Solution: Quiz 4

1. Let $P(n)$ be the predicate defined as

$$P(n) \;:\; \sum_{i=1}^{n}(4i-1) \;=\; 2n^2+n.$$

(a) Show that $P(4)$ is true.

(b) Prove by mathematical induction that $P(n)$ is true for any positive integer n.

Solution:

(a) For $n=4$,

$$\begin{aligned}(i)\ \sum_{i=1}^{n}(4i-1) &= \sum_{i=1}^{4}(4i-1)\\ &= [4(1)-1]+[4(2)-1]+[4(3)-1]+[4(4)-1] = 3+7+11+15 = 36,\end{aligned}$$

and

(ii) $2n^2+n = 2(4^2)+4 = 36.$

From (i) and (ii), $\sum_{i=1}^{4}(4i-1) = 2(4^2)+4$, which implies P(4) is true. ■

(b) We want to show that $P(n)$ is true for $n \geq 1$.

(I) **Basis step:** $P(1)$: $\sum_{i=1}^{1}(4i-1) \;=\; 2\cdot 1^2+1$ is true because $\sum_{i=1}^{1}(4i-1) = (4(1)-1) = 3$ and $2\cdot 1^2+1 = 3$.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.

Assume that $P(k)$ is true: $\sum_{i=1}^{k}(4i-1) \;=\; 2k^2+k.$ ————($\bigstar$) "inductive hypothesis"

We want to show that $P(k+1)$: "$\sum_{i=1}^{k+1}(4i-1) \;=\; 2(k+1)^2+(k+1).$ "

Note that $2(k+1)^2+(k+1) = (2(k+1)+1)(k+1) = (2k+3)(k+1) = 2k^2+5k+3$. So, we can show that $P(k+1)$ is true by showing that:

$$P(k+1) \;:\; \sum_{i=1}^{k+1}(4i-1) = 2k^2+5k+3. \tag{1}$$

Consider the left-hand side of $P(k+1)$:

$$\begin{aligned}\sum_{i=1}^{k+1}(4i-1) &= \underbrace{\sum_{i=1}^{k}(4i-1)}_{(\bigstar)} + [4(k+1)-1]\\ &= [2k^2+k]+[4(k+1)-1] \text{ by } (\bigstar)\\ &= [2k^2+k]+[4k+4-1] = 2k^2+5k+3,\end{aligned}$$

which is the same as 1. That is, $P(k+1)$ is true.

Hence, from (I) basis step and (II) inductive step, $P(n)$ is proved for any integer $n \geq 1$ by mathematical induction. ■