Chapter 2

Probability

2.1 Background

The theory of probability could be extracted from the roots of where gambling started. Mathematics was brought into explaining the concept of probability, where it has been introduced by Pierre de Fermat and Blaise Pascal (1654) when Chevalier de Méré was troubled by the concept of tossing the dice and therefore, relayed his question to both individuals. Later in the 18th century, Jacob Bernoulli and Abraham de Moivre had combined Mathematics with probability in a much more complex way, where Bernoulli has proved the 'Law of Large Number'. The development of the latter continued on, through various mathematicians like Carl Friedrich Gauss, in the 19th century. At this point, Ludwig Boltzmann and J. Willard Gibbs have both introduced the theory where probability has its roots from statistics; the theory includes the characteristics of gas, for instance, the temperature in the term of random motion many particles have. In the 20th century, probability and statistics became significant in the process of hypothesis testing, introduced by R.A Fisher and Jerzy Neyman. The latter led to what we refer to as 'Probability Distribution', in case the null hypothesis is true. At this time period, the axiom of probability has been introduced, where it is sometimes referred to as the 'Kolmogorov axioms' derived from the name 'Andrey Kolmogorov', an axiom that holds a strong importance in probability.

2.2 Sample Space and Events

The theory of probability will only regard events where the results are randomized, which often refers to as an '**Experiment**'. Although the results are not known beforehand, all the possible results are to be known. We call the set of these possible outcomes '**Sample Space**', which will be symbolized as 'S'.

Example 1 Write down the sample space for the following experiment

1. Suppose the way to work will encounter a total of 3 intersections. Each intersection has a stop light (s) and a go light (c). Provide the possible traffic light types a driver could possibly encounter.

   Answer:

2. In horse racing, 3 horses are represented as: H1, H2, and H3. Provide the possible sequence where 3 horses reach the finish line.

   Answer:

3. An experiment that strives to measure the lifespan (in hours) of a light bulb

   Answer:

4. A family has 2 kids. Consider the genders: (M or F)

   Answer:

5. The number of people waiting in line at the counter-front

   Answer:

If the object of interest is a subset of S, we call the set an '**Event**', which is represented with an E.

From example 1 – 5, answer the following questions

6. An event where the car stops at the first intersection

   Answer:

7. An event where the first horse (H1) will not reach the finish line first.

   Answer:

8. An event where the light bulb has a lifespan of not over 48 hours

   Answer:

9. An event where this family could give birth to 2 kids of 2 different genders

   Answer:

10. An event where the number of people wait in line at the counter-front does not exceed 3 people

    Answer:

For any two events, let A and B, which are the subsets of S, be the representation of any two events.

$A \cup B$ is an event that consists of members being in set A or B, or both areas

$A \cap B$ is an event that consists of members being in set A and B

A' is an event that consists of members not being in set A (but in S)

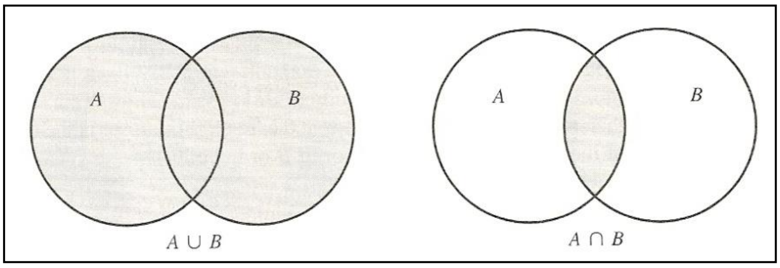

Figure 2.1 Venn Diagram for $A \cup B$ and $A \cap B$

The events: A, B, and C are still in accordance to the rule

1. Commutative law

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. Associative law

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. Distributive law

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

4. De Morgan's law

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Furthermore, there is an important definition related to the set, which is of the following:

Definition 1 Set A and B will be referred to as mutually exclusive sets or disjoint set when $A \cap B = \varnothing$

For A_1, A_2, … will be referred to as disjoint sets when $A_i \cap A_j = \varnothing$ for all i and j, and $i \neq j$.

Example 2 An experiment involves 2 dices; let E represent an event where the results from dice tossing has the sum that is odd number and not less than 7, let F represent an event where at least 1 dice shows a numeral of '3', and let G represent an event where the sum of the numerals on the dice equals to 5. Find the following:

1. $E \cap F$

2. $F \cap G$

3. $E \cap F \cap G$

4. $E \cup F$

5. Are E and F mutually exclusive events?

6. Are E and G mutually exclusive events?

7. Are F and G mutually exclusive events?

8. Are E, F, and G events considered mutually exclusive events?

2.3 Axioms in Probability

One way to define probability for an event is to define the term of relative probability, by assuming that an experiment has a sample space S which will be repeatedly experimented under similar condition. For E, let n(E) represent the number of times that the event E happened in a total of n times the experiment has been done. The probability of the event E is substituted by P(E), and will be defined as follows:

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

It can be seen that P(E) can be calculated from the limit of the number of times the event E happened.

