Optimization theory: single variable Calculus II

EE320

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Where we are headed today

- Provide tools needed for solving the extreme-point problem.
- Applying the tool for understanding the optimization problem in economics.

List of applications

- Cost minimization problem.
- Profit maximization problem.
- Revenue-maximization problem.

Mathematical tools for optimization

- Characteristic of functions by derivative
- Nature of extreme points problem
- Solution method

Characteristic of functions by derivative

- Increasing v.s. decreasing
- Concave and Convex
- Monotonic function

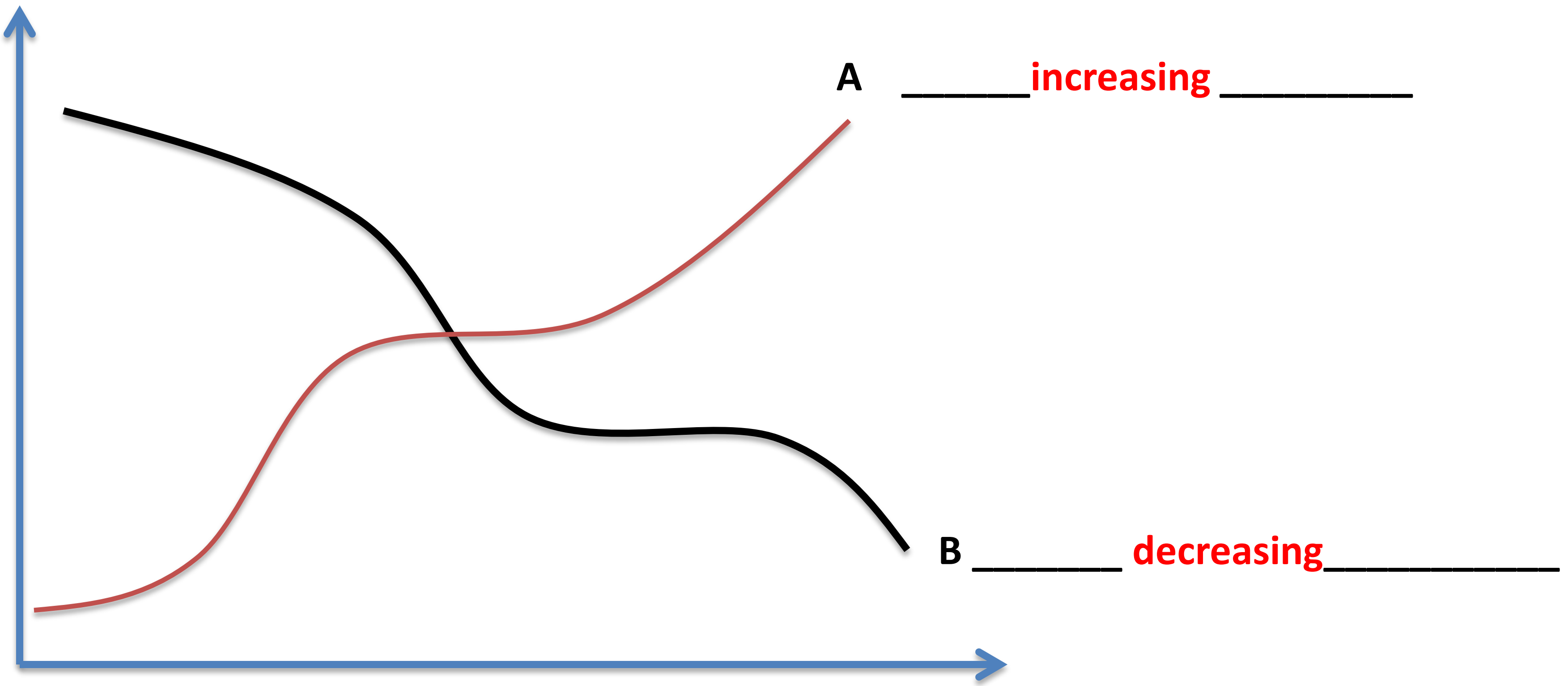
Increasing v.s. decreasing function

- Definition: A function is said to be increasing if and only if for any $x_1 > x_2$, we have that $f(x_1) \geq f(x_2)$, and vice versa.
  - Strictly increasing = the condition holds with strict inequality.
- Definition: A function is said to be decreasing if and only if for any $x_1 < x_2$, we have that $f(x_1) \geq f(x_2)$, and vice versa.
  - Strictly decreasing= the condition holds with strict inequality.

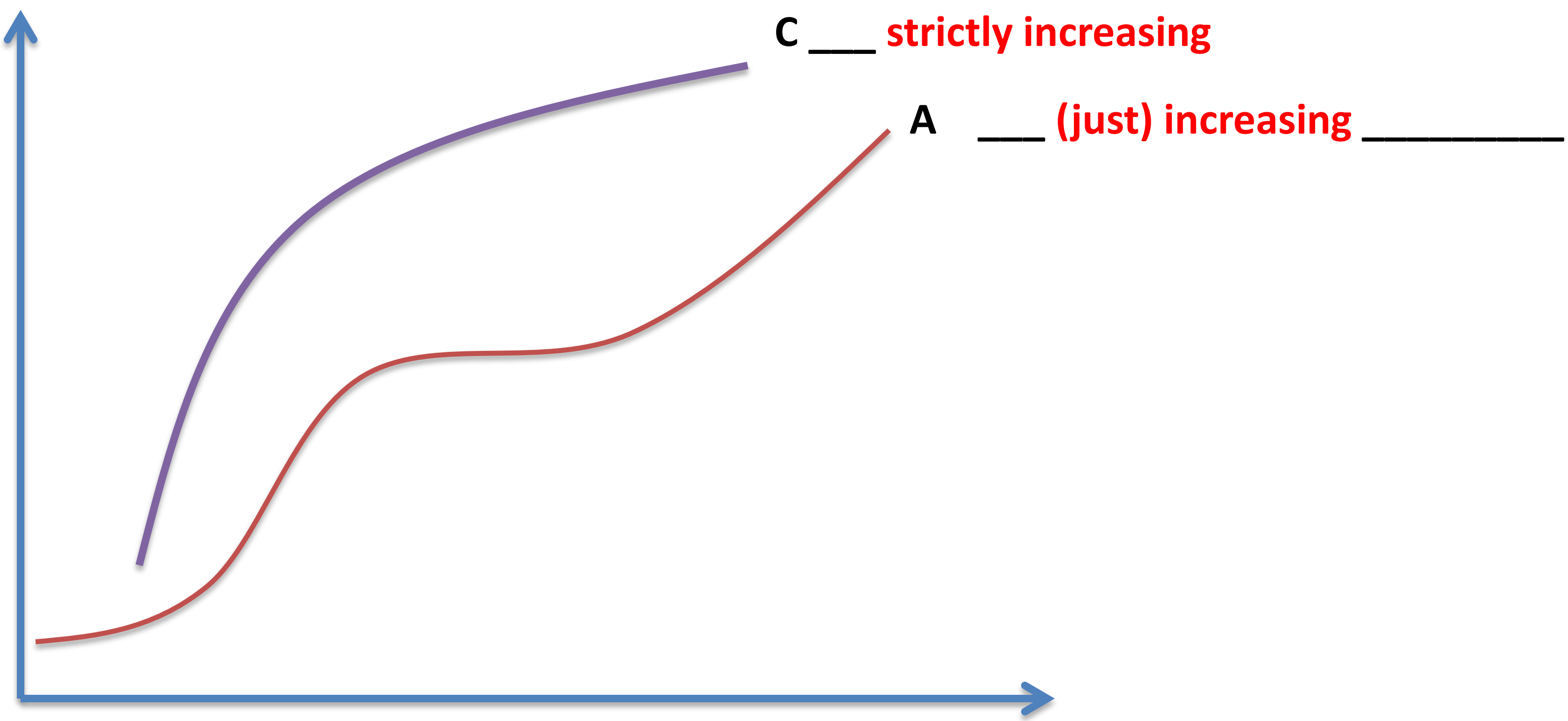
Increasing v.s. decreasing function

- Equivalent results for increasing (decreasing) function:
  - Slope is positive (negative)
  - $f'(x) \geq (\leq) 0$ (with strict inequality for "strictly" increasing/decreasing function)

Decreasing and increasing.



