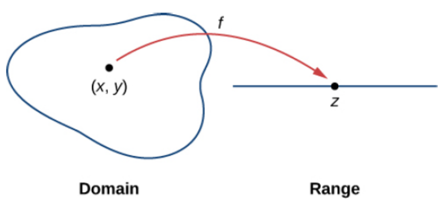
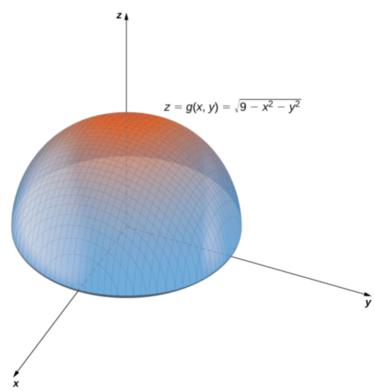
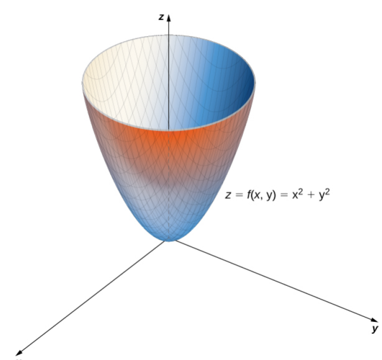
Differentiation of Functions of Several Variables

1 Functions of Several Variables

Definition 1.1. A function of two variables $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in the following figure.







Example 1.1. Find the domain of each of the following functions:

(a) $f(x, y) = 3x + 5y + 2$, (b) $g(x, y) = \sqrt{9 - x^2 - y^2}$

2 Partial Derivatives

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. The notation for the derivative is $\frac{dy}{dx}$, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable.

Definition 2.1. Let $f(x, y)$ be a function of two variables.

- The partial derivative of f with respect to x, written as $\frac{\partial f}{\partial x}$ or f_x, is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

  $\frac{\partial f}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x} f(x, y)$, $\frac{\partial}{\partial x} f(x, y)$, $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$, $f_x(x, y)$ or $D_x f$

- The partial derivative of f with respect to y, written as $\frac{\partial f}{\partial y}$ or f_y, is defined as

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

  $\frac{\partial f}{\partial y}$ is also denoted by $\frac{\partial}{\partial y} f(x, y)$, $\frac{\partial}{\partial y} f(x, y)$, $\frac{\partial f}{\partial y}$, $\frac{\partial z}{\partial Y}$, $f_y(x, y)$ or $D_y f$

The partial derivative of $z = f(x, y)$ with respect to x and y at (a, b), which is in the domain of f will be denoted by $\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)}$ $f_x(a, b)$ and $\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)}$ $f_y(a, b)$, respectively.

Example 2.1. Find $f_x(x, y)$ for $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$ by using the limit definition.

First, calculate $f(x+h,\ y)$.

$$\begin{aligned} f(x+h,\ y) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h} \\ &= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12\right) - \left(x^2 - 3xy + 2y^2 - 4x + 5y - 12\right)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 - 3hy - 4h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h - 3y - 4)}{h} \\ &= \lim_{h \to 0} (2x + h - 3y - 4) \\ &= 2x - 3y - 4. \end{aligned}$$

> Remark: The main concept of calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable.

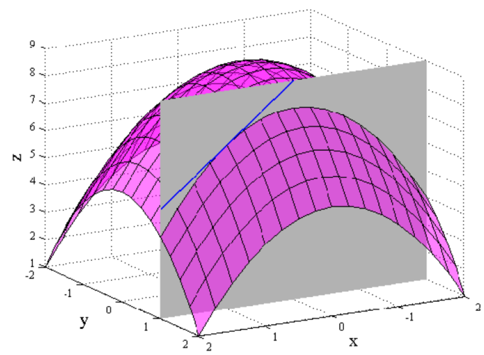
To see why this remark is true, first fix y and define $g(x) = f(x, y)$ as a function of x. Then