However, the above definition will be in accordance to common sense although a significant downside would be n(E) n, where it is required to converge in order to find a constant value which has to remain constant when n continuously increases, which is what we find unsure. After probability has been defined, the concept of defining axioms has been applied alongside the concept of probability itself. As a result, we refer the latter as axioms in probability or the ‘Kolmogorov Axioms’.

In an experiment with sample space S; for each event E, S will result in P(E) which is in accordance to the 3 axioms as follows.

Axiom 1

$$0 \le P(E) \le 1$$

This axiom states that the probability of an event ranges from the 0 to 1

Axiom 2

P (S) = 1

This axiom states that the results obtained must be a member of S with a probability equals to 1

Axiom 3 As for the chronological order of mutually exclusive events E_1, E_2, ...

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

This axiom states that the probability of at least 1 event will result from E1, E2, ... chronologically

Example 3 Suppose event A and event B are mutually exclusive events, where P(A) = 0.3 and P (B) = 0.5, find the probability that

1. At least 1 event happens

2. Event A happens but B does not

3. Both event A and B happens

Example 4 60% of students in a school is found not to wear both rings and necklaces, where 20% of students wear rings and 30% of the others wear necklaces. If one student is to be picked randomly, find the probability that the student will either wear a ring or a necklace.

Solution:

In case we suppose each result (member) in S, which is a limited set, has a probability that it will occur at a similar rate; let the number of the members in S be represented with n(S) and let A be an event where there are members of n(A).

$$P(A) = \frac{n(A)}{n(S)}$$

For example, suppose $S = \{w_1, w_2, ..., w_k\}$ and $P(\{w_i\}) = \frac{1}{k}$ for i = 1, 2,… k

Therefore, if event $A = \{u_1, u_2, ..., u_m\}$ consists of a sample point of m, where $m \le k$ then

$$P(A) = \sum_{i=1}^{m} P(\{u_i\}) = \underbrace{\frac{1}{k} + \frac{1}{k} + \cdots + \frac{1}{k}}_{\text{m terms}} = \frac{m}{k}$$

Example 5 In an event where a coin is tossed accurately 3 times, find the probability of the following scenarios

1. A head is up once ; represented by event A, Find P(A).

   Solution:

2. A head is up for at least once ; represented by event B, Find P(B).

   Solution:

However, in complex experiments, writing out all members of S to find n(S) is deemed complicated and is a waste of time. Therefore, the next part will be discussing the principles of counting, permutation, and combination, which are the basic principles in finding n(S) and n(E).

2.4 Combinatorial Analysis

This section will discuss the principle of counting as for the results obtained in a sample space or an event of interest. In mathematics, the latter is referred to as 'combinatorial analysis'.

2.4.1 Basic Principle of Counting or Multiplication Principle

If the experiment is repeated m times, where the 1^{st} experiment resulted in a possible outcome of n_1 ways, the 2^{nd} experimentation has a possible outcome of n_2 ways, the 3^{rd} experimentation led to a possible outcome of n_3 ways, and this continues on, where the total number of probable outcomes would be $n_1 \times n_2 \times n_3 \times \cdots \times n_m$ ways.

Example 6 Find the number of ways that 4 representatives out of 15, could be selected as the members of the student committee. Freshmen consists of 4 students, sophomore consists of 3 students, junior consists of 5 students, and the senior consists of 3 students, where the student committee members are to be selected from all academic years.

Solution:

Example 7 A car's license plate has 7 digit placements, where the first 2 digits are of letters ranging from A-Z and the other digits are of numbers ranging from 0-9. Find the following:

1. All the possible number of license plates

Solution:

2. All the possible number of license plates, where there are no repeated letters and numerals

Solution:

Example 8 In tossing 1 coin for a total of 5 times, find the following:

1. The total number of possible outcomes

   Solution:

2. The total number of possible outcomes if the coin lands head up in the last two trials

   Solution:

2.4.2 Principle of Addition

If the experiments are repeated m times, where the 1st experimentation has a possible outcome of n_1 ways, the 2nd experimentation results in a possible outcome of n_2 ways, and the 3rd experimentation gives off a possible outcome of n_3 ways, consecutively and continues on under a condition of where each experiment cannot be done together at the same instance, the total number of possible outcomes would be $n_1 + n_2 + n_3 \ldots + n_m$ ways.

