Vector spaces and subspaces

- Vectors and Vector equations
- Subspaces of R^n
- Null spaces, Column spaces
- Linear indepenence, Spanning sets
- Basis and Dimension
- Rank
- Linear transformations, change of basis

Vector: A matrix with only one column.

Vectors in $\mathbf{R}^n$ (vectors with n entries):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Geometric Description of $\mathbf{R}^2$

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane.

$\mathbf{R}^2$ is the set of all points in the plane.

Parallelogram rule for addition of two vectors:
If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are given below:

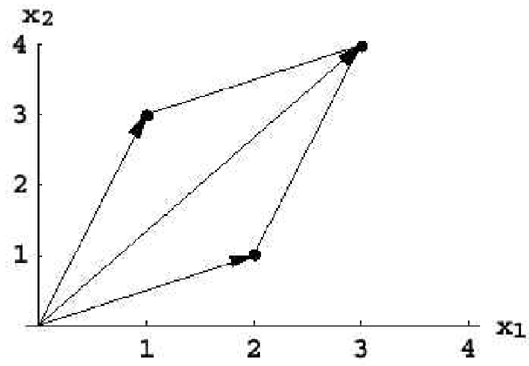
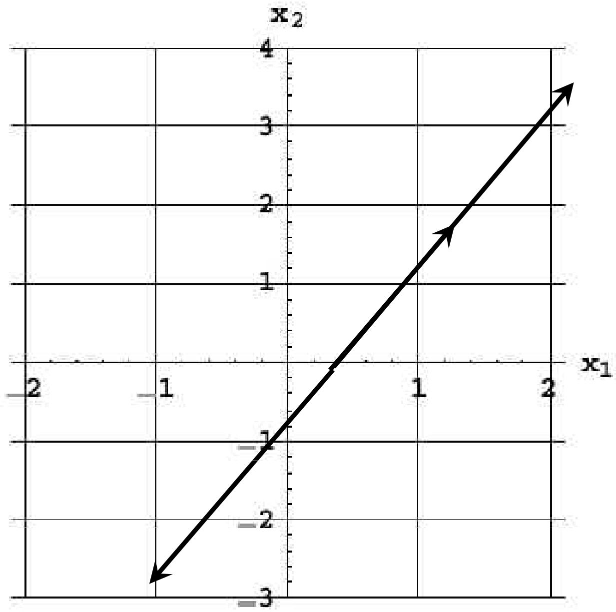


Illustration of the Parallelogram Rule

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.



in R^2

MA332 LINEAR ALGEBRA

Linear Combinations

DEFINITION

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using weights $c_1, c_2, \ldots, c_p$.

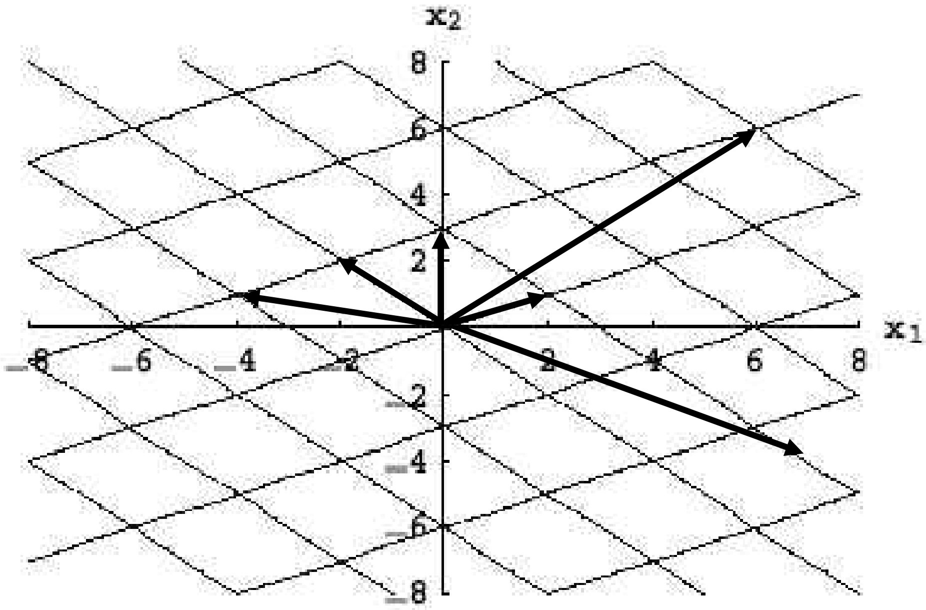
Examples of linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$3\mathbf{v}_1 + 2\mathbf{v}_2, \qquad \frac{1}{3}\mathbf{v}_1, \qquad \mathbf{v}_1 - 2\mathbf{v}_2, \qquad \mathbf{0}$$