Increasing v.s. strictly increasing function



Characteristic of functions by derivative

- Increasing v.s. decreasing
- Concave and Convex
- Monotonic function

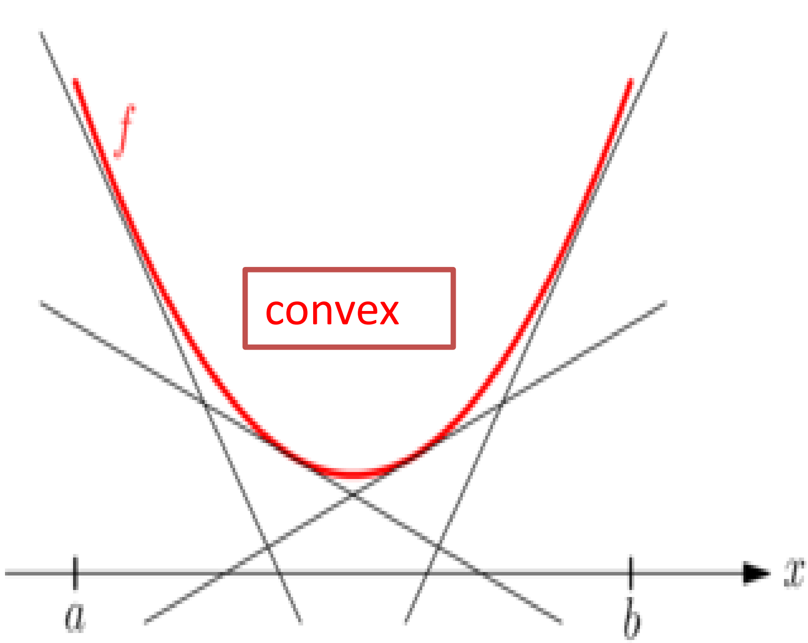
Concave and Convex function

- A function is said to be concave if
  - Slope of f(x) decreases as X increases.
  - Tangent line is above the curve.
  - $f''(x) \leq 0$.
- If to be a “strict” concave function, the last condition is modified to hold for only “strict” inequality, i.e. $f''(x) < 0$

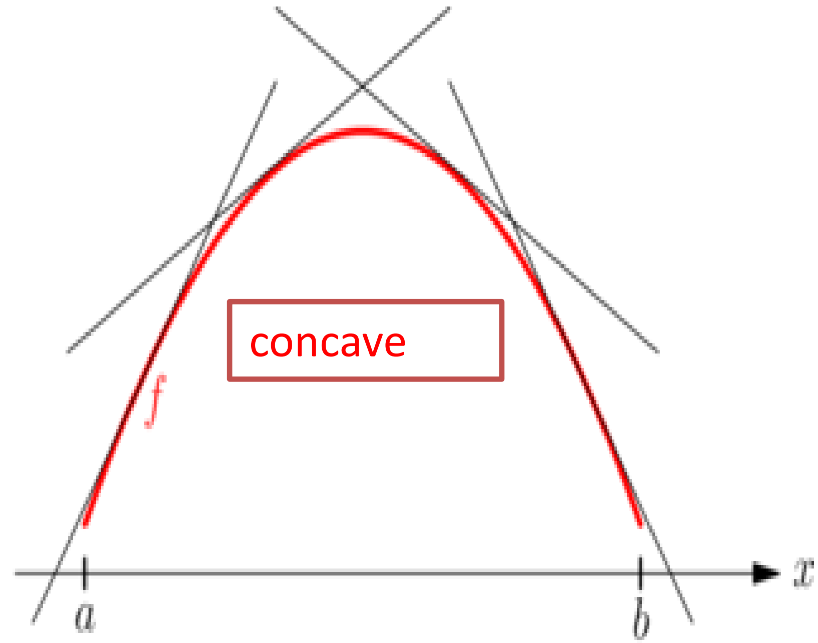
Concave and Convex function

- A function is said to be convex if
  - Slope of f(x) increases as X increases.
  - Tangent line is below the curve.
  - $f''(x) \geq 0$.
- If to be a "strict" convex function, the last condition is modified to hold for only "strict" inequality, i.e. $f''(x) > 0$

Concave v.s. convex

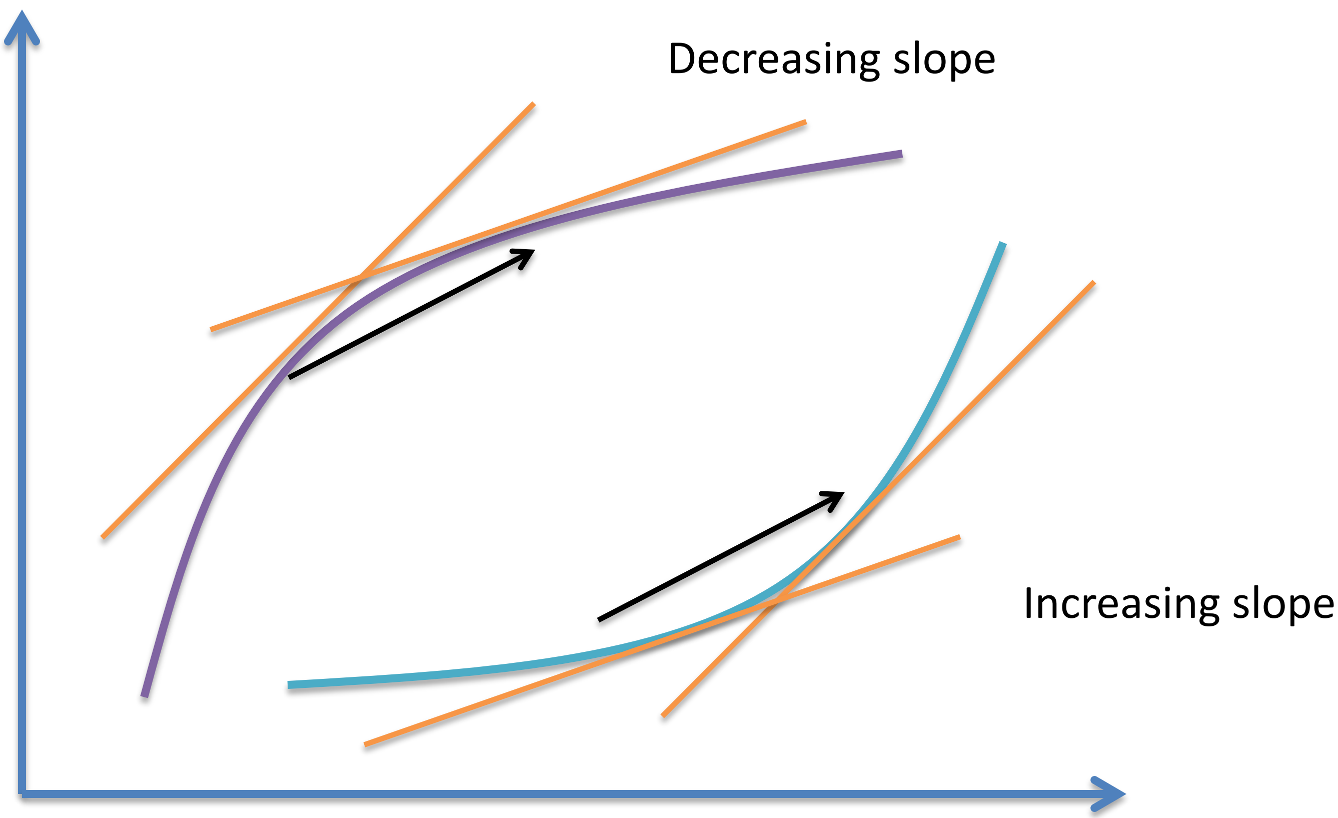


all straight lines are tangents to f and lies below f where f' is increasing



all straight lines are tangents to f and lies above f and f' is decreasing

Which one is which one?



