Name: .... ID: ....

Quiz 5:

1. Consider the production function $Q = (K^{\frac{1}{3}} + L^{\frac{1}{3}})^3$.
   a) Show that this function exhibits constant returns to scale.
   b) Set up the firm's cost minimization problem, solve the problem, and check for the second-order condition.
   c) show that the cost function can be expressed as unit cost multiplied with quantity produced.

Solution

a) From $Q(K,L) = (K^{\frac{1}{3}} + L^{\frac{1}{3}})^3$, for any constant $j > 0$ we have:

$$\begin{aligned} Q(jK, jL) &= \left[(jK)^{\frac{1}{3}} + (jL)^{\frac{1}{3}}\right]^3 \\ &= \left[j^{\frac{1}{3}}\,(K^{\frac{1}{3}} + L^{\frac{1}{3}})\right]^3 \\ &= j(K^{\frac{1}{3}} + L^{\frac{1}{3}})^3 \\ &= jQ(K,L). \end{aligned}$$

The function is homogeneous of degree 1, which implies that the production function exhibits constant returns to scale.

b) Set up the Lagrangian equation:

$$Z = rK + wL + \mu\left[Q_0 - (K^{\frac{1}{3}} + L^{\frac{1}{3}})^3\right]$$

FOC:

$$\frac{\partial Z}{\partial K} = r - \mu\left[3\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2\left(\frac{1}{3}\right)K^{-\frac{2}{3}}\right] = 0 \qquad (1)$$

$$\frac{\partial Z}{\partial L} = w - \mu\left[3\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2\left(\frac{1}{3}\right)L^{-\frac{2}{3}}\right] = 0 \qquad (2)$$

$$\frac{\partial Z}{\partial \mu} = Q_0 - (K^{\frac{1}{3}} + L^{\frac{1}{3}})^3 = 0. \tag{3}$$

From the first two equations:

$$\frac{r}{w} = \left(\frac{L}{K}\right)^{\frac{2}{3}} \rightarrow K = \left(\frac{w}{r}\right)^{\frac{3}{2}} L. \tag{4}$$

Substitute (4) into (3), we have:

$$Q_0 - \left[\left(\frac{w}{r}\right)^{\frac{1}{2}} L^{\frac{1}{3}} + L^{\frac{1}{3}}\right]^3 = 0$$

$$\left[\left\{1 + \left(\frac{w}{r}\right)^{\frac{1}{2}}\right\} L^{\frac{1}{3}}\right]^3 = Q_0$$

$$\left[1 + \left(\frac{w}{r}\right)^{\frac{1}{2}}\right]^3 L^* = Q_0$$

$$\left[\frac{r^{1/2}+w^{1/2}}{r^{1/2}}\right]^3 L^* = Q_0$$

$$L^* = \left[\frac{r^{1/2}}{r^{1/2}+w^{1/2}}\right]^3 Q_0$$

$$L^* = \frac{r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3} Q_0 \tag{A}$$

Substitute (5) into (4), we get:

$$K^* = \frac{w^{3/2}}{r^{3/2}} \frac{r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3} Q_0$$

$$K^* = \frac{w^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3} Q_0 \tag{B}$$

SOC:

Note that for $K, L, \mu > 0$:

$$\frac{\partial^2 Z}{\partial K^2} = -\mu\left[\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2\left(-\frac{2}{3}\right)K^{-\frac{5}{3}} + K^{-\frac{2}{3}}(2)\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(\frac{1}{3}\right)K^{-\frac{2}{3}}\right]$$

$$= -\mu\left[\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(-\frac{2}{3}\right)\left(K^{-\frac{4}{3}} + K^{-\frac{5}{3}}L^{\frac{1}{3}} - K^{-\frac{4}{3}}\right)\right]$$

$$= \mu\left(\frac{2}{3}\right)\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(K^{-\frac{5}{3}}L^{\frac{1}{3}}\right) > 0$$

$$\frac{\partial^2 Z}{\partial L^2} = -\mu\left[\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2\left(-\frac{2}{3}\right)L^{-\frac{5}{3}} + L^{-\frac{2}{3}}(2)\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(\frac{1}{3}\right)L^{-\frac{2}{3}}\right]$$

$$= -\mu\left[\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(-\frac{2}{3}\right)\left(K^{\frac{1}{3}}L^{-\frac{5}{3}} + L^{-\frac{4}{3}} - L^{-\frac{4}{3}}\right)\right]$$

$$= \mu\left(\frac{2}{3}\right)\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(K^{\frac{1}{3}}L^{-\frac{5}{3}}\right) > 0$$

$$\frac{\partial^2 Z}{\partial K \partial L} = -\mu\left[(2)\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(\frac{1}{3}\right)L^{-\frac{2}{3}}K^{-\frac{2}{3}}\right]$$

$$= -\mu\left(\frac{2}{3}\right)\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)\left(K^{-\frac{2}{3}}L^{-\frac{2}{3}}\right) < 0$$

Therefore, the bordered Hessian is:

$$|\overline{H}| = \begin{vmatrix} 0 & \left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2 K^{-\frac{2}{3}} & \left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2 L^{-\frac{2}{3}} \\ \left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2 K^{-\frac{2}{3}} & \frac{\partial^2 Z}{\partial K^2} & \frac{\partial^2 Z}{\partial K \partial L} \\ \left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^2 L^{-\frac{2}{3}} & \frac{\partial^2 Z}{\partial K \partial L} & \frac{\partial^2 Z}{\partial L^2} \end{vmatrix}$$

$$|\bar{H}| = 2\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^4 K^{-\frac{2}{3}} L^{-\frac{2}{3}} \frac{\partial^2 Z}{\partial K \partial L} - \left[\left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^4 L^{-\frac{4}{3}} \frac{\partial^2 Z}{\partial K^2} + \left(K^{\frac{1}{3}} + L^{\frac{1}{3}}\right)^4 K^{-\frac{4}{3}} \frac{\partial^2 Z}{\partial L^2}\right] < 0.$$

This implies that the solutions represent the minimum point.

c) To construct the cost function, we substitute (5) and (6) back into the objective function $rK + wL$. Then the cost function C is:

$$C = r\left[\frac{w^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3} Q_0\right] + w\left[\frac{r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3} Q_0\right]$$

$$C = \left[\frac{r\,w^{3/2}+w\,r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3}\right] Q_0.$$

Observe that we can see $\left[\frac{r\,w^{3/2}+w\,r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3}\right]$ as the cost per unit of production. For example:

If the firm produces $Q_0 = 1$, the (minimum) cost of production is $\left[\frac{r\,w^{3/2}+w\,r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3}\right]$.

If the firm produces $Q_0 = 2$, the (minimum) cost of production is $2\left[\frac{r\,w^{3/2}+w\,r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3}\right]$.

If the firm produces $Q_0 = 3$, the (minimum) cost of production is $3\left[\frac{r\,w^{3/2}+w\,r^{3/2}}{\left(r^{1/2}+w^{1/2}\right)^3}\right]$.