Solution: Quiz 4

1. Find the solution set for

$$\left|\frac{1+x}{x-|x|}\right| \le \frac{1}{|x|}.$$

Solution: Notice that

$$\left|\frac{1+x}{x-|x|}\right| \le \frac{1}{|x|} \quad \Leftrightarrow \quad -\frac{1}{|x|} \le \frac{1+x}{x-|x|} \le \frac{1}{|x|}.$$

So we will consider 2 cases based on the definition of absolute value: $|x| = \begin{cases} -x, & x<0 \\ x, & x \ge 0. \end{cases}$

(I) For $x < 0$ or $x \in (-\infty, 0)$,

$$\begin{aligned}
-\frac{1}{|x|} &\le \frac{1+x}{x-|x|} \le \frac{1}{|x|} \\
-\frac{1}{-x} &\le \frac{1+x}{x-(-x)} \le \frac{1}{-x} \\
\frac{1}{x} &\le \frac{1+x}{2x} \le -\frac{1}{x} \\
\text{Multiplying by } x<0 \qquad 1 &\ge \frac{1+x}{2} \ge -1 \\
2 &\ge 1+x \ge -2 \\
2-1 &\ge x \ge -2-1 \quad \Rightarrow \quad x \in [-3, 1]
\end{aligned}$$

and the solution set in this case is $[-3,1] \cap (-\infty, 0) = [-3, 0)$.

(II) For $x \ge 0$, we have $|x| = x$ implying $x - |x| = 0$. However the term $x - |x|$ is the denominator and cannot be zero. I.e. $x - |x| \ne 0 \quad \Leftrightarrow \quad |x| \ne x \quad \Leftrightarrow \quad x \not\ge 0 \quad \Leftrightarrow \quad x \notin [0, \infty)$.
So there is no solution in this case.

From (I) and (II), the solution set only comes from (I) and it is given by $[-3, 0)$. ■