Characteristic of functions by derivative

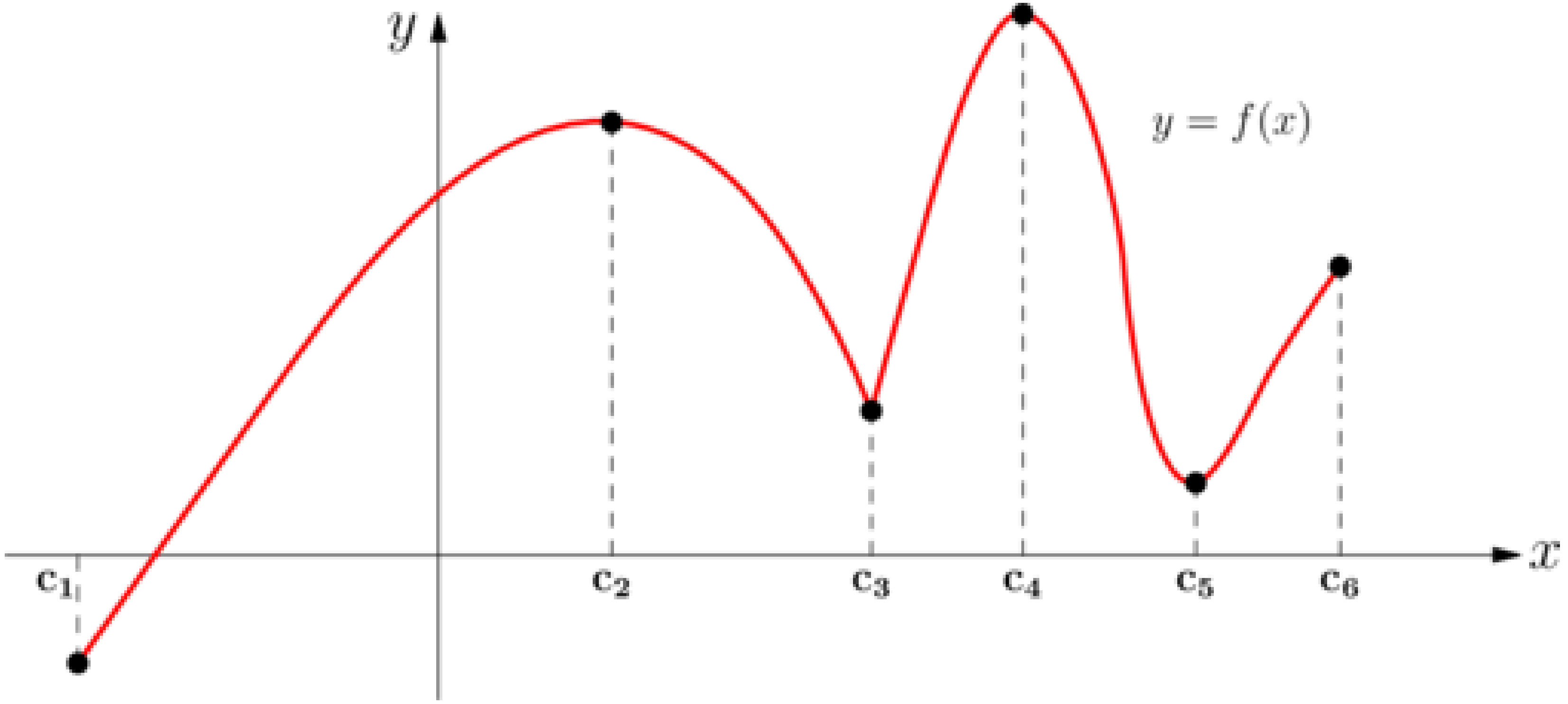
- Increasing v.s. decreasing
- Concave and Convex
- Monotonic function

Monotonic function.

- Monotonically increasing function → only increasing function for the whole domain set.

- Generally, monotone = one single property for the function.

Non-monotone function



Mathematical tools for optimization

- Characteristic of functions by derivative
- Nature of extreme points problem
- Solution method

Extreme point problem

Suppose $f(x)$ is an objective function

Mathematical form of the extreme point problem:

$$\max(min)_x \quad f(x) \quad ; \qquad x \in D$$

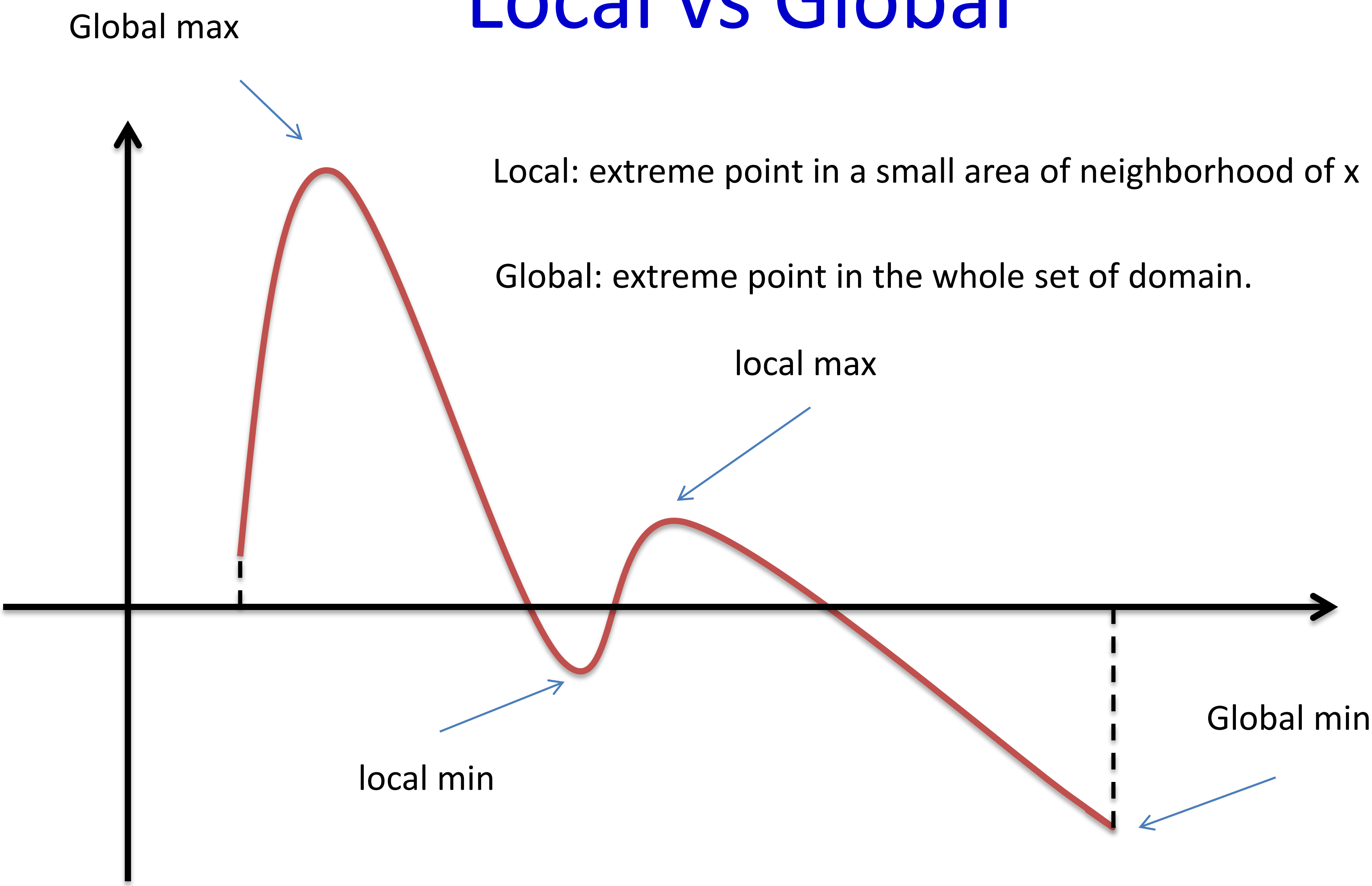
Type of extreme points: local vs Global

- Local
  - For all the “X” locating in the neighborhood of x*, the x* is a local optimizer if
    - Local max: $f(x^*) > f(x)$.
    - Local min: $f(x^*) < f(x)$.
- Global
  - For all the “X” in the domain defining “f”, the x* is the global optimizer if
    - Global max: $f(x^*) > f(x)$
    - Global min: $f(x^*) < f(x)$

Local vs Global

Local: extreme point in a small area of neighborhood of x

Global: extreme point in the whole set of domain.



Mathematical tools for optimization

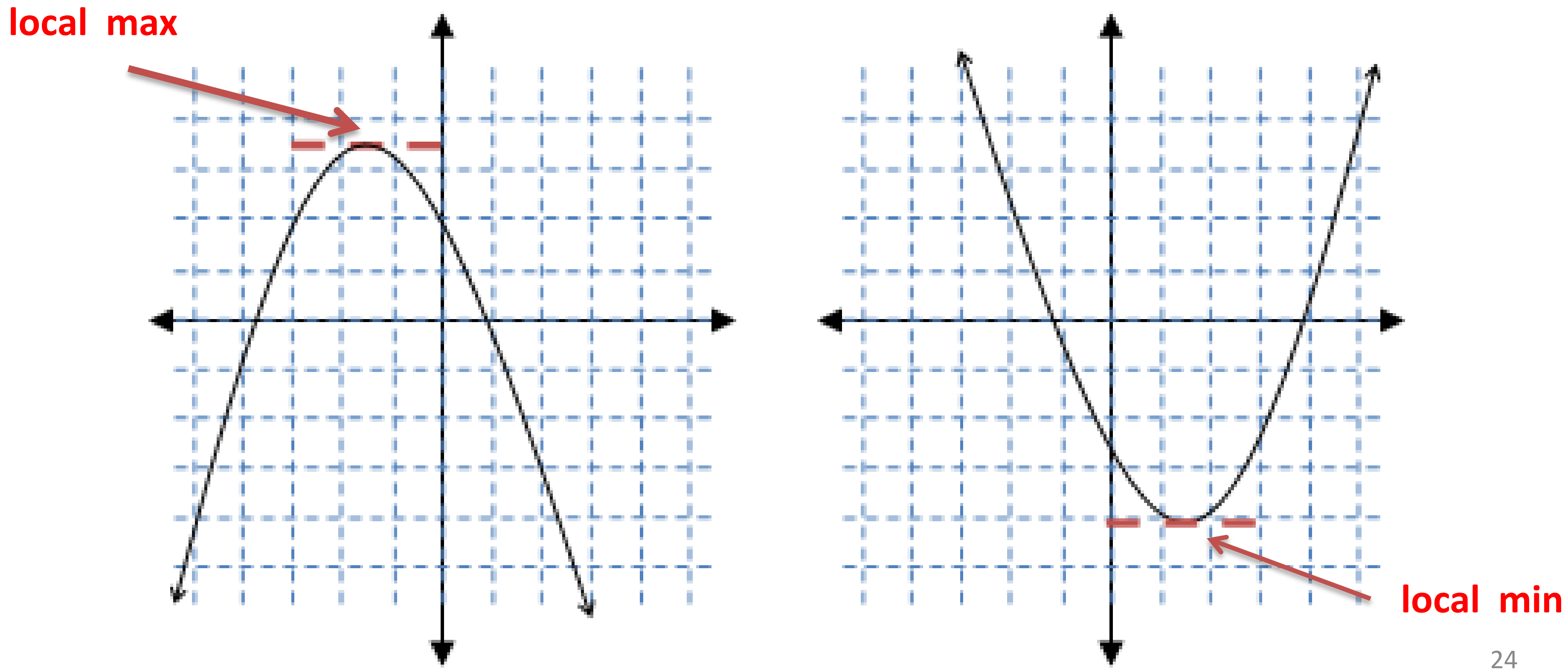
- Characteristic of functions by derivative
- Nature of extreme points problem
- Solution method