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}$$

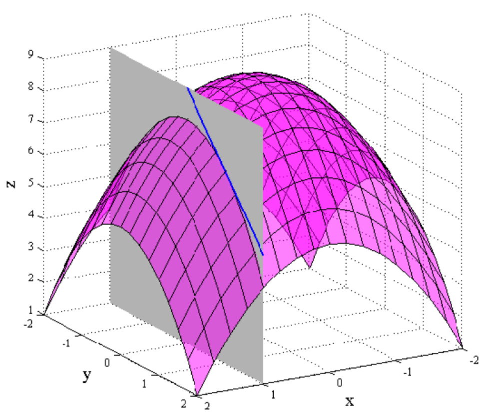
For calculating the partial derivative of f with respect to y, we fix x and define $w(y) = f(x, y)$ as a function of y. Then

$$w'(x) = \lim_{h\to 0} \frac{w(y+h) - w(y)}{h} = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} = \frac{\partial f}{\partial y}$$

The x partial derivative of the function $f(x, y) = 9 - x^2 - y^2$



The y partial derivative of the function $f(x, y) = 9 - x^2 - y^2$



Example 2.2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$.

Example 2.3. Find the partial derivatives of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = \sin(x^2y - 2x + 4)$.

Example 2.4. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 2.5. Let $f(x, y) = x^4e^{xy}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
Hint: Use "Product Rule" for $\frac{\partial f}{\partial x}$

2.1 Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition 2.2. Let $w = f(x, y, z)$ be a function of three variables.

- Partial derivative of f with respect to x

$$\frac{\partial}{\partial x} f(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

Notation: $\frac{\partial}{\partial x} f(x, y, z)$, $\frac{\partial}{\partial x} f(x, y, z)$, $\frac{\partial f}{\partial x}$, $\frac{\partial w}{\partial x}$, $f_x(x, y)$, $D_x f$

- Partial derivative of f with respect to y

$$\frac{\partial}{\partial y} f(x, y, z) = \lim_{h \to 0} \frac{f(x, y + h, z) - f(x, y, z)}{h}$$

Notations: $\frac{\partial}{\partial y} f(x, y, z)$, $\frac{\partial}{\partial y} f(x, y, z)$, $\frac{\partial f}{\partial y}$, $\frac{\partial w}{\partial y}$, $f_y(x, y, z)$, $D_y f$

- Partial derivative of f with respect to z

$$\frac{\partial}{\partial z} f(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h}$$

Notations: $\frac{\partial}{\partial z} f(x, y, z)$, $\frac{\partial}{\partial z} f(x, y, z)$, $\frac{\partial f}{\partial z}$, $\frac{\partial w}{\partial z}$, $f_z(x, y, z)$,$D_z f$

Example 2.6. Let $f(x, y, z) = e^{xy} \ln(z)$. Find f_x, f_y, f_z and $f_z(1, 0, 1)$. Note: No need to use limit definition.

Example 2.7. (Exercise) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ for $f(x, y, z) = \sin(x^2 y - z) + \cos(x^2 - yz)$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\sin\left(x^2 y - z\right) + \cos\left(x^2 - yz\right)\right] \\
&= \left(\cos\left(x^2 y - z\right)\right)\frac{\partial}{\partial x}\left(x^2 y - z\right) - \left(\sin\left(x^2 - yz\right)\right)\frac{\partial}{\partial x}\left(x^2 - yz\right) \\
&= 2xy \cos\left(x^2 y - z\right) - 2x \sin\left(x^2 - yz\right) \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\sin\left(x^2 y - z\right) + \cos\left(x^2 - yz\right)\right] \\
&= \left(\cos\left(x^2 y - z\right)\right)\frac{\partial}{\partial y}\left(x^2 y - z\right) - \left(\sin\left(x^2 - yz\right)\right)\frac{\partial}{\partial y}\left(x^2 - yz\right) \\
&= x^2 \cos\left(x^2 y - z\right) + z \sin\left(x^2 - yz\right) \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[\sin\left(x^2 y - z\right) + \cos\left(x^2 - yz\right)\right] \\
&= \left(\cos\left(x^2 y - z\right)\right)\frac{\partial}{\partial z}\left(x^2 y - z\right) - \left(\sin\left(x^2 - yz\right)\right)\frac{\partial}{\partial z}\left(x^2 - yz\right) \\
&= -\cos\left(x^2 y - z\right) + y \sin\left(x^2 - yz\right)
\end{aligned}
$$

3 Higher-Order Partial Derivatives

From the previous sections, each of the partial derivatives, e.g. f_x and f_y, is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these**second-order derivatives, third-order derivatives**, and so on. In general, they are referred to as **higher-order partial derivatives**.
There are four second-order partial derivatives for any function (provided they all exist):

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right], \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right],$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right], \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right],$$

Example 3.1. Calculate all four second partial derivatives for the function $f(x,y) = xe^{-3y} + \sin(2x - 5y)$.