Dr. Julaluk Carmai

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix},\ \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



MA332 LINEAR ALGEBRA

EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$,

and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution: Vector $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

Corresponding System:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -1 \\ 2x_2 + 6x_3 &= 8 \\ 3x_1 + 14x_2 + 10x_3 &= -5 \end{aligned}$$

Corresponding Augmented Matrix:

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = \_\_\_ \\ x_2 = \_\_\_ \\ x_3 = \_\_\_ \end{matrix}$$

Review of the last example: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{matrix}$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}.$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

The span of a set of vectors

Definition

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbf{R}^n$; then

Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ = set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

Stated another way: **Span**$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written as

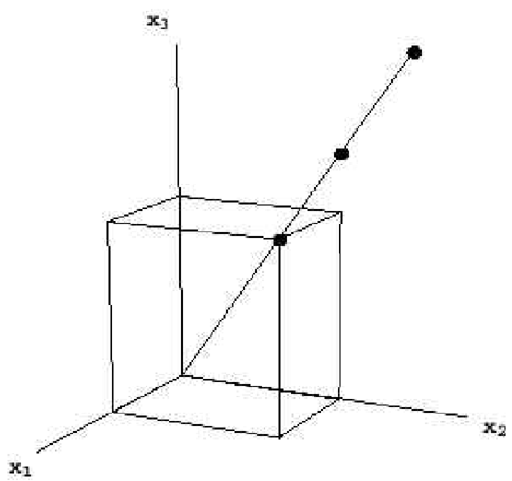
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$$

where $x_1, x_2, \ldots, x_p$ are scalars.

A Geometric description of Span {v} and Span {u,v}

The Span of a Set of Vectors

EXAMPLE: Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Label the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ together with $\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ on the graph below.

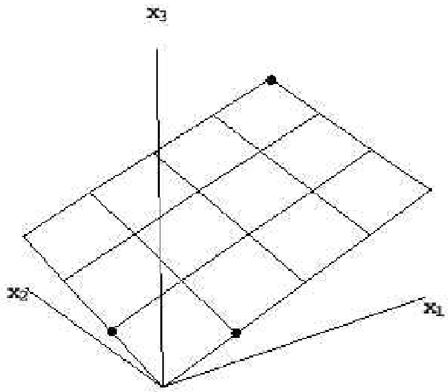


$\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ all lie on the same line.
Span$\{\mathbf{v}\}$ is the set of all vectors of the form $c\mathbf{v}$.
Here, **Span**$\{\mathbf{v}\}$ = a line through the origin.

- $\underline{u}$ and $\underline{v}$ are nonzero vector in R^3 with $\underline{v}$ is not a multiple of $\underline{u}$
- Span $\{\underline{u},\underline{v}\}$ is a plane in R^3 that contains $\underline{u}$, $\underline{v}$ and $\underline{0}$
- Span {u,v} contains
  the line in R^3 through $\underline{u}$ and $\underline{0}$
  and the line in R^3 through $\underline{v}$ and $\underline{0}$

u, **v**, **u** + **v** and 3**u** +4**v** all lie in the same plane.
Span{**u**, **v**} is the set of all vectors of the form $x_1\mathbf{u} + x_2\mathbf{v}$.
Here, **Span**{**u**, **v**} = a plane through the origin.

