Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

Full name Goatchasorn Atcharanuwul **Student ID.** 6004640477

Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean $(\mu) = 0$ and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

The model will be weakly stationary if
1) $E(r_t) = \mu$ (constant)
2) $var(r_t) = \sigma^2 = E[(r_t - \mu)^2]$ is constant.
3) $cov(r_t, r_{t-j}) = \gamma_j$

$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t = 0.25 + 1.5BR_t - 0.9B^2R_t + a_t$$

$$R_t - 1.5\,\cancel{B}R_t + 0.9B^2R_t\,] = 0.25 + a_t$$

$$[1 - 1.5B + 0.9B^2]R_t = 0.25 + a_t$$

Reverse characteristic equation: $x^2 - 1.5x + 0.9 = 0.$

$$\lambda^2 - \phi_1\lambda - \phi_2 = 0.$$

$$\lambda = \frac{-(-1.5) \pm \sqrt{(-1.5^2) - 4(1)(+0.9)}}{2(1)}$$

$\lambda_1 = 0.75 + 0.58i \rightarrow$ modulus $= \sqrt{(0.75^2) + (0.58^2)} = 0.948.$

$\lambda_2 = 0.75 - 0.58i \rightarrow$ modulus $= \sqrt{0.75^2 + (0.58)^2} = 0.948$

$\therefore$ which are less than 1 Hence R_t is weakly stationary

Question 1.2 (10 points)

Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$

conditional mean: $R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t$

$E[r_t|\cdot] = 0.25 + 1.5R_{t-1} - 0.9R_{t-2}$ #

unconditional mean:

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t$. and as $E[R_t] = \frac{\phi_0}{[1-\phi_1-\phi_2]}$

$E[R_t] = \frac{0.25}{(1-1.5+0.9)} = 0.625$ #

Question 1.3 (10 points) Your score.....................

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

- conditional variance: $var[R_t | F_{t-1}]$

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t$

$var(R_t|\cdot) = var[0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t|\cdot]$

$= var(1.5R_{t-1}|\cdot) + var(-0.9R_{t-2}|\cdot) + \underbrace{var(a_t|\cdot)}_{\sigma_a^2} + 2cov(1.5R_{t-1}, -0.9R_{t-2}|\cdot)$
$+ 2cov(0.9R_{t-2}, a_t|\cdot) + 2cov(1.5R_{t-2}, a_t|\cdot)$

$= 1.5^2 var(\cancel{R_{t-1}}|\cdot) + (-0.9^2) var(\cancel{R_{t-2}}|\cdot) + var(a_t|\cdot) + 2cov$ —〃——)
$+ 2cov(\cancel{0.9R_{t-2}}, a_t|\cdot) + 2cov(1.5\cancel{R_{t-2}}, a_t)\,r)$

$= \sigma_a^2 = 0.25$

- unconditional variance:

$(R_t - u) = 1.5(R_{t-1} - u) - 0.9(R_{t-2} - u) + a_t + 0.25$

$var(R_t) = E[(R_t - u)^2]$

$(R_t - u)^2 = 0.25^2 + (1.5)^2(R_{t-1} - u)^2 + (-0.9^2)(R_{t-2} - u)^2 + a_t^2 + 2(0.25)(1.5)(R_{t-1} - u)$
$+ 2[(0.25)(-0.9)(R_{t-2} - u)] + 2[(0.25)(a_t)] + 2[(1.5)(-0.9)(R_{t-1} - u)(R_{t-2} - u)]$
$+ 2[(1.5)(R_{t-1} - u)(a_t)] + 2[(-0.9)(R_{t-2} - u)(a_t)]$

$E[(R_t - u)^2] = 0.0625 + 2.25 var(R_T) + 0.81 var(R_T) + E[a_t^2] + 0.75 E(\cancel{R_{t-1}} - u)$
$- 0.45 E[R_{t-2} - u] + 0.5 \cancel{E[a_t]} - 2.7\gamma_1 + 3 E[(\cancel{R_{t-1}} - u) a_t]$
$- 1.8 E[(\cancel{R_{t-2} - u}) a_t] 0.$

$var(R_t) - 2.25 var(R_t) - 0.81 var(R_t) = \sigma^2 - 2.7\gamma_1 + 0.0625$

$(1 - 2.25 - 0.81) var(R_t) = \sigma^2 - 2.7\gamma_1 + 0.0625$

$var(R_t) = \dfrac{\sigma^2 - 2.7\gamma_1 + 0.0625}{-2.06}$

and as $\gamma_j = 1.5\gamma_{j-1} - 0.9\gamma_{j-2}$

$\gamma_1 = 1.5\gamma_0 - 0.9\gamma_1$

$1.9\gamma_1 = 1.5\gamma_0$

$\gamma_1 = \dfrac{1.5\gamma_0}{1.9}$

$[\ \rho_j = corr(r_t, r_{t-j}) = \dfrac{cov(r_t, r_{t-j})}{std(r_t)\,std(r_{t-j})} = \dfrac{\gamma_j}{var(R_t)}$

so $\gamma_1 = 0.8 var(R_t) \rightarrow var(R_t) = \dfrac{0.3125 - 2.7(0.8 var(R_t))}{-2.06}$

$-2.06 var(R_t) + 2.7(0.8 var(R_t)) = 0.3125$

$0.1 \operatorname{var}(R_t) = 0.3125$

$\operatorname{var}(R_t) = 3.125$#

Question 1.4 (10 points) Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

$R_t - u = 1.5(R_t - u) - 0.9(R_{t-2} - u) + a_t$

$E[(R_t - u | R_{t-j} - u)] = 1.5 E[(R_t - u)(R_{t-j} - u)] - 0.9 E[(R_{t-2} - u)(R_{t-j} - u)] + E[a_t \cdot (R_{t-j} - u)]$

$\gamma_j = 1.5 R_{j-1} - 0.9 R_{j-2} \rightarrow \frac{\gamma_j}{\gamma_0} = \frac{1.5\gamma_{j-1}}{\gamma_0} - \frac{0.9\gamma_{j-2}}{\gamma_0}$

$\rho_j = 1.5 \rho_{j-1} - 0.9 \rho_{j-2}$

$j = 1 \rightarrow \rho_1 = 1.5\left(\overset{1}{\cancel{\frac{\gamma_0}{\gamma_0}}}\right) - 0.9\left(\frac{\gamma_{-1}}{\gamma_0}\right) \Rightarrow \rho_1 = 1.5 - 0.9\rho_1 \rightarrow 1.9\rho_1 = 1.5 \rightarrow \rho_1 = \frac{1.5}{1.9} = 0.8$#

$j = 2 \rightarrow \rho_2 = 1.5\left(\frac{\gamma_1}{\gamma_0}\right) - 0.9\left(\overset{1}{\cancel{\frac{\gamma_0}{\gamma_0}}}\right) \Rightarrow \rho_2 = 1.5\rho_1 - 0.9 \Rightarrow \rho_2 = 1.5(0.8) - 0.9 = 0.3$#

Question 1.5 (10 points)

Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

n: current time

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t$

$R_{n+1} = 0.25 + 1.5R_n - 0.9R_{n-1} + a_{n+1}$

$\therefore E[R_{n+1} | F_n] = \hat{R}_n(1)$

$E[0.25|\cdot] + E[1.5R_n|\cdot] + E[-0.9R_{n-1}|\cdot] + E[a_{n+1}|\cdot]$

$\hat{R}_n(1) = 0.25 + 1.5R_n - 0.9R_{n-1} + 0.$

* 1 stepahead.

$\hat{R}_{1000}(1) = 0.25 + 1.5R_{1000} - 0.9R_{999}$

$= 0.25 + 1.5(0.01) - 0.9(0.02)$

Then $\hat{R}_{1000}(1) = 0.247$ - point estimator - 1

2 Forecasting error:

$R_{n+1} - \hat{R}_n(1) = e_n(1) = a_{n+1}$

$R_{1001} - \hat{R}_{1000}(1) = e_{1001}$

3 variance : $var(e_n(1)|\cdot) = var(a_{1001}|\cdot) = \sigma_a^2 = 0.25$

4 Interval estimation: $\hat{R}_n(1) \pm Z_{\frac{\alpha}{2}} \sqrt{var(a_{1001}|\cdot)}$

$\hat{R}_{1000} \pm 1.96\sqrt{0.25} = 0.247 \pm 0.98 \Rightarrow -0.73 \leq R_{1001} \leq 1.23$

* 2 stepahead:

$R_{n+2} = 0.25 + 1.5R_{n+1} - 0.9R_{n+2} + a_{n+2}$

$E[R_{n+2}|\cdot] = \hat{R}_n(2) = 0.25 + 1.5E[R_{n+1}|\cdot] \underbrace{-0.9E[R_n|\cdot]}_{-0.9R_{1000}} + E[a_{n+2}|\cdot]$

$\hat{R}_{1000}(2) = 0.25 + 1.5(0.247) - 0.9(0.01) + 0.$

$\hat{R}_{1000}(2) = 0.612$

2 Forecasting error:

$e_{1000}(2) = R_{1002} - \hat{R}_{1000}(2) = 1.5R_{1001} - 1.5\hat{R}_{1001} + a_{n+2}$

$= 1.5(R_{1001} - \hat{R}_{1001}) + a_{n+2}$

$= 1.5(a_{1001}) + a_{n+2}$

3 variance for forecasting error:

$var(e_{1000}(2)|\cdot) = var[1.5a_{1001} + a_{1002}|\cdot]$

$$var(e_{1000}(2)|\cdot) = 1.5^2 var(a_{1001}|\cdot) + (0.9^2)\, var(a_{1002}|\cdot) + 2cov(1.5a_{1001}, \cancel{a_{1002}})|\cdot)$$

$$= 1.5^2 \sigma_a^2 + \sigma_a^2 = (1.5^2+1)\sigma_a^2 = 3.25(0.25)$$

$$= 0.8125 \ \#$$

4 Interval forecasting:

$$\hat{R}_{1000}(2) \pm 1.96\sqrt{0.8125}$$

$$= 0.612 \pm 1.77 \Rightarrow -1.158 \leq R_{1002} \leq 2.382$$

* Infinity stepahead

$$\hat{R}_t(\infty) = \phi_0 + \sum_{i=1}^{p} 1.5\hat{R}_t(\infty - i)$$

$$\hat{R}_t(\infty) = E(R_t)$$

$$var(e_t \infty) = var(R_t) = \sigma^2 = 0.25$$

Interval forecasting : $E(R_t) \pm 1.96\sqrt{0.25}$

$$= 0.625 \pm (0.98)$$

$$= -0.355 \leq R_\infty \leq 1.605 \ \#$$