Tools for solving for local extreme point…

- Calculus only provides tool set for solving *local extreme point*.
  - If one wants to solve for global extreme points, one needs to check all possible local extreme points, and boundary values of the domain set that defines the function.
- Under some regularities conditions, local extreme point is warranted to be global extreme point.

Theorem: local optimizer

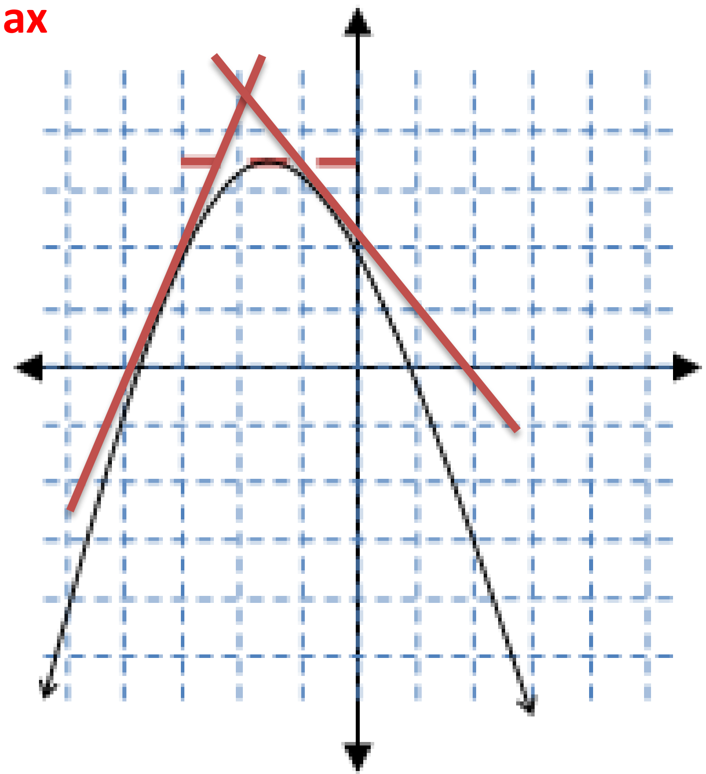
- If "f" is differentiable, i.e. no kinked point
  - First-order derivative: $f'(x^*) = 0$

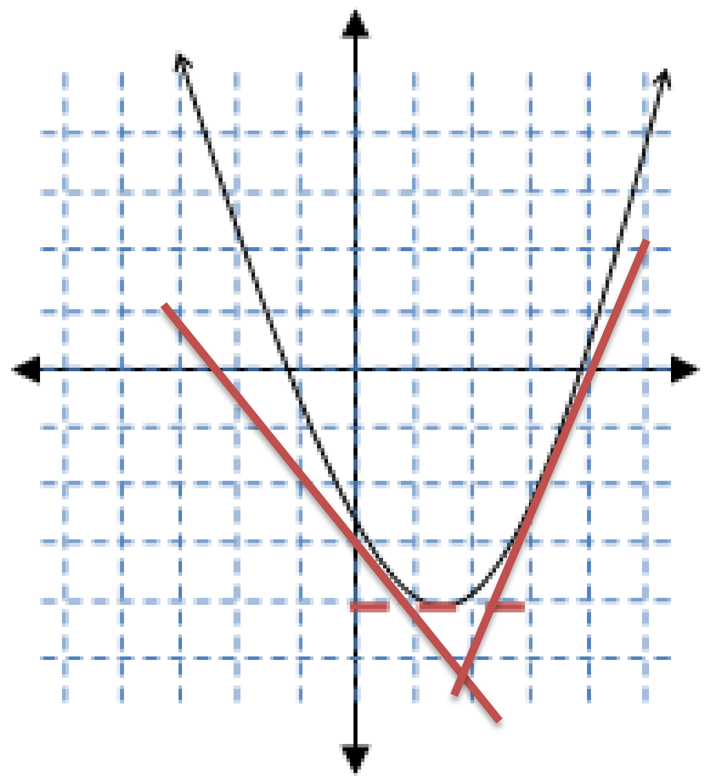


Theorem: local optimizer

- If *"f"* is differentiable,
  - First-order derivative: $f'(x^*) = 0$

local max





local m

Cooked procedure: Max versus Min

- **Step 1:** find the value of x, such that

  $f'(x^*) = 0$ ➔ x^*: stationary/critical point

- **Step 2:** check the slope in the neighborhood of x obtained from step 1.
  - Local max: positive slope becomes negative slope. (decreasing slope)
  - Local min: negative slope becomes positive slope. (increasing slope)

Cooked procedure: Max versus Min

- If *"f"* is twice-differentiable function, i.e. $f''(*)$ exists for all the values of x,
  - First-order condition: $f'(x^*) = 0$
  - Second-order condition
    - Maximizer: $f''(x^*) < 0$ (decreasing slope)
    - Minimizer: $f''(x^*) > 0$ (increasing slope)

What is the intuition of the second-order condition?

- Intuition:
- Local max: positive slope becomes negative slope.
  - That is, the change in slope is negative.
  - Slope is decreasing.
  - Slope is 1^{st} derivative, measuring change in f(x).
  - Thus, change in slope is 2^{nd} derivative, measuring change in f'(x).
  - Thus, this means that local max occurs in the region that function has negative value of second-order derivative. That is, it's concave.

Example

- $y = x^3 - 3x$

Using the two-step procedure

First order derivative

$$\frac{dy}{dx} = 3x^2 - 3 = 0$$

$$x = \pm 1.$$

(Two possible candidate points)