<u>**Remark:**</u>

Theorem 3.1 (Clairaut's Theorem: Equality of Mixed Partial Derivatives)**.** Suppose that $f(x,y)$ is defined on an open disk $D \in \mathbb{R}^2$ that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $\boxed{f_{xy} = f_{yx}}$.

Example 3.2. Let $f(x, y) = y^2 e^x + y$. Find $\dfrac{\partial^3 f}{\partial y^2 \partial x}$.

Example 3.3. (Exercise) Show that $u(x, y, t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$ is a solution to the wave equation:

$$u_{tt} = 4(u_{xx} + u_{yy}).$$

$$\begin{aligned}
u_{tt} &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\
&= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\
&= \frac{\partial}{\partial t}[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)] \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{xx} &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial}{\partial x}[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)] \\
&= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{yy} &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\
&= \frac{\partial}{\partial y}[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)] \\
&= \frac{\partial}{\partial y}[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)] \\
&= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t).
\end{aligned}$$

$$\begin{aligned}
4(u_{xx} + u_{yy}) &= 4\left(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)\right) \\
&= 4\left(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)\right) \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
&= u_{tt}.
\end{aligned}$$

4 Chain Rule

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule.

Theorem 4.1 (Chain Rule for 1 Independent Variable)**.** Suppose that $x = x(t)$ and $y = y(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Example 4.1. Find $\dfrac{dz}{dt}$ if $\quad z = f(x, y) = 4x^2 + 3y^2, \quad x = x(t) = \sin(t), \quad y = y(t) = \cos(t).$

Remark:

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$\begin{aligned} z &= f(x, y) \\ &= f(x(t), y(t)) \\ &= 4(x(t))^2 + 3(y(t))^2 \\ &= 4\sin^2 t + 3\cos^2 t. \end{aligned}$$

Then

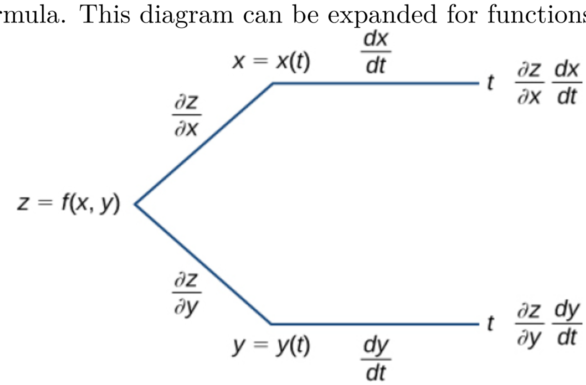
$$\begin{aligned} \frac{dz}{dt} &= 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) \\ &= 8\sin t\cos t - 6\sin t\cos t \\ &= 2\sin t\cos t, \end{aligned}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

Example 4.2. Find $\dfrac{dz}{dt}$ if

$$z = f(x, y) = \sqrt{x^2 - y^2}, \qquad x = x(t) = e^{2t}, \qquad y = y(t) = e^t.$$

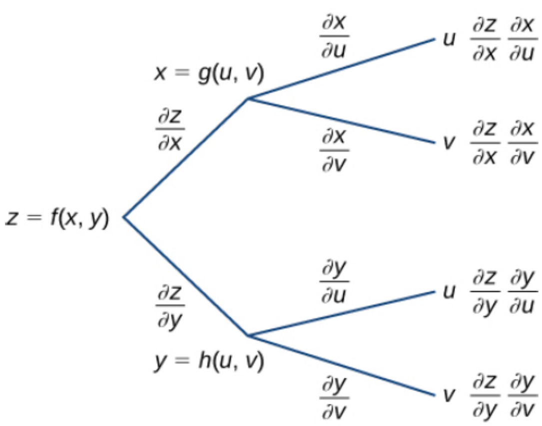
Remark: It is often useful to create a visual representation for the chain rule $\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula. This diagram can be expanded for functions of more than one variable, as we shall see very shortly.



