning set containing no unnecessary vectors. In this case, we would consider the linearly independent sets  $\{\mathbf{v}_1, \mathbf{v}_2\}$  and  $\{\mathbf{v}_1, \mathbf{v}_3\}$  to both be examples of basis sets or bases (plural for basis) for  $H$ .

### DEFINITION

Let  $H$  be a subspace of a vector space  $V$ . An indexed set of vectors  $\beta = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$  in  $V$  is a basis for  $H$  if

- (i)  $\beta$  is a linearly independent set, and  
 (ii)  $H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ .

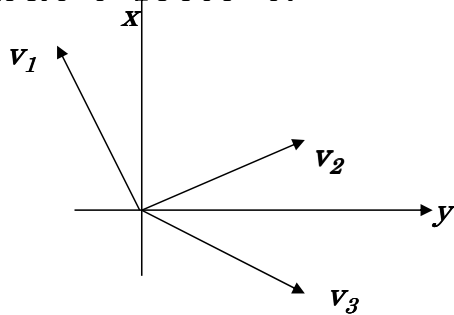
In another word  $\rightarrow$  A *basis* for a vector space is a set of vectors having two properties at once:

1. They are linearly independent.
2. They span the space.

Dr. Julaluk Carmai

## Basis

- Every vector in the space is a combination of the basis vectors, because they span.
- The combination is unique.
- There is one and only one way to write  $v$  as a combination of the basis vectors.



The vector  $v_1$  by itself is \_\_\_\_\_  
but fails to \_\_\_\_\_  $\mathbb{R}^2$

The vectors  $v_1, v_2, v_3$  span  $\mathbb{R}^2$   
but are not \_\_\_\_\_

Any two of these vectors have  
both properties  $\rightarrow$  form a basis.

## Basis

### Example Space in $\mathbb{R}^3$

One basis is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Are these vectors independent?

The null space of identity matrix is only zero. Therefore columns are \_\_\_\_\_

Another basis  $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$

$v_1$  &  $v_2$  :they are independent but  
may not span  $\mathbb{R}^3$ ( there can be vector in  $\mathbb{R}^3$   
that is not combination of those vectors)

If  $v_3 = (3,3,7)^T \rightarrow$  dependent it lies on the  
same plane as  $v_1, v_2$   
 $v_3$  must be any vector that does not lies  
in the same plane as  $v_1, v_2$

*How to check it?*

Square matrix is invertible  $\rightarrow$  columns are independent  $\rightarrow$  a basis for  $\mathbb{R}^n$

# MA332 LINEAR ALGEBRA

**EXAMPLE:** Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ .

Is  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  a **basis** for  $\mathbf{R}^3$ ?

*Solution:* Again, let  $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ . Using row

reduction,

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Dr. Julaluk Carmai

# MA332 LINEAR ALGEBRA

**EXAMPLE:** Find a basis for  $\text{Nul } A$  where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

*Solution:* Row reduce  $[A \ \mathbf{0}]$ :

$$\begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$x_1 = -2x_2 - 13x_4 - 33x_5$$

$$x_3 = 6x_4 + 15x_5$$

$x_2, x_4$  and  $x_5$  are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

$\uparrow$                        $\uparrow$                        $\uparrow$   
 $\mathbf{u}$                        $\mathbf{v}$                        $\mathbf{w}$

Dr. Julaluk Carmai

**Basis****Example**

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The four columns span the column space but not \_\_\_\_\_

The columns that contain pivots are a basis for the column space.

The columns of a matrix span its column space.

If they are independent  $\rightarrow$  a basis for the column space

(whether matrix is square or rectangular)

The columns to be a basis for the space  $\mathbb{R}^n$ , the matrix must be square and invertible.

Dr. Julaluk Carmai

**Bases for Col  $A$** 

**EXAMPLE:** Find a basis for Col  $A$ , where

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

*Solution:* Row reduce:

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4] \sim \dots \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \ \mathbf{b}_4]$$

Dr. Julaluk Carmai

Note that

$$\mathbf{b}_2 = \underline{\quad} \mathbf{b}_1 \quad \text{and} \quad \mathbf{a}_2 = \underline{\quad} \mathbf{a}_1$$

$$\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3 \quad \text{and} \quad \mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$$

$\mathbf{b}_1$  and  $\mathbf{b}_3$  are not multiples of each other

$\mathbf{a}_1$  and  $\mathbf{a}_3$  are not multiples of each other

Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

Therefore  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$  and  $\{\mathbf{a}_1, \mathbf{a}_3\}$  is a basis for  $\text{Col } A$ .

Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ . Find a basis for  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .

*Solution:* Let  $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$  and note that  $\text{Col } A = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .

By row reduction,  $A \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ . Therefore a basis

for  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is  $\left\{ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \right\}$ .

**The pivot columns of a matrix  $A$  form a basis for  $\text{Col } A$**

Review:

1. To find a basis for  $\text{Nul } A$ , use elementary row operations to transform  $[A \ \mathbf{0}]$  to an equivalent reduced row echelon form  $[B \ \mathbf{0}]$ . Use the reduced row echelon form to find parametric form of the general solution to  $A\mathbf{x} = \mathbf{0}$ . The vectors found in this parametric form of the general solution form a basis for  $\text{Nul } A$ .

2. A basis for  $\text{Col } A$  is formed from the pivot columns of  $A$ .

**Warning:** Use the pivot columns of  $A$ , not the pivot columns of  $B$ , where  $B$  is in reduced echelon form and is row equivalent to  $A$ .

Dr. Julaluk Carmai

## Dimension of a vector space

**A vector space has infinitely many different bases but something in common → The same number of vectors**

The x-y plan → 2 vectors in every basis → dimension=2

$\mathbf{R}^3$  → Number of vectors = 3

$\mathbf{R}^n$  → Number of vectors = n

### DEFINITION of dimension of a vector space

If  $V$  is spanned by a finite set, then  $V$  is said to be **finite-dimensional**, and the **dimension** of  $V$ , written as  $\dim V$ , is the number of vectors in a basis for  $V$ . The dimension of the zero vector space  $\{\mathbf{0}\}$  is defined to be 0. If  $V$  is not spanned by a finite set, then  $V$  is said to be **infinite-dimensional**.

**EXAMPLE:** The standard basis for  $\mathbf{R}^n$  is  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  where  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are the columns of  $I_n$ . So, for example,  $\dim \mathbf{R}^3 = 3$ .

Dr. Julaluk Carmai

## Dimension of a vector space

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What is a basis for the column space of  $U$ ?

Column \_\_\_\_\_ and Column \_\_\_\_\_  
which are the \_\_\_\_\_ column.

The column space of  $U$  has dimension 2  
"Two dimensional subspace of  $\mathbb{R}^3$ "

### Example

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

**EXAMPLE:** Find a basis and the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} : a, b, c, d \text{ are real} \right\}.$$

*Solution:* Since  $\begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$

$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

# MA332 LINEAR ALGEBRA

- Note that  $\mathbf{v}_3$  is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , so by the Spanning Set Theorem, we may discard  $\mathbf{v}_3$ .
- $\mathbf{v}_4$  is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . So  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$  is a basis for  $\mathcal{W}$ .
- Also,  $\dim \mathcal{W} = \underline{\hspace{2cm}}$ .

**EXAMPLE: Dimensions of subspaces of  $\mathbb{R}^3$**

**0-dimensional subspace** contains only the zero vector  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

**1-dimensional subspaces.**  $\text{Span}\{\mathbf{v}\}$  where  $\mathbf{v} \neq \mathbf{0}$  is in  $\mathbb{R}^3$ .

These subspaces are                                  through the origin.

**2-dimensional subspaces.**  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  where  $\mathbf{u}$  and  $\mathbf{v}$  are in  $\mathbb{R}^3$  and are not multiples of each other.

These subspaces are                                  through the origin.

**3-dimensional subspaces.**  $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  where  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  are linearly independent vectors in  $\mathbb{R}^3$ . This subspace is  $\mathbb{R}^3$  itself because the columns of  $A = [\mathbf{u} \ \mathbf{v} \ \mathbf{w}]$  span  $\mathbb{R}^3$  according to the IMT.

Dr. Julaluk Carmai

# MA332 LINEAR ALGEBRA

Let  $H$  be a subspace of a finite-dimensional vector space  $\mathcal{V}$ . Any linearly independent set in  $H$  can be expanded, if necessary, to a basis for  $H$ . Also,  $H$  is finite-dimensional and  $\dim H \leq \dim \mathcal{V}$ .