Second order derivative

$$\frac{d^2y}{dx^2} = 6x$$

$$x = 1 \Rightarrow \frac{d^2y}{dx^2} = 6(1) > 0 \Rightarrow convex$$

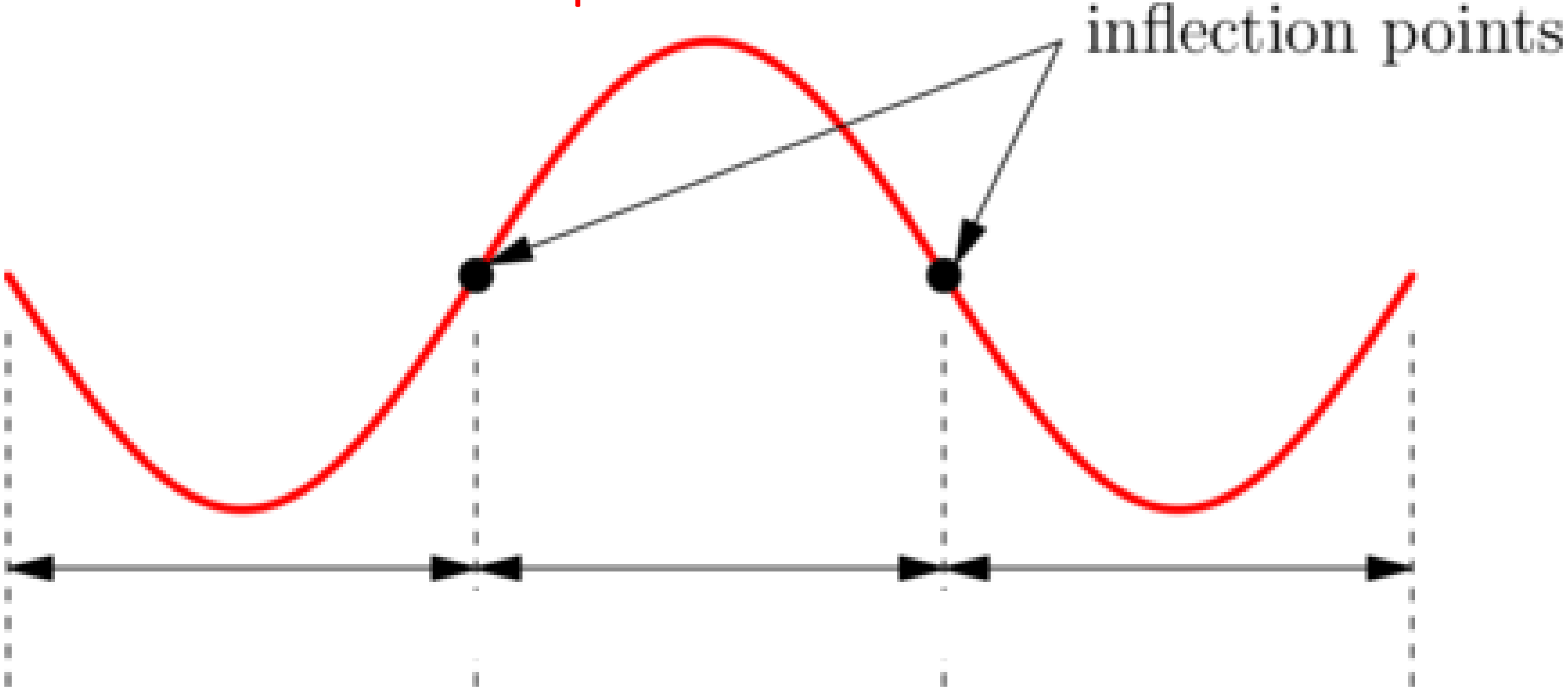
=> Local min

$$x = -1 \Rightarrow \frac{d^2y}{dx^2} = 6(-1) < 0 \Rightarrow concave$$

=> Local max

An inflection point…

- Changing from concave to convex, and vice versa….
- At that point, $f''(x^{inflection}) = 0$.
- Extreme value for the slope function.



Example

- $y = x^4 - 2x^2 + 1,$ find local optimizer and inflection point

Locating the optimizers

First order derivative

$$\frac{dy}{dx} = 4x^3 - 4x = 0$$

$$4x(x-1)(x+1) = 0$$

$$x = \pm 1, 0$$

(Three possible candidate points)

Second order derivative

$$\frac{d^2y}{dx^2} = 12x^2 - 4$$

- $x = 1 \Rightarrow \frac{d^2y}{dx^2} = 8 > 0 \Rightarrow convex$
  => Local min
- $x = -1 \Rightarrow \frac{d^2y}{dx^2} = 8 > 0 \Rightarrow convex$
  => Local min
- $x = 0 \Rightarrow \frac{d^2y}{dx^2} = -4 < 0 \Rightarrow concave$
  => Local max

Example

- $y = x^4 - 2x^2 + 1$, find local optimizer and inflection point

Locating the inflection points

From the Second order derivative

$$\frac{d^2y}{dx^2} = 12x^2 - 4 = 0$$

- $x = \frac{\sqrt{3}}{3}$ and $-\frac{\sqrt{3}}{3}$

Check the third-order derivative:

$$\frac{d^3y}{dx^3} = 24x$$

$$x = \frac{\sqrt{3}}{3} \quad (concave - to - convex)$$

$$x = -\frac{\sqrt{3}}{3} \quad (convex - to - concave)$$

$\frac{d^2y}{dx^2}$

+	-	+
Convex	**Concave**	**Convex**

$-\frac{\sqrt{3}}{3}$ $\quad$ $+\frac{\sqrt{3}}{3}$

N-derivative test

- For some functions, it's possible that x^*

$$f'(x^*) = 0$$

$$f''(x^*) = 0$$

$$x^* =?\, optimizer \;\; or \;\; inflection\, point$$

Example: consider $y = x^5 + 1$

- $f'(0) = 0$
  - X = 0 is a critical value. So, one may have tempted to think that this is a candidate for local optimizer.
- $f''(0) = 0$
  - But, this is also a possible candidate for an inflection..
- Then, what is it actually?
- N-th order derivative test

The n-th order derivative test..

- For the x^* with $f'(x^*) = 0$, and other higher-order derivatives are all also zero, but **the N-th**
  - If N is an even number,
    - » x^* is local max if $f^N(x^*) \neq 0 < 0$.
    - » x^* is local min if $f^N(x^*) \neq 0 > 0$.
  - If N is odd number, x^* is an inflection point.

Example

- $y = x^5 + 1$ v.s. $y = x^4 + 1$

$f'(x) = 5x^4$; $x^* = 0$ *is a critical point*

$f''(x^* = 0) = 20x^{*3} = 0 \Rightarrow inconclusive$
$f^3(x^* = 0) = 60x^{*2} = 0 \Rightarrow inconclusive$
$f^4(x^* = 0) = 1200x^{*2} = 0 \Rightarrow inconclusive$
$f^5(x^* = 0) = 240 \neq 0 \Rightarrow STOP\ here!$

Since "N" is an odd number, $x^* = 0$ is in fact the inflection point.

Since, the N-th derivative is greater than zero, we can tell further that the curvature would change from concave to convex at $x^* = 0$.

$f'(x) = 4x^3$; $x^* = 0$ *is a critical point*

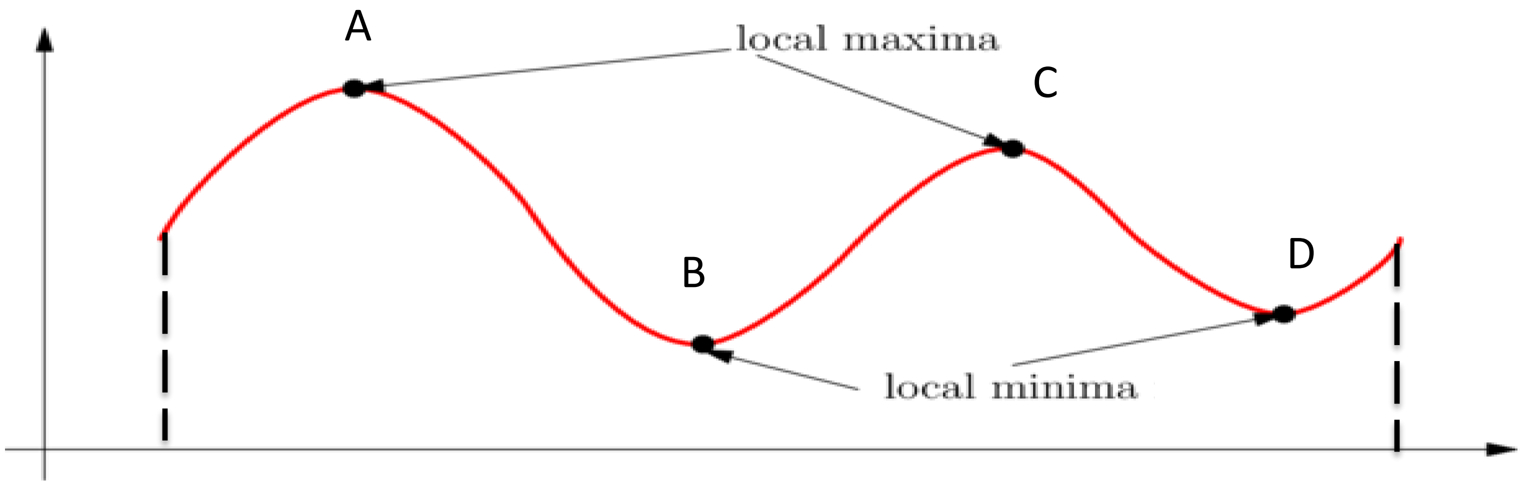
$f''(x^* = 0) = 12x^{*2} = 0 \Rightarrow inconclusive$
$f^3(x^* = 0) = 24x^* = 0 \Rightarrow inconclusive$
$f^5(x^* = 0) = 24 \neq 0 \Rightarrow STOP\ here!$

Since "N" is an even number, $x^* = 0$ is a local Optimizer.

Since, the N-th derivative is greater than zero, we can tell further that it is local minimizer.

Global optimizer

- Calculate the value of all the local optimizers and the two-value at the boundary.



Example

- $y = x^3 - 3x; \quad x \in [-2,2]$

1. Locate all local optimizers:
Using the two-step procedure

First order derivative

$$\frac{dy}{dx} = 3x^2 - 3 = 0$$

$$x = \pm 1.$$

(Two possible candidate points)

Second order derivative:

$$\frac{d^2y}{dx^2} = 6x$$

$$x = 1 \Rightarrow \frac{d^2y}{dx^2} = 6(1) > 0 \Rightarrow convex$$

=> Local min
f(1) = -2

$$x = -1 \Rightarrow \frac{d^2y}{dx^2} = 6(-1) < 0 \Rightarrow concave$$

=> Local max
f(-1) = 2.