Example 9 There are 5 digits which are: 0, 1, 2, 3, 4,

1. To make a 3 digit number with no repetitions, how many ways are there?

   Solution:

2. To make an even number of 3 digits with no repetitions, how many ways are there?

   Solution:

3. To make an odd number of 3 digits with no repetitions, how many ways are there?

   Solution:

2.4.3 Permutation

Finding the number of ways to arrange the letters A, B, and C in the sample splace will result in ABC, ACB, BAC, BCA, CAB, CBA or a total of 6 outcomes, where each sequence method is referred to as the method of 'Permutation'. In case where there are many letters involved, the calculation can be time-consuming. To solve this problem, finding the number of ways to arrange the letters can be done with the principles of counting, where the first digit can have 3 letters selected, the next digit with only 2 selected letters, and the last digit with only 1. Therefore, the number of possible ways to arrange the letters equals to 3 x 2 x 1 = 6

Normally, if there are n different objects, and were to be arranged, the number of possible ways of arranging them would be equals to

$$n(n-1)(n-2)\cdots(3)(2)(1) = n!$$

Example 10 In a singing competition, there are 7 contestants in the final round, which consists of 4 males and 3 females. After the competition, the judges will evaluate the scores and further rank the 7 remaining contestants. The contestant with the highest score will be ranked first (1^{st}), while the contestant with the lowest score will be ranked as the last (7^{th}). Find the following.

1. The number of possible results of the competition

   Solution:

2. The number of possible results of the competition, where the winner is a male

   Solution:

3. The number of possible results of the competition, where the male with the highest score is ranked the 3^{rd} place

   Solution:

Example 11 Lisa is going to arrange 10 books onto the shelf, by classifying her books into the following categories: 4 books in Mathematics, 3 books on Economics, 2 English books, and 1 Thai language book, where the same subjects must be arranged next to each other. How many possible ways are there for Lisa to arrange her books?

Solution:

Example 12 10 college students are standing in straight line for a picture, where 4 students are females. Find the following.

1. Find the number of possible ways the students can stand in a line (no conditions)

   Solution:

2. Find the number of possible ways the students can stand in a line, where those with the same gender will stand next to each other

   Solution:

3. If there are 2 college students who do not get along well, they will not stand next to each other. Find the number of possible ways they could stand to avoid standing next to each other.

   Solution:

Example 13 Eight people will be sitting in line. Find the number of all the possible ways they can sit.

1. Lila and Ryan has to sit next to each other

   Solution:

2. If there are 4 couples, each couple has to sit next to each other

   Solution:

3. In the amount of 8 people, there are 4 males and 4 females. Find the number of possible ways they can sit where people of the same gender cannot be seated next to each other.

   Solution:

In case where objects are to be selected in the number of r from the total of n, and are to be arranged, the following are the results:

$$n(n-1)(n-2)\times\cdots\times(n-(r-1)) = \frac{n!}{(n-r)!} = {}^{n}P_{r}$$

Example 14 In a class where there are 20 students, the homeroom teacher wants to select a class president, a vice president, and a treasurer. Find the number of possible ways they can be selected.

Solution:

In case where similar objects are switched or replaced, for instance: arranging the letters of the word “STAT”, which can be arranged in a total of 4! = 24 ways if the letters are to be viewed as different from each other.

$$\begin{array}{ll}
\left.\begin{matrix} SAT_1T_2 \\ SAT_2T_1 \end{matrix}\right\} SATT & \left.\begin{matrix} T_1SAT_2 \\ T_2SAT_1 \end{matrix}\right\} TSAT \\
\left.\begin{matrix} ST_1AT_2 \\ ST_2AT_1 \end{matrix}\right\} STAT & \left.\begin{matrix} T_1AST_2 \\ T_2AST_1 \end{matrix}\right\} TAST \\
\left.\begin{matrix} ST_1T_2A \\ ST_2T_1A \end{matrix}\right\} STTA & \left.\begin{matrix} T_1T_2SA \\ T_2T_1SA \end{matrix}\right\} TTSA \\
\left.\begin{matrix} AST_2T_1 \\ AST_1T_2 \end{matrix}\right\} ASTT & \left.\begin{matrix} T_1T_2AS \\ T_2T_1AS \end{matrix}\right\} TTAS \\
\left.\begin{matrix} AT_2ST_1 \\ AT_1ST_2 \end{matrix}\right\} ATST & \left.\begin{matrix} T_1AT_2S \\ T_2AT_1S \end{matrix}\right\} TATS \\
\left.\begin{matrix} AT_2T_1S \\ AT_1T_2S \end{matrix}\right\} ATTS & \left.\begin{matrix} T_1ST_2A \\ T_2ST_1A \end{matrix}\right\} TSTA
\end{array}$$