EXAMPLE: Label **u**, **v**, **u** + **v** and 3**u** +4**v** on the graph below.



EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) Find a vector in **Span**{$\mathbf{v}_1, \mathbf{v}_2$}.

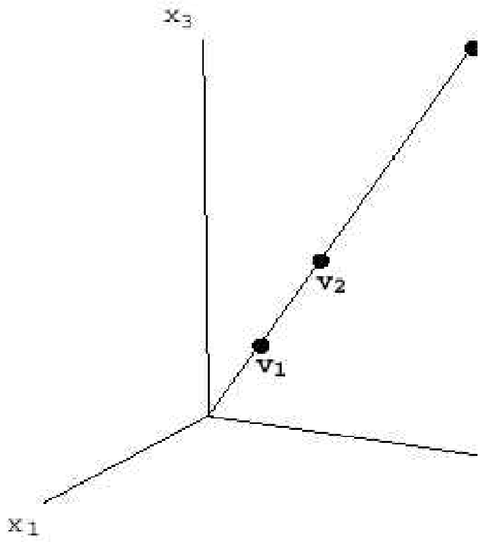
$$x_1\underline{v}_1 + x_2\underline{v}_2 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Vectors in R^2

(b) Describe **Span**{$\mathbf{v}_1, \mathbf{v}_2$} geometrically.

A line through the origin.

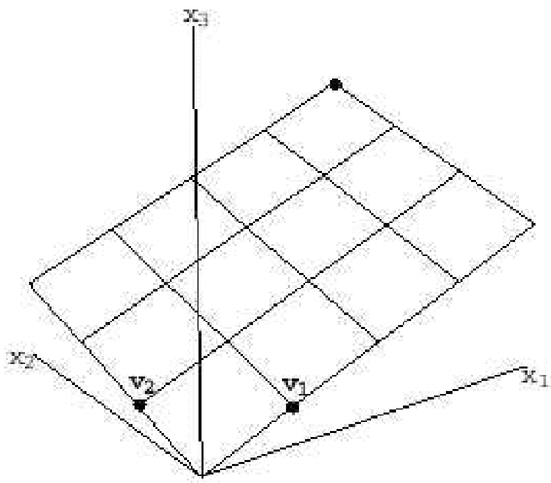
Spanning Sets in R^3



$\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ = **Span**$\{\mathbf{v}_1\}$ = **Span**$\{\mathbf{v}_2\}$

(line through the origin)



$\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ = plane through the origin

MA332 LINEAR ALGEBRA

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$. Is **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?

A line $\quad \underline{v}_2 = \dfrac{3}{2}\underline{v}_1$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is $\mathbf{b}$ in the plane spanned by the columns of A?

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Do x_1 and x_2 exist so that

Corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{array}\right]$$

So $\mathbf{b}$ is not in the plane spanned by the columns of A

Is b a linear combination of columns of A? If it is $x_1\underline{a}_1 + x_2\underline{a}_2 = \underline{b}$ must have solution.

Vector Spaces and Subspaces

Many concepts concerning vectors in $\mathbf{R}^n$ can be extended to other mathematical systems.

We can think of a *vector space* in general, as a collection of objects that behave as vectors do in $\mathbf{R}^n$. The objects of such a set are called *vectors*.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

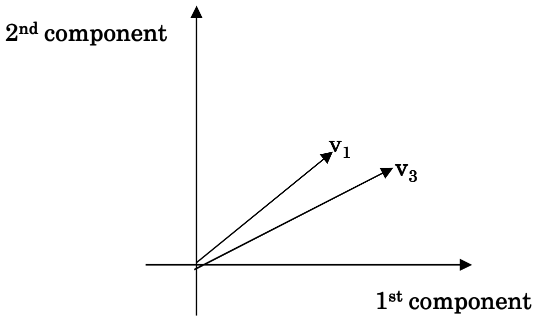
The space R^n consists of all column vectors with n components

A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers.
The resulting vector must be within the space.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Example

R^2 ➔ all 2D real vectors



$$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}$$

Vector addition?
Scalar multiplication?