Example

- $y = x^3 - 3x; \quad x \in [-2,2]$

2. Check the value at boundary point

f(-2) = -2;
f(2) = 2;

3. Rank all the valued function at each point

f(2) = 2=f(-1) ➔ global max
f(1) = -2 = f(-2) ➔ Global min

Local optimizer ➔ Global optimizer

- Assume some nice properties of the function to ensure that local optimizer is automatically global optimizer.
  - If function is *monotonically* concave, local maximizer is then global maximizer.
  - If function is *monotonically* convex, local minimizer is then global minimizer.

Example

- Example:
  - $f(x) = 2x^2 - 2x + 1$
  - $x^* = \frac{1}{2}$ ➔ $f''(x) = 4 > 0.$ (always positive)
  - Monotonically convex ➔ global min.
- Example
  - $f(x) = -e^x + x$
  - $f'(x) = -e^x + 1 \Rightarrow x^* = 0$
  - $f''(x) = -e^x < 0$ (always negative)
  - Monotonically concave ➔ global max.

Applications

- Firm's optimization.
  - Property of cost minimization
  - Profit-maximizing output.
  - Individual supply curve.
- Competitive equilibrium
  - Market supply curve.
  - Short-run competitive equilibrium.
- Monopoly equilibrium
  - Profit-maximizing output
- Revenue-maximizing taxation.

Cost minimization problem (CMP)

Suppose that cost function takes the following equation: $C(Q) = 40 + 8Q - 2Q^2 + Q^3$

- Find the expression of marginal cost.
- Find the level of Q that results in minimum average variable cost. Also how much is the value of MC at that Q. Draw a conclusion between the two.

Find the expression of marginal cost.

$$MC(Q) = \frac{dC}{dQ} = 8 - 4Q \quad + 3Q^2$$

- MC has the lowest value when Q = 4/6.
- (In this example, MC is a quadratic function.)

Find the level of Q that results in minimum average variable cost. Also how much is the value of MC at that Q. Draw a conclusion between the two.

$$AVC = \frac{TVC(Q)}{Q} = \frac{8Q - 2Q^2 + Q^3}{Q} = 8 - 2Q + Q^2$$

Min AVC: Q such that

$$\frac{\mathrm{d}AVC}{dQ} = 0 \Rightarrow -2 + 2Q = 0 \Rightarrow Q = 1$$

Verifying using the second-order derivative

$$\frac{\mathrm{d}^2AVC}{dQ^2} = 2 \Rightarrow monotonically\ convex$$

$$\Rightarrow Global\ min$$

Find the level of Q that results in minimum average variable cost. Also how much is the value of MC at that Q. Draw a conclusion between the two.

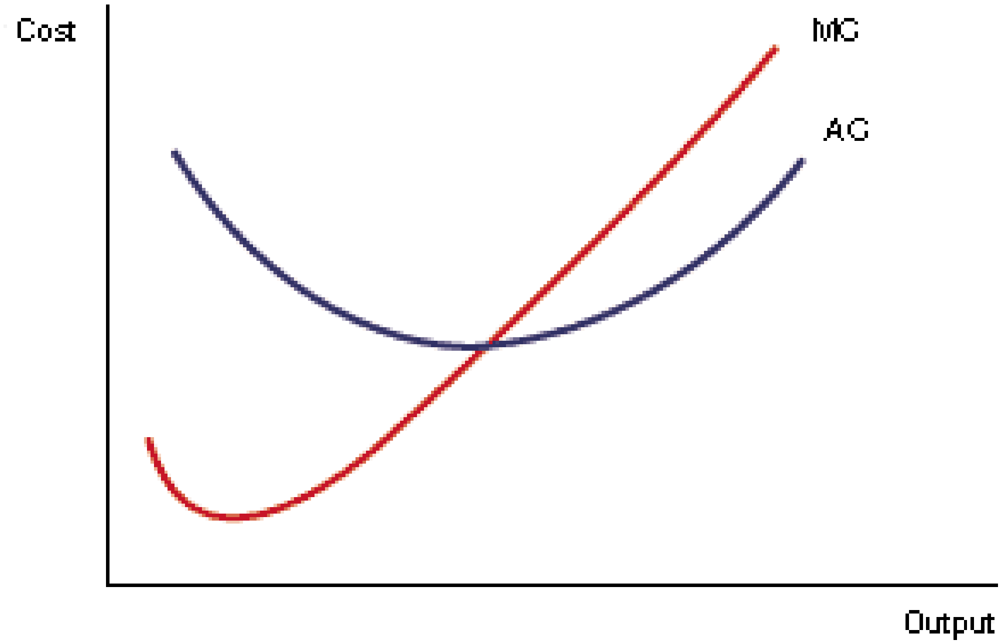
$$When\ Q = 1, the\ minimized\ level\ of\ AVC = 7.$$

$$When\ Q = 1, the\ level\ of\ MC = 7.$$

Numerically, we now see that at the minimum level of AVC, MC must be equal to AVC.

Proof is given in the next slide.

Generality…



- $AC = \frac{TC}{Q}$
- $\min_{Q} \; AC(Q)$

$$-AC'(Q) = \frac{[Q * C'(Q) - C(Q)]}{Q^2} = \frac{C'(Q)}{Q} - \frac{AC(Q)}{Q}$$

$$-\frac{C'(Q)}{Q} = \frac{AC(Q)}{Q} \Rightarrow C'(Q) = AC(Q)$$

$$C'(Q) = MC(Q) = AC(Q)$$

Profit maximization problem (PMP)

- Firm's problem is to choose for Q that maximizes profit.

$$\pi = R(Q) - C(Q)$$

Profit maximization problem (PMP)

- First-order optimality condition

$$\pi'(Q^*) = 0$$

$$R'^{(Q)} - C'^{(Q)} = 0 \Rightarrow MR(Q^*) = MC(Q^*)$$

- Second order

$$\pi''(Q^*) < 0$$

$$MR'(Q^*) - MC'(Q^*) < 0$$

Profit maximization problem (PMP)

- $R(Q)$ differs across market structure.

- Can firm set price? Is the firm taking price as given?
  - Competitive firms: $\mathrm{R}(Q) = PQ$
  - Monopoly: $\mathrm{R}(Q) = P(Q) * Q$

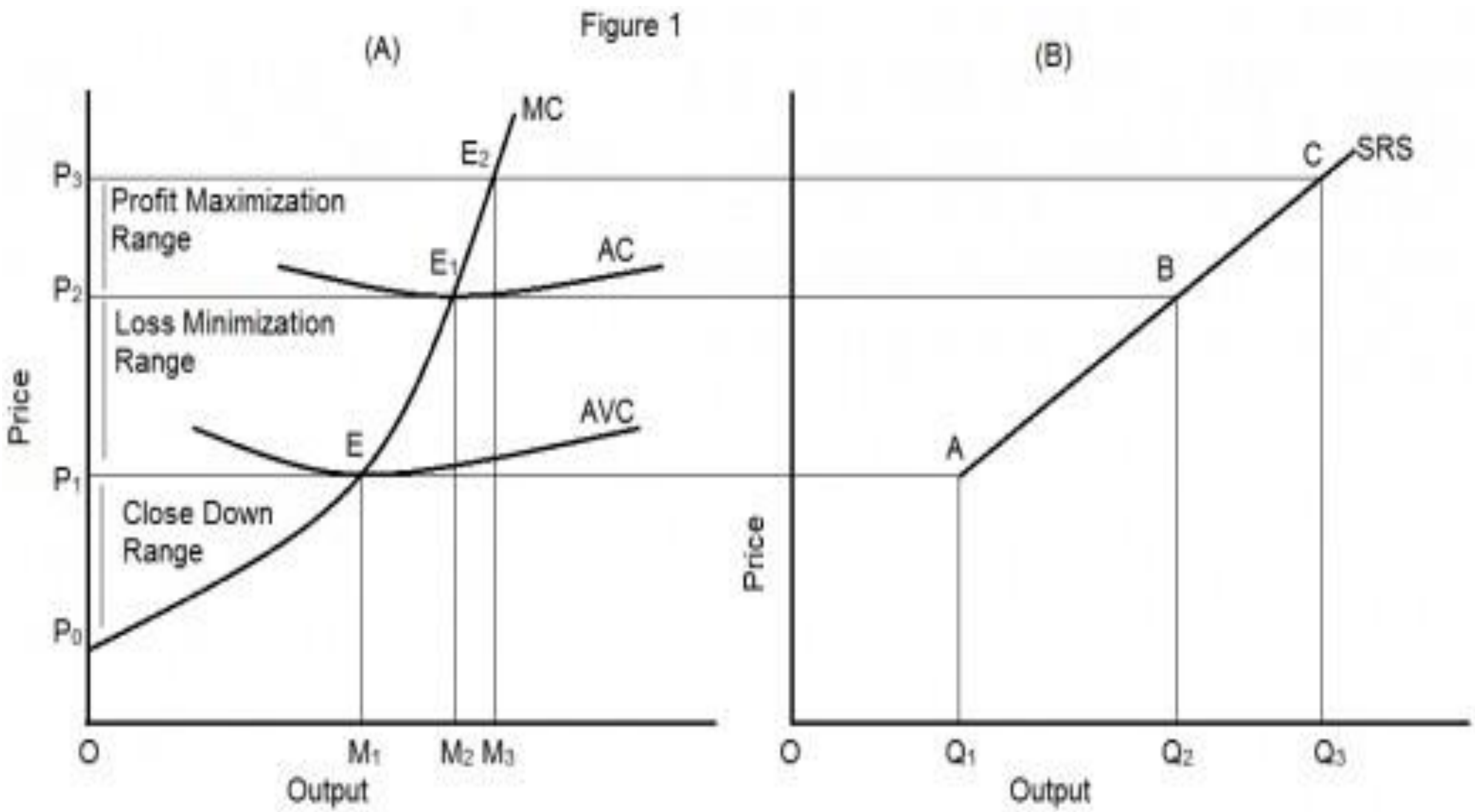
PMP in the *competitive market*

- Under competitive market, price is given.
- $MR(Q) = AR(Q) = P$
- This yields us

$$P = MC(Q)$$

- The equation tells us different level of Q chosen for different level of market price, P.
- What is this relationship: the Individual supply

PMP in competitive market



PMP in the *competitive market*

$$MR'(Q^*) = 0$$

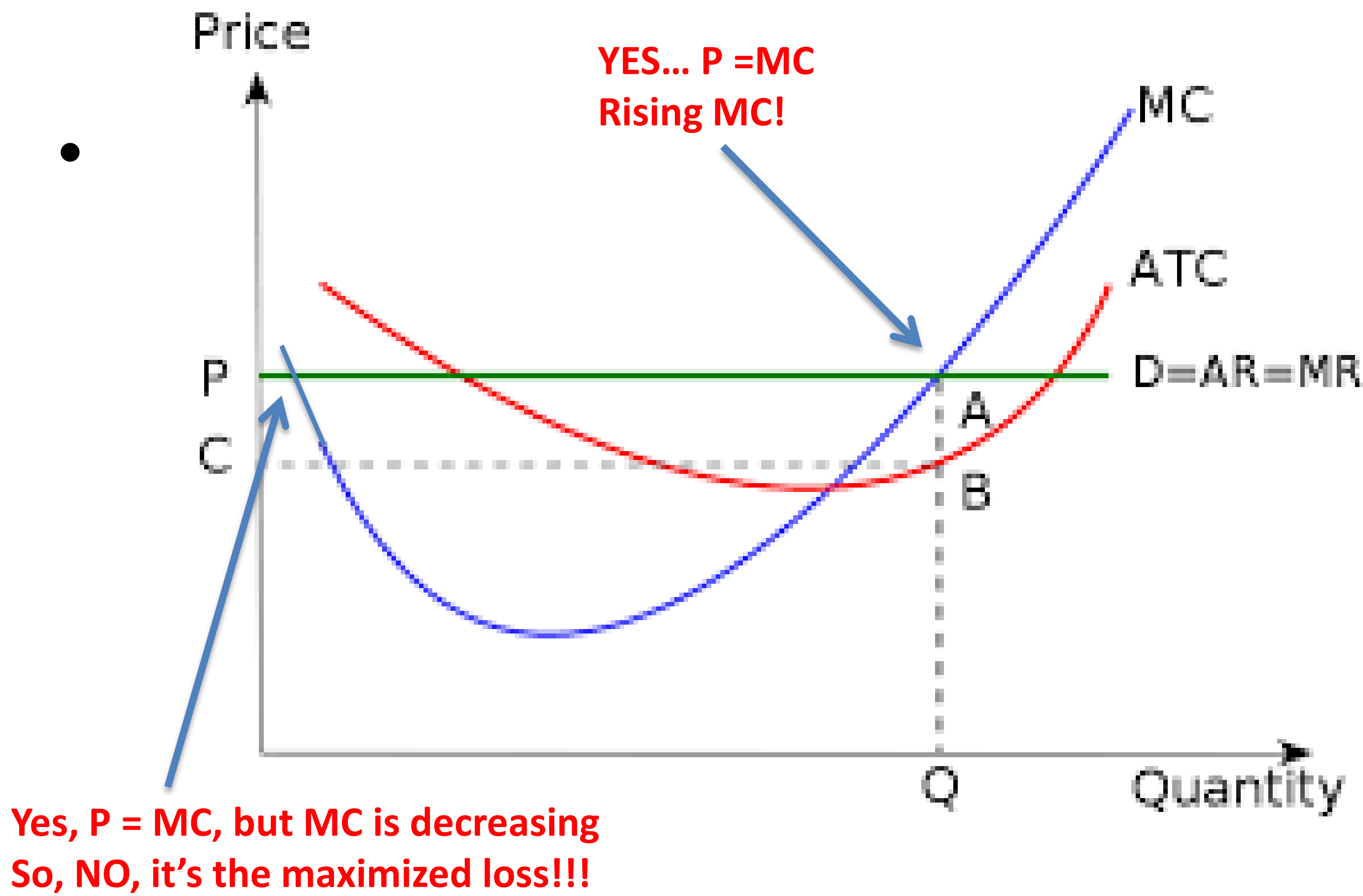
From the second-order condition:

$$0 - MC'(Q^*) < 0$$

$$MC'(Q^*) > 0$$

- Price intersects with marginal cost, for the part under which marginal cost is rising.

marginal cost must be rising, graphically illustrated…



Individual supply: *numerical example*

Suppose that cost function takes the following equation: $AC(Q) = 0.2Q + 4 + \frac{400}{Q}$

- What is the level of profit-maximizing output when P = 8?
- Derive the individual supply equation.

What is the level of profit-maximizing output when P = 8?

- I will solve the problem by treating “P” as a variable which can take any arbitrary values.
- Doing so would allow us to obtain the general solution of profit-maximizing level of output.
- To pin down the optimal level of output when price is $8, just plug in P=8 into the derived equation.

What is the level of profit-maximizing output when P = 8?

$$R(Q) = PQ$$

$$C(Q) = AC(Q) * Q = 0.2Q^2 + 4Q + 400$$

– Note “400” is the fixed cost.

$$\pi(Q) = PQ - (0.2Q^2 + 4Q + 400)$$

What is the level of profit-maximizing output when P = 8?

From the profit function,

$$\pi(Q) = PQ - (0.2Q^2 + 4Q + 400)$$

The optimal level Q can be solved by using the two-step procedure;

$$\pi'^{(Q)} = P - \left(0.4Q \quad + 4\right) = 0$$

So, when P = 8 ➔ Q = 10.

What is the level of profit-maximizing output when P = 8?

- This Q = 10 needs to be verified that it is actually the global optimizer.
- To do this, we check for the second derivative of the profit function.

$$\pi'' = -0.4 < 0$$

- So the profit function is monotonically concave.
- This implies that Q = 10 is the optimal level of output that generates the highest level of profit, under which P is equal $8.

Derive the individual supply equation.

- Back to the first-order condition, we know that optimal Q must satisfy this equation:

$$\pi'(Q) = P - (0.4Q + 4) = 0$$

  - Interpretably, different values of P results in different value of Q.
  - So, optimal condition suggests that Q is determined by P.
- If we rewrite "Q" in terms of P, we yield an expression that

$$Q = \frac{P-4}{0.4}.$$

Derive the individual supply equation.

- This function $Q = \frac{P-4}{0.4}$ is resemble to the mathematical expression of the individual supply function.
  - The function is increasing in "P".
- But, the function results in negative value of Q for the level of P which is lower than 4.
- So, this function only describe a porttion/part of the individual supply function. When price is lower than 4, firm would produce NOTHING, and thus resulting in Q equal to zero.

Derive the individual supply equation.

- A more proper way to define the individual supply function is to use the following representation:

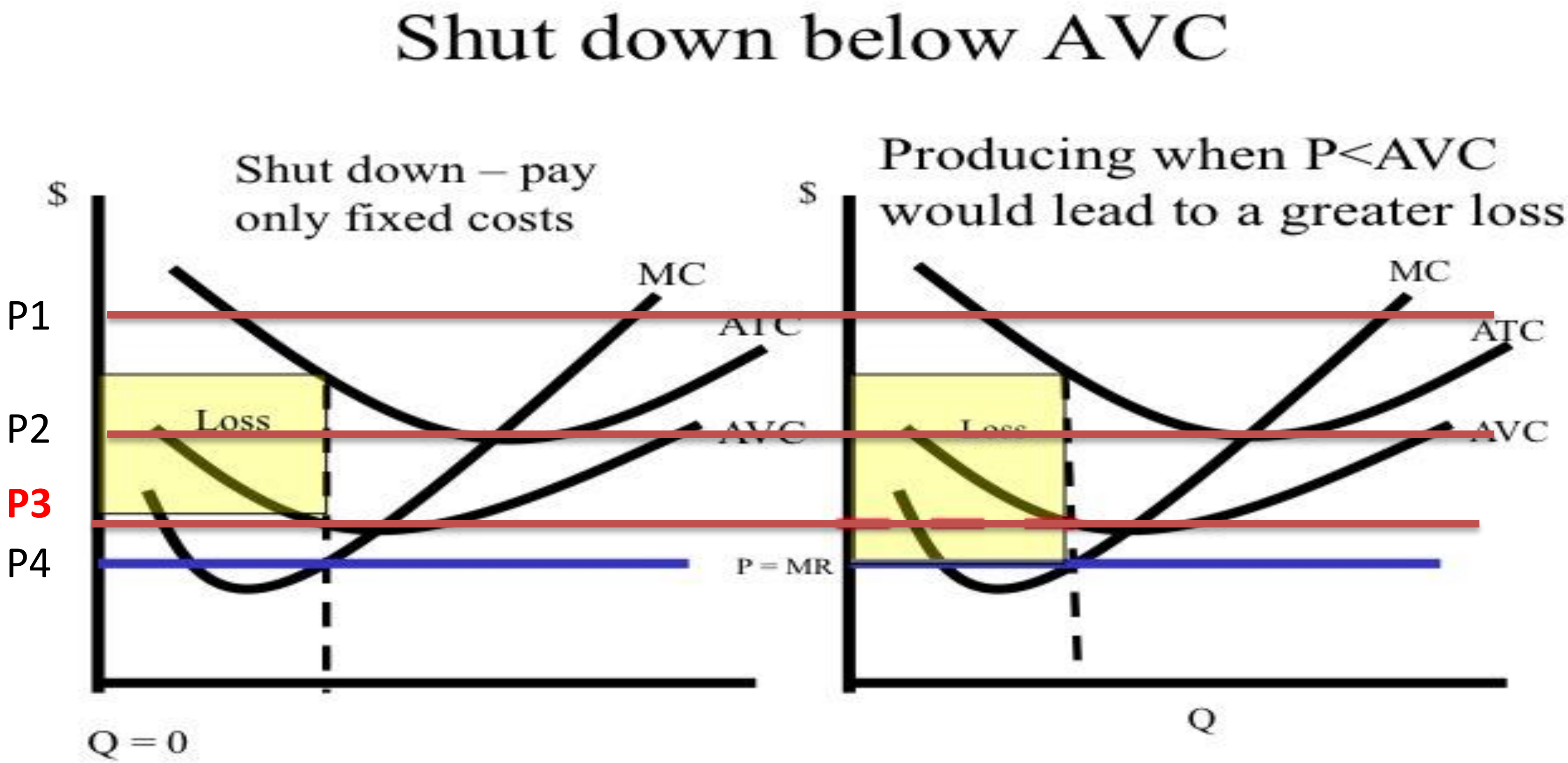
$$Q = \begin{cases} \dfrac{P-4}{0.4} & ; \quad \mathrm{P} \geq 4 \\ 0 & ; \quad 0 \leq P < 4 \end{cases}$$

- Economically, firm shut-down their plant when price is lower than \$4. At the level of P = \$4, firm is indifferent between (i) shutting-down plant and (ii) continue to produce with strict positive amount of output.

Individual supply curve and Shut-down point.

- If price declines, firms choose to lower the production because marginal cost is increasing.
- How low can the price go, while still keeping firm active in the market. Threshold to stop?
- The threshold is called the "shut-down price"
- It's the point where P = min(AVC)
  - Firm still produces even if price is below the break-even price, i.e. zero profit. (fixed cost is paid upfront.)

Shut-down point graphically illustrated.



Individual supply: shut-down point

Suppose that cost function takes the following equation: $TC(Q) = 40 + 8Q - 2Q^2 + Q^3$

– Finding the shut-down point.

$$\textit{Firm will shut down if price drops below the minimized level of AVC.}$$

From the beginning part, we've already seen that the level of minimized AVC is \$7.

Thus, if price is below \$7, firm will not produce anything. (Recall that at that level of Minimized AVC, firm will produce Q = 1 unit.

Individual supply: shut-down point

- So, setting P equal to MC, we can derive the supply function.

- In this example, the individual supply function, written in terms of Q, can be represented by;

$$P = 8 - 4Q \quad + 3Q^2; \quad Q \geq 1$$

Individual supply curve and Market supply curve

- Quantity supplied of each individual firm combined.
- From the example, individual supply curve is given by the equation ___see page 62_______.
- Active range of market price __P > 4_____.
- Suppose we have N identical firms, market supply is then $Q = \begin{cases} N\left(\frac{P-4}{0.4}\right) & ; \quad P \geq 4 \\ 0 & ; \quad 0 \leq P < 4 \end{cases}$

Example: Wrap up.

- Given P as the market price and $TC(Q) = 0.1Q^2 + 10$. Consider the following problem
  - Find the break-even price.
  - Find the shut-down price
  - Find the individual supply curve.
  - Suppose that there are "N" identical suppliers in the market, what is the equation that characterizes the market supply curve.

Find the shut-down price

Minimized level of AVC.

AVC = 0.1Q

AVC is minimized when Q = 0.

- When price is lower than zero, quantity supplied is equal to zero.
- Firm always stay active for the entire positive domain of price.

Individual supply

Firm will produce at the level of Q where P = MC or marginal profit is equal to zero.

$$P = 0.2Q$$

This is supply function.

Market supply with N firm

- Rewrite the supply function in Q-equal form and aggregate up the term for N times, we yield that

$$Q = N(5P) = 5NP$$

Competitive equilibrium: sketch

- At what price people trade?
- Short-run
  - Each firm takes price as given, and determine individual supply.
  - For a fixed number of firms, say “N”, market supply can be derived from an aggregation of each N-firm individual supply.
  - To derive the competitive equilibrium Q and P, we set *market supply and market demand* equal to each other.

Example

- Suppose that $TC(Q) = 0.1Q^2 + 10$ Consider the following problem
  - Derive individual supply equation.
  - Suppose that there are 5 identical firms. Derive market supply equation.

  Given further that two identical consumers in the market with individual demand equation given by P = 10 – Q.
  - Find the market equilibrium.
  - How much is the profit that each firm earns in the equilibrium.

Individual supply and Market supply

- From what we've seen before, market supply is given by

$$Q = N(5P) = 5NP$$

- With N = 5, the supply function is then,

$$Q = 5 * 5 * P = 25P$$

Market demand and Equilibrium

Setting demand equal to supply:

$P = 10 – Q$

$P = 10 – 25P$ (plug in the supply)

$P = 10/26$ (market price)

$Q = 250/26$ (market Q)

$q = 50/26$ (individual supply $= Q / N = Q / 5$)

Competitive equilibrium: just an idea..

- How about the long-run?
  - Zero profit will be attained. That is, P = MC = AC.
  - This is driven by entry-and-exit of firms in the market.
  - "N" will be changed until zero profit condition can be attained.
  - That is long-run equilibrium price would always be at the break-even price.
  - At this level of break-even price, we know (i) total quantity of demand and (ii) quantity of each individual supply. We can then figure out the number of firms in the long-run.

Profit maximization problem for Monopoly (PMP)

- First-order optimality condition

$$\pi'(Q^*) = 0$$
$$R'^{(Q)} - C'^{(Q)} = 0 \Rightarrow MR(Q^*) = MC(Q^*)$$

- Second order

$$\pi''(Q^*) < 0$$
$$MR'(Q^*) - MC'(Q^*) < 0$$

Profit maximization problem for Monopoly (PMP)

$$R(Q) = P(Q) * Q$$

$$Q + P(Q)\frac{dQ}{dQ} \Rightarrow \frac{dP}{dQ}Q + P(Q)$$

$$\Rightarrow P\left(\frac{dP}{dQ}\frac{Q}{P} + 1\right)$$

$$\varepsilon_d = \frac{dQ}{dP}\frac{P}{Q} \quad \rightarrow \quad \frac{dP}{dQ}\frac{Q}{P} = \frac{1}{\varepsilon_d} < 0$$

Profit maximization problem for Monopoly (PMP)

- $MR(Q) = P\left(\frac{dP}{dQ}\frac{Q}{P} + 1\right) = P\left(1 - \frac{1}{|\varepsilon_d|}\right)$
- Economically, the first order condition is

$$P\left(1 - \frac{1}{|\varepsilon_d|}\right) = MC(Q)$$

Monopoly :
optimal mark-up pricing
The more inelastic, the higher mark-up

$$\frac{P}{MC} = \frac{|\varepsilon_d|}{1 - |\varepsilon_d|}$$

PMP for monopoly

- Given market demand is p = 10 – Q and

$$TC = \frac{1}{3}Q^3 - Q^2 + 6Q$$

Find the level of profit-maximizing output, and price that producer would charge.

PMP for monopoly

- What happen if government imposes tax on the producer by \$3 baht for each unit of production. Determine the level of tax collected.

PMP for monopoly: sketch of work

- If the government wishes to gain the highest revenue from imposing the unit tax, what would be the appropriate level of unit tax?