In reality, T_1 and T_2 switching places is considered as not having any difference. Therefore, any switching in placements of T_1 and T_2 will only be counted once. Hence, it can be seen that the number of ways they can be arranged in different sequences will be of $\frac{4!}{2!}$ = 12

Normally, if there is n objects, where there are n_1 identical objects, n_2 identical objects, and so on until n_r identical objects, the total number of possible ways they can be arranged are as follows:

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Example 15 Find the total number of possible ways 10 flags can be arranged; there are 3 red flags, 4 white flags, and 3 indigo flags.

Solution:

Example 16 Find the number of ways to arrange the letters of the word "THAMMASAT"

Solution:

2.4.4 Combination

Suppose there are n different objects, where r objects are to be selected and the number of possible different groups are to be taken interest in, for example: letters A, B, C, D, and E. From the latter, select 3 letters and group them into the following: ABC, ACB, BAC, BCA, CAB, and CBA. These arrangements of letters are considered of the same group, as they consist of the same letters. This means that in each arrangement or combination of the letters A, B, and C result in 3! outcome and is to be counted only once.

Therefore in general cases, where there are n different objects, and r objects are to be selected, with the above counting principle: the 1^{st} time will result in the n number of ways objects can be selected, the 2^{nd} time of (n-1) ways, and the 3^{rd} time of (n-2) ways and so on until when r has n- (r-1) ways. Therefore, the total number of possible ways equals to:

$$n(n-1)(n-2)(n-3)\cdots(n-(r-1)) = \frac{n!}{(n-r)!}$$

However, at each switching or re-arrangement of objects in the number of r, which equals to r!, will only be counted once. Therefore, the number of possible groups that are different can be calculated as follows:

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-(r-1))}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r} = {}^{n}C_{r}$$

Example 17 To select 4 members of the student committee from the total number of 20 students; 4 freshmen, 5 sophomores, 6 juniors, and 5 seniors.

1. Find the number of possible ways the students can be selected without any conditions

   Solution:

2. Find the probability of getting all the student committee members from the same academic year

   Solution:

3. Find the probability of getting all the student committee members from all academic years

   Solution:

Example 18 Renee has a total of 8 friends. She wants to invite 4 of her friends to watch a movie.

1. If there are 2 of her friends that do not get along well with each other and does not want to watch the movie together, find the number of possible ways she can invite her friends with the previously stated condition in consideration.

   Solution:

2. If there are 2 friends who would always go everywhere together, find the number of possible ways she can invite her friends.

   Solution:

Example 19 There are 12 balls in a box; 3 are white, 4 are black, and 5 are red balls. If 3 balls are to be picked randomly from the box, find the following:

1. Randomly pick 3 balls from the box at the same time. Find the probability of getting 3 different colored balls.

   Solution:

2. Randomly pick 3 balls from the box at the same time. Find the probability of getting at least 1 white colored ball.

   Solution:

3. Randomly pick 3 balls from the box at the same time. Find the probability of getting the same colored balls.

   Solution:

4. Randomly pick 1 ball at a time without replacement. Find the probability of getting 3 different colored balls.

   Solution:

5. Randomly pick 1 ball at a time without replacement. Find the probability of getting only 1 red ball.

   Solution:

6. Randomly pick 1 ball at a time, then returning it back into the box before the next turn. Find the probability of getting all different colored balls.

   Solution:

7. Randomly pick 1 ball at a time with replacement. Find the probability of getting only 1 red ball.

   Solution:

2.5 Conditional Probability and Independent Events

To find the probability of an event where there is a previously occurred event, it will take interest in the event and will be applied alongside conditions. For example: tossing 1 coin twice will result in S = {HH, HT, TH, TT}, where each sample point has a possibility of happening at the same rate and equals to ¼ . If the possibility of an event where heads are up twice, is of interest and is taken into account, the result will be P ({HH}) = ¼. However, if it is previously known that the 1st time landed on the head, the sample space will be S' = {HH, HT} and is sometimes referred to as Reduced Sample Space. Therefore, the probability of an event where the coin landed twice on head equals to ½.

Let A represent an event where heads are up twice

B represent an event where head is up on the first trial

The probability of an event where the coin lands on heads twice, the first trial where the coin lands on head is represented with P (A I B) and can be interpreted generally as "the probability of an event A occurring knowing that event B had already occurred beforehand", where the definition is as below.