**EXAMPLE:** Let  $H = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ . Then  $H$  is a subspace of  $\mathbb{R}^3$  and  $\dim H < \dim \mathbb{R}^3$ .

We could expand the spanning set  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$  to  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  to form a basis for  $\mathbb{R}^3$ .

## THE BASIS THEOREM

Let  $\mathcal{V}$  be a  $p$ -dimensional vector space,  $p \geq 1$ . Any linearly independent set of exactly  $p$  vectors in  $\mathcal{V}$  is automatically a basis for  $\mathcal{V}$ . Any set of exactly  $p$  vectors that spans  $\mathcal{V}$  is automatically a basis for  $\mathcal{V}$ .

Dr. Julaluk Carmai

**Dimensions of Col A and Nul A**

Recall our techniques to find basis sets for column spaces and null spaces.

**EXAMPLE:** Suppose  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$ . Find  $\dim \text{Col } A$  and  $\dim \text{Nul } A$ .

*Solution*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$  is a basis for  $\text{Col } A$  and  $\dim \text{Col } A = 2$ .

Now solve  $AX = 0$  by row-reducing the corresponding augmented matrix. Then we arrive at

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 7 & 8 & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2x_2 - 4x_4$$

$$x_3 = 0$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So  $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for  $\text{Nul } A$  and  
 $\dim \text{Nul } A = 2$ .

## Dimension of a vector space

**Dim Col  $A$  = number of pivot columns of  $A$  (Rank)**

**Dim Nul  $A$  = number of free variables of  $A$  ( $n-r$ )**

•Any linearly independent set  $V$  can be extended to a basis, by adding more vectors if necessary.

•Any spanning set  $V$  can be reduced to a basis, by discarding vectors if necessary.

•A basis is a maximal independent set.

It cannot be made larger without losing independence

•A basis is also a minimal spanning set.

It cannot be made smaller and still span the space.

~~Basis of a matrix  
Rank of a space  
Dimension of a basis~~

Dimension of the space  
Rank of the matrix

## Rank

The set of all linear combinations of the row vectors of a matrix  $A$  is called the row space of  $A$  and is denoted by  $\text{Row } A$ .

**EXAMPLE:** Let

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \mathbf{r}_1 &= (-1, 2, 3, 6) \\ \mathbf{r}_2 &= (2, -5, -6, -12) \\ \mathbf{r}_3 &= (1, -3, -3, -6) \end{aligned}$$

$$\text{Row } A = \text{Span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\} \text{ (a subspace of } \mathbf{R}^4\text{)}$$

$$\boxed{\text{Col } A^T = \text{Row } A}.$$

When we use row operations to reduce matrix  $A$  to matrix  $B$ , we are taking linear combinations of the rows of  $A$  to come up with  $B$ . We could reverse this process and use row operations on  $B$  to get back to  $A$ . Because of this, the row space of  $A$  equals the row space of  $B$ .

### THEOREM 13

If two matrices  $A$  and  $B$  are row equivalent, then their row spaces are the same. If  $B$  is in echelon form, the nonzero rows of  $B$  form a basis for the row space of  $A$  as well as  $B$ .

Dr. Julaluk Carmai

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are row equivalent. Find a basis for row space, column space and null space of  $A$ . Also state the dimension of each.

Basis for  $\text{Row } A : \{ \quad \quad \quad \}$

$\dim \text{Row } A : \underline{\quad}$

Basis for  $\text{Col } A : \left\{ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \right\}$

$\dim \text{Col } A : \underline{\quad}$

Dr. Julaluk Carmai

# MA332 LINEAR ALGEBRA

To find  $\text{Nul } A$ , solve  $AX = 0$  first:

$$\begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 2 & -5 & -6 & -12 & 0 \\ 1 & -3 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -6 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 + 6x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for  $\text{Nul } A$  :  $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  and  $\dim \text{Nul } A = \underline{\hspace{2cm}}$

Dr. Julaluk Carmai

# MA332 LINEAR ALGEBRA

Note the following:

$$\dim \text{Col } A = \# \text{ of pivots of } A = \# \text{ of nonzero rows in } B = \dim \text{Row } A.$$

$$\dim \text{Nul } A = \# \text{ of free variables} = \# \text{ of nonpivot columns of } A.$$

## DEFINITION

The **rank** of  $A$  is the dimension of the column space of  $A$ .

$$\boxed{\text{rank } A = \dim \text{Col } A = \# \text{ of pivot columns of } A = \dim \text{Row } A}$$

$$\begin{array}{ccccc} \underbrace{\text{rank } A} & + & \underbrace{\dim \text{Nul } A} & = & \underbrace{n} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \left\{ \begin{array}{c} \# \text{ of pivot} \\ \text{columns} \\ \text{of } A \end{array} \right\} & & \left\{ \begin{array}{c} \# \text{ of nonpivot} \\ \text{columns} \\ \text{of } A \end{array} \right\} & & \left\{ \begin{array}{c} \# \text{ of} \\ \text{columns} \\ \text{of } A \end{array} \right\} \end{array}$$

Dr. Julaluk Carmai

**THE RANK THEOREM**

The dimensions of the column space and the row space of an  $m \times n$  matrix  $A$  are equal. This common dimension, the rank of  $A$ , also equals the number of pivot positions in  $A$  and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n.$$

Since  $\text{Row } A = \text{Col } A^T$ ,

$$\text{rank } A = \text{rank } A^T.$$

**EXAMPLE:** Suppose that a  $5 \times 8$  matrix  $A$  has rank 5. Find  $\dim \text{Nul } A$ ,  $\dim \text{Row } A$  and  $\text{rank } A^T$ . Is  $\text{Col } A = \mathbf{R}^8$ ?

*Solution:*

$$\begin{array}{ccc} \underbrace{\text{rank } A} & + & \underbrace{\dim \text{Nul } A} = \underbrace{n} \\ \downarrow & & \downarrow \quad \downarrow \\ 5 & & ? \quad 8 \end{array}$$

$$5 + \dim \text{Nul } A = 8 \quad \Rightarrow \quad \dim \text{Nul } A = \underline{\hspace{2cm}}$$

$$\dim \text{Row } A = \text{rank } A = \underline{\hspace{2cm}} \quad \Rightarrow \quad \text{rank } A^T = \text{rank } \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

Since  $\text{rank } A = \#$  of pivots in  $A = 5$ , there is a pivot in every row. So the columns of  $A$  span  $\mathbf{R}^5$

Hence  $\text{Col } A = \mathbf{R}^5$ .

**EXAMPLE:** For a  $9 \times 12$  matrix  $A$ , find the smallest possible value of  $\dim \text{Nul } A$ .

*Solution:*

$$\text{rank } A + \dim \text{Nul } A = 12$$

$$\dim \text{Nul } A = 12 - \underbrace{\text{rank } A}_{\text{largest possible value}} = \underline{\hspace{2cm}}$$

$$\text{smallest possible value of } \dim \text{Nul } A = \underline{\hspace{2cm}}$$

# MA332 LINEAR ALGEBRA

## Visualizing Row $A$ and Nul $A$

**EXAMPLE:** Let  $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$ . One can easily verify the following:

Basis for Nul  $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$  and therefore Nul  $A$  is a plane in  $\mathbf{R}^3$ .

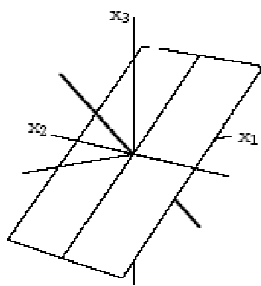
Basis for Row  $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$  and therefore Row  $A$  is a line in  $\mathbf{R}^3$ .

Basis for Col  $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$  and therefore Col  $A$  is a line in  $\mathbf{R}^2$ .

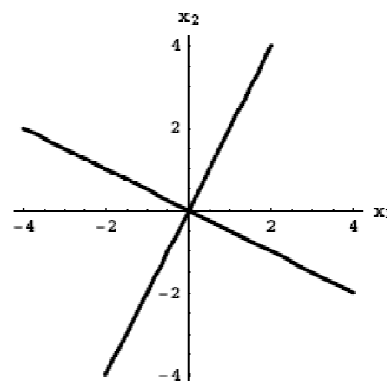
Basis for Nul  $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$  and therefore Nul  $A^T$  is a line in  $\mathbf{R}^2$ .

Dr. Julaluk Carmai

# MA332 LINEAR ALGEBRA



Subspaces Nul  $A$  and Row  $A$



Subspaces Nul  $A^T$  and Col  $A$

Dr. Julaluk Carmai