Mathematical Induction I

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer.

Suppose the following two statements are true:

1. $P(a)$ is true.

2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement " for all integers $n \geq a$, $P(n)$" is true.

The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the principle of mathematical induction rather than as a theorem.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true."

To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

where k is any particular but arbitrarily chosen integer with $k \geq a$.
Then show that $P(k+1)$ is true.

Definition
If n and d are integers and $d \neq 0$ then n is divisible by d if, and only if, n equals d times some integer. The notation $d|n$ is read "d divides n." Symbolically, if n and d are integers and $d \neq 0$

$$d|n \Leftrightarrow \exists \text{ an integer } k \text{ such that } n = dk.$$

Instead of "n is divisible by d," we can say that

- n is a multiple of d, or
- d is a factor of n, or
- d is a divisor of n, or
- d divides n.

Example Use mathematical induction to prove that for all integers $n \geq 1$.

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Example(Sum of a Geometric Sequence)
Use mathematical induction to prove that for any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Example Use mathematical induction to prove that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Example Use mathematical induction to prove that $n < 2^n$ for all non-negative integers n.