Thursday, August 18, 2016 8:10 AM

$$y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \qquad i = 1, 2, \ldots, n$$

$$\begin{array}{l} i=1 \\ i=2 \\ \vdots \\ i=n \end{array} \quad \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \begin{array}{l} = \beta_1 + \beta_2 X_{21} + \beta_3 X_{31} + \cdots + \beta_k X_{k1} + u_1 \\ = \beta_1 + \beta_2 X_{22} + \beta_3 X_{32} + \cdots + \beta_k X_{k2} + u_2 \\ \\ = \beta_1 + \beta_2 X_{2n} + \beta_3 X_{3n} + \cdots + \beta_k X_{kn} + u_n \end{array}$$

$$\underset{n\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} 1 & X_{21} & X_{31} & \cdots & X_{k1} \\ 1 & X_{22} & X_{32} & \cdots & X_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & X_{3n} & \cdots & X_{kn} \end{bmatrix}} \cdot \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{u}$$

Min.

$$\underset{1\times n}{\hat{u}'} \cdot \underset{n\times 1}{\hat{u}} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_n \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \sum_{i=1}^{n} \hat{u}_i^2$$

$$Y = X\hat{\beta} + \hat{u}$$

$$\hat{u} = Y - X\hat{\beta}$$

$$\hat{u} = Y - X\hat{\beta}$$

$$\hat{u}'\hat{u} = (Y - X\hat{\beta})'(Y - X\hat{\beta}) = Y'Y + \underset{1\times k}{\hat{\beta}'}\underset{k\times n}{X'}\underset{n\times k}{X}\underset{k\times 1}{\hat{\beta}} - 2\hat{\beta}'X'Y$$

$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} = 0 + 2X'X\hat{\beta} - 2X'Y = 0$$

$$\cancel{2}X'X\hat{\beta} = \cancel{2}X'Y$$

$$\cancel{(X'X)^{-1}X'X}\hat{\beta} = (X'X)^{-1}X'Y$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\underset{n\times 1}{u}\cdot\underset{1\times n}{u'} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1u_1 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2u_2 & \cdots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \cdots & u_nu_n \end{bmatrix}$$

$$E\left[(X_i - E(X_i))(Y_i - E(Y_i))\right] = \mathrm{cov}(X,Y)$$

$$E\left[(X_i - E(X_i))^2\right] = \mathrm{var}(X)$$

$$E(uu') = \begin{bmatrix} E(u_1^2) & E(u_1u_2) & \cdots & E(u_1u_n) \\ E(u_2u_1) & E(u_2^2) & \cdots & E(u_2u_n) \\ \vdots & \vdots & \ddots & \vdots \\ E(u_nu_1) & E(u_nu_2) & \cdots & E(u_n^2) \end{bmatrix}_{n\times n}$$

$$E(u_1^2) = E\left[(u_1 - E(u_1))^2\right] = \sigma_1^2$$

$$E(u_1u_2) = E\left[(u_1 - \overset{0}{E(u_1)})(u_2 - \overset{0}{E(u_2)})\right] = \sigma_{12}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & & & \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

cov

$$= \begin{bmatrix} \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} \text{var.} = \begin{bmatrix} 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

$$E(uu')_{OLS} = \sigma^2 \underset{n \times n}{I} = \underset{n \times n}{\Sigma}$$

$$\hat{\beta}_{OLS} = (X'X)^{-1}X'Y$$

$$\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$$

In OLS $\Sigma = \sigma^2 I$

$$\hat{\beta}_{\substack{GLS \\ OLS}} = (X'(\sigma^2 I)^{-1}X)^{-1}X'(\sigma^2 I)^{-1}Y$$

$$= \cancel{\sigma^2}\left(\frac{1}{\cancel{\sigma^2}}\right)(X'I^{-1}X)^{-1}X'I^{-1}Y$$

$$= (X'X)^{-1}X'Y$$

$$= (X'X)^{-1} X'Y$$