Every vector spaces got zero vectors in it.

MA332 LINEAR ALGEBRA

Not a vector space
Consider all vectors whose components are positive or zero.
If the original space is the x-y plane R^2

Vector addition?

Scalar Multiplication? Multiplying a vector (1,3) by –2➔
This ¼ is not closed under scalar multiplication.

The distinction between a subset and a subspace
✓Can you add vectors? and
✓Can you multiply by scalars without leaving the space?

Vector spaces have to be closed by addition and scalar multiplication.

Dr. Julaluk Carmai

Subspaces

Vector spaces may be formed from subsets of other vectors spaces. These are called *subspaces*.

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

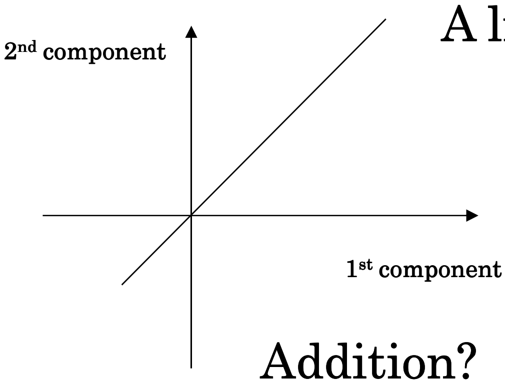
b. For each **u** and **v** are in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

A subspace is a subset which is closed under addition and multiplication.

A vector space inside R^2 ➔ a subspace of R^2



A line through the origin

Addition?

Multiplication?

The zero dimensional space → a subspace contains only one vector, the zero vector. (the smallest subspace)

Vector addition → 0+0=0 → within the subspace

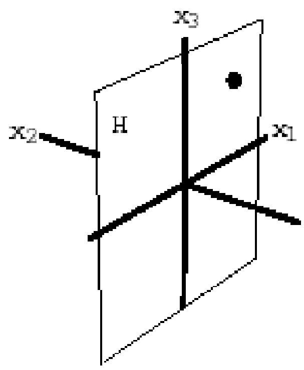
Scalar multiplication→ c 0=0→ within the subspace

The largest subspace is the whole of the original space.

Subspaces of R^2
- R^2 itself
- Any lines through the zero vector (the origin)
- The zero vector

Subspaces of R^3
- R^3 itself
- Any plane through the zero vector (the origin)
- Any line through the zero vector
- The zero vector

EXAMPLE: Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.



Solution: Verify properties a, b and c of the definition of a subspace.

a. The zero vector of $\mathbf{R}^3$ is in H (let a = _______ and b = _______).

b. Adding two vectors in H always produces another vector whose second entry is ______ and therefore the sum of two vectors in H is also in H. (H is closed under addition)

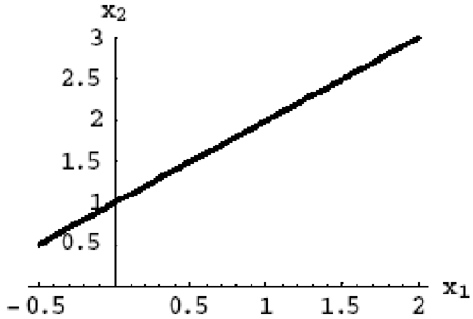
c. Multiplying a vector in H by a scalar produces another vector in H (H is closed under scalar multiplication).

Since properties a, b, and c hold, H is a subspace of $\mathbf{R}^3$.

Note: Vectors $(a, 0, b)$ in H look and act like the points (a, b) in $\mathbf{R}^2$.

EXAMPLE: Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ is real} \right\}$ a subspace of ?

I.e., does H satisfy properties a, b and c?



Graphical Depiction of H

All three properties must hold in order for H to be a subspace of $\mathbf{R}^2$.