Definition

If P(B) > 0

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Notice where P (A' I B) = 1 – P (A I B) and P (A I B) and is in accordance to the three axioms of probability

Example 20 A box contains 10 yellow balls, 6 red balls, and 7 white balls

1. Randomly pick a ball. Find the probability of getting a white ball knowing that a red ball will not be picked.

   Solution:

2. Randomly pick up 2 balls at the same time. Find the probability that 1 white ball will be picked knowing that at least 1 red ball will be picked.

   Solution:

Example 21 From a conducted research involving 500 married couples inquiring their income, data obtained are as follows:

Wife	Husband: Less than 100,000 Baht	Husband: More than 100,000 Baht
Less than 100,000 Baht	210	190
More than 100,000 Baht	30	70

If a couple is selected at random,

1. Find the probability that a husband earns more than 100,000 baht

   Solution:

2. In a random trial, it is known beforehand that the result will come off as where the wife earns more than 100,000 baht. Find the probability that a husband earns less than 100,000 baht.

   Solution:

Example 22 In tossing a coin accurately 6 times, find the probability that the coin will land on heads prime number of times under a condition where the coin will result in heads up for at least 5 times.

Solution:

From the definition of $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$ once multiplied with P(B), both sides will result in

$$P(A\cap B)=P(B)P(A|B)$$

which can mean that the probability of both event A and B occurring equals to the probability of event B multiplied with the probability of event A, once event B has already occurred beforehand. In a general case,

$$P(E_1\cap E_2\cap E_3\cap\ldots\cap E_n)=P(E_1)P(E_2|E_1)P(E_3|E_1\cap E_2)\cdots P(E_n|E_1\cap E_2\cap\ldots\cap E_{n-1})$$

which can easily be proved by using the definition of conditional probability

$$P(E_1)\frac{P(E_2\cap E_1)}{P(E_1)}\frac{P(E_3\cap E_1\cap E_2)}{P(E_2\cap E_1)}\cdots\frac{P(E_n\cap E_1\cap E_2\cap\ldots\cap E_{n-1})}{P(E_1\cap E_2\cap\ldots\cap E_{n-1})}$$

Example 23 The 1st jar contains 2 white marbles and 4 red marbles. The 2^{nd} jar has 2 white marbles and 1 red marble. Randomly pick a marble from the 1^{st} jar and put it into the 2^{nd} jar. Then, randomly pick a marble from the 2^{nd} jar.

1. Find the probability of getting a white marble from the 2^{nd} jar

   Solution:

2. Find the probability of getting a white marble from the 1^{st} jar and a red marble from the 2^{nd} jar

   Solution:

1.5.1 Probability of Two Independent Events

Generally, P (A I B) is not equal to P(A), which means that once P(B) is known, the probability of event A occurring will therefore change. However, if P (A I B) equals to P(A), A and B will be referred to as two independent events. Event B does not provide any insight or information regarding event A, and therefore, will be defined as follows:

Definition $P(A|B)=P(A)$ and $P(B|A)=P(B)$ but $P(A|B)=\frac{P(A\cap B)}{P(B)}$. Therefore, $P(A\cap B)=P(A)P(B)$ if and only if A and B are independent events.

In a similar matter, 3 events of A, B, and C will all be independent when

$$P(A\cap B\cap C)=P(A)P(B)P(C)$$

$$P(A\cap B)=P(A)P(B)\text{ , }P(A\cap C)=P(A)P(C)\text{ and }P(B\cap C)=P(B)P(C)$$

Notice that 4 conditions must be satisfied.

If A, B, and C own specific characteristics of $P(A\cap B)=P(A)P(B)$, $P(A\cap C)=P(A)P(C)$, and $P(B\cap C)=P(B)P(C)$ they will all be referred to as being 'Pairwise Independent'.

Example 24 Suppose an equal-sided dice has 4 sides with 1, 2, 3, and 4 on it. Tossing it once, results in S = {1, 2, 3, 4}, where A = {1, 2}, B = {1, 3}, C = {1, 4}. Prove that these 3 events are pairwise independent but not independent of each other.

Solution:

Theorem If A and B are independent events.

1. A’ and B will be independent events and
2. A and B’ will be independent events and
3. A’ and B’ will also be independent events

(We will omit the prove here.)

Example 25 Suppose in each married couple, the probability of giving birth to a female and a male is of equal value. In each child birth, the gender of the baby born is independent to that of other attempts of child birth. In a family of 5 children, find the probability that

1. All children are of the same gender

   Solution:

2. 3 of the children are males

   Solution:

3. There is at least 1 female child

   Solution: