Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$J(c_t, t) = \max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \delta^T \left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income ($y_t = 0\ \forall t$) and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1} \underbrace{\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)}_{u(C_t,t)} + \underbrace{\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)}_{B(W_T,T)}\right]$$

$$u_c = E_{T-1}[B_w R_{T-1}]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}[\delta^T W_{T-1}^{-\gamma} R_{T-1}]$$

$$\delta^{T-1} = \delta^T E_{T-1}[R_{T-1}](W_{T-1} - C_{T-1})^{-\gamma}$$

$$\text{Thus, } C^*_{T-1} = \frac{(\delta^T E_{T-1}[R_{T-1}])^{\frac{1}{\gamma}}}{1 + (\delta^T E_{T-1}[R^{\gamma}_{T-1}])^{\frac{1}{\gamma}}} W_{T-1} \quad \#$$

$$\text{Note: } C_1 = \frac{a_1}{(1+a_1)} \quad ; \quad a_1 = (\delta^T E_{T-1}[R_{T-1}])^{\frac{1}{\gamma}}$$

For weights (w^*_{T-1})

we know that $E_{T-1}[B_W R_{r,T-1}] = R_f E_{T-1}[B_W]$

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score.....................

current wealth

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1}, T-1) = u(C_{T-1}, T-1) + E_{T-1}[B(W_T, T)]$$

$$= \delta^{T-1} C_{T-1}^{-\gamma} + \delta^T E_{T-1}[R_{T-1}](W_{T-1} - C_{T-1})^{-\gamma}$$

$$= \delta^{T-1} b_1 W^{-\gamma}$$

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$u_c(C_{T-2}, T-2) = E_{T-2}[J(W_{T-1}, T-1) R_{T-2}]$$

$$C_{T-2}^{-\gamma} = \delta^{T-1} E_{T-2}[b_1 W_{T-2}^{-\gamma} R_{T-2}]$$

$$= \delta^{T-1} b_1 E_{T-2}[R_{T-2}](W_{T-2} - C_{T-2})^{-\gamma}$$

Thus, $C^*_{T-2} = \dfrac{(\delta b_1 E_{T-2}[R_{T-2}])^{\frac{1}{\gamma}}}{1+(\delta b E_{T-2}[R_{T-2}])^{\frac{1}{\gamma}}} W$

For weights (w^*_{T-2}),

we know that $E_{T-2}[J_W R_{r,T-2}] = R_f E_{T-2}[J_W]$

$$\max_{C_s,\omega_s,\forall t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = U(C_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)]$$

$$= \delta^{T-2} C^{-\gamma} + E_{T-1}[\delta^{T-1} b_1 W^{-\gamma}]$$

$$= \delta^{T-2} b_2 W_{T-2}^{-\gamma} \quad *$$

∴ The optimal consumption and weight at any dates can be expressed as

$$C^*_{T-t} = c_t W_{T-t} \; ; \; c_t = \frac{a_1 c_{t-1}}{(1 + a_1 c_{t-1})}$$