Property (a) is not true because

____. Therefore H is not a subspace of $\mathbf{R}^2$.

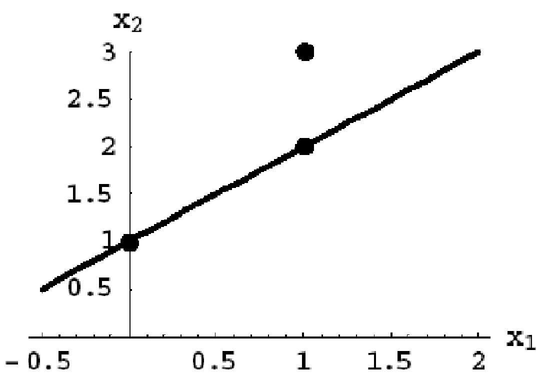
MA332 LINEAR ALGEBRA

Another way to show that H is not a subspace of $\mathbf{R}^2$:

Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and so $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is not____ in H. So property (b) fails and so H is not a subspace of $\mathbf{R}^2$.



Property (b) fails

- **A line in R^2 not through the origin is not a subspace of R^2**
- **A plane in R^3 not through the origin is not a subspace of R^3**

$$A=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}$$

Columns are in_______

The **column space** of an $m \times n$ matrix A (Col A) is the set of all linear combinations of the columns of A.

If $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Column space of A is a subspace of $\underline{\ \ R^m\ \ \ \ \ \ }$

What are in this subspace?

MA332 LINEAR ALGEBRA

Subspaces are tied directly to matrix A and they give information about the system Ax=b

Connection with linear system $Ax=b$

- Does $Ax=b$ have solution for every b?

- Which RHS allow this system to be solved?

$$\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix} \longrightarrow u\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+v\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+w\begin{bmatrix}2\\3\\4\\5\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

Find numbers u,v,w that multiply col1, col2, col3 to produce the vector *b*. The system is solvable exactly when such coefficient exist.

The subset of attainable RHS b is the set of all combinations of the columns of A.

The equations Ax=b can be solved if and only if b lies in *the column space of A*

Recall that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the columns of A. Therefore

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}$$

We can also describe the result geometrically. Ax=b can be solved if and Only if b lines in the plane that is spanned by 2 calcium vectors

EXAMPLE: Find a matrix A such that $W = Col\ A$ where

$$W = \left\{ \begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} : x, y \text{ in } \mathbf{R} \right\}.$$

Solution:

$$\begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore $A = \begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}$.

The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Nul $A = \{\mathbf{x} : \mathbf{x}$ is in $\mathbf{R}^n$ and $A\mathbf{x} = \mathbf{0}\}$ (set notation)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Null space is in __________

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since _________, $\mathbf{0}$ is in

___________.

Therefore

$A(\mathbf{u}+\mathbf{v})$ = _______ + _______ = ______ + ______ = ___.

Property (b) If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, show that $\mathbf{u}+\mathbf{v}$ is in Nul A. Since $\mathbf{u}$ and $\mathbf{v}$ are in Nul A,

__________ and __________.

Property (c) If **u** is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \underline{\quad} A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

Solving $A\mathbf{x} = \mathbf{0}$ yields an ***explicit description*** *of Nul A.*

EXAMPLE: Find an explicit description of Nul A where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$$

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

MA332 LINEAR ALGEBRA

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

The Contrast Between Nul A and Col A

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of A is a subspace of $\mathbf{R}^k$ where k =_____.

(b) The null space of A is a subspace of $\mathbf{R}^k$ where k =_____.

(c) Find a nonzero vector in Col A. (There are infinitely many possibilities.)

$$\_\_\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \_\_\begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

MA332 LINEAR ALGEBRA

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_2$

x_2 is free

$x_3 = 0$

Let $x_2 =$ ___ and then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Dr. Julaluk Carmai

Review

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each **u** and **v** in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x - y = 0 \\ y + z = 0 \end{matrix} \right\}$

Solution: Rewrite $\begin{matrix} x - y = 0 \\ y + z = 0 \end{matrix}$ as

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So V =Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

MA332 LINEAR ALGEBRA

(c) $S = \left\{ \begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

One *Solution:* Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

S =span$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$; therefore S is a vector ace by 1.

Another Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

S =Col A where $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$; therefore S is a vector space,

since a column space is a vector space.