Theorem 4.2 (Chain Rule for 2 Independent Variables)**.** Suppose $x = g(u,v)$ and $y = h(u,v)$ are differentiable functions of u and v, and $z = f(x,y)$ is a differentiable function of x and y. Then, $z = f(g(u,v),h(u,v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Tree diagrams:



Example 4.3. Calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when

$$z = f(x, y) = 3x^2y + y^2, \qquad x = x(u, v) = 3u + 2v, \qquad y = y(u, v) = 4u - v.$$

4.1 Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the generalized chain rule states.

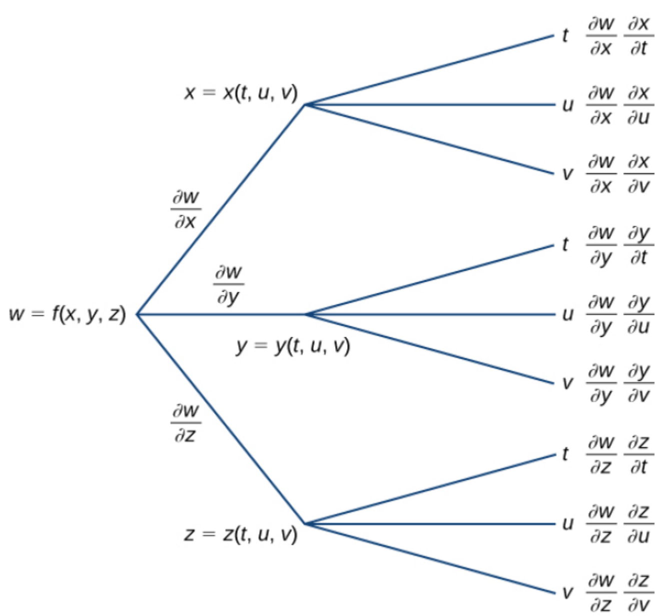
Theorem 4.3 (Generalized Chain Rule)**.** Let$w = f(x_1, x_2, , x_m)$ be a differentiable function of m independent variables, and for each $i \in \{1, ..., m\}$, let x_i, $i = 1, ..., m$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \qquad j = 1, 2, ..., n$$

Example 4.4. Create a tree diagram for the case when

$$w = f(x, y, z), \qquad x = x(t, u, v), \qquad y = y(t, u, v), \qquad z = z(t, u, v)$$

and write out the formulas for the three partial derivatives of w.



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

Example 4.5. Let

$$w = f(x, y, z) = 3x^2 - 2xy + 4z^2$$

$$x = x(u, v) = e^u \sin(v), \qquad y = y(u, v) = e^u \cos(v) \qquad z = z(u, v) = e^u.$$

(a) Find $\dfrac{\partial w}{\partial u}$.

(b) Find $\dfrac{\partial w}{\partial v}$ (Exercise).

5 Implicit Differentiation

Theorem 5.1 (Implicit Differentiation of a Function of Two or More Variables)**.** Suppose the function $z = f(x,y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x,y) = 0$. Then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

as long as $\frac{\partial f}{\partial y} = f_y(x,y) \neq 0$

If the equation $f(x,y,z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{f_x}{f_z} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{f_y}{f_z}$$

as long as $\frac{\partial f}{\partial z} = f_z(x,y,z) \neq 0$

Remark: Derivation:
If the equation $f(x,y,z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\begin{aligned} \frac{\partial}{\partial x} f(x,y,z) &= \frac{\partial 0}{\partial x} \\ \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \end{aligned}$$

where we have used $\frac{dx}{dx} = 1$.
The formula for $\frac{\partial z}{\partial y}$ can be derived in a similar way.

Example 5.1. Calculate $\frac{dy}{dx}$ if y is defined implicitly as a function of x via the equation
$3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$.
Find the equation of the tangent line to the graph of this curve at point $(2, 1)$.

Example 5.2. Suppose $x^2e^y - yze^x = 0$. Calculate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.