EE320 Chapter 7
Derivatives of More Than One Independent Variable Function

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1 Partial Differentiation

1.1 First-order partial derivatives

Consider $y = f(x_1, x_2, \cdots, x_n)$ where $x_1, x_2, ..., x_n$ are independent of one another. Suppose only x_1 changes by Δx_1 , y will change accordingly by Δy (other x_s remain the same)

So, $\frac{\Delta y}{\Delta x_1} = \frac{f(x_1+\Delta x_1, x_2, \cdots, x_n) - f(x_1, x_2, ..., x_n)}{\Delta x_1}$

If $\Delta x \to 0$, consider a small change in x_1

$$f_1 \equiv \frac{\partial y}{\partial x} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x} \text{ called partial differentiation.}$$

Partial differentiation helps us ask: How much y will change when x_1 changes, holding other x constant?

ex. 1 $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, find $f_1(1,3)$and $f_2(1,3)$

$f_1 = \frac{\partial f}{\partial x_1}$

$f_2 = \frac{\partial f}{\partial x_2}$

ex. 2 $z = 9 - x^2 - y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x} =$

$\frac{\partial z}{\partial y} =$

ex. 3 $y = (2x_1^2x_2 + x_1)(x_2 + 1)$ find $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$

$\frac{\partial y}{\partial x_1} =$

$=$

$=$

$\frac{\partial y}{\partial x_2} =$

$=$

$=$

$=$

ex. 4 $Q = Q(K, L) = 6K^{\frac{1}{2}}L^{\frac{1}{2}}$

$\frac{\partial Q}{\partial K} = MP_K =$

$\frac{\partial Q}{\partial L} = MP_L =$

Gradient vector

The gradient vector of the function $f(x_1, x_2, ..., x_n)$ is an n-vector of all partial derivatives:

$$\nabla f(x_1, x_2, ..., x_n) = (f_1 \; f_2 \cdots f_n)$$

where $f_i \equiv \frac{\partial f}{\partial x_i}$

<u>ex.</u> $Q = aK^b L^{1-b}$ where $a > 0$ and $0 < b < 1$

$$\nabla Q(K, L) =$$

Application : Demand & Supply Model

$$\begin{aligned} Q^d &= a - bP \qquad a, b > 0 \\ Q^s &= -c + dP \quad c, d > 0 \\ \Rightarrow P^* &= \tfrac{a+c}{b+d} \\ Q^* &= \tfrac{ad-bc}{b+d} \end{aligned}$$

Conduct comparative-static derivative:

[a] $\frac{\partial P^*}{\partial a} =$

$\frac{\partial Q^*}{\partial a} =$

[b] $\frac{\partial P^*}{\partial b} =$

$\frac{\partial Q^*}{\partial b} =$

$=$

$$[c]\ \frac{\partial P^*}{\partial c} =$$

$$\frac{\partial Q^*}{\partial c} =$$

$$[d]\ \frac{\partial P^*}{\partial d} =$$

$$\frac{\partial Q^*}{\partial d} =$$

Application : Elasticity

Consider a function $z = f(x, y)$, partial elasticity of z with respect to x and y are

$$\varepsilon_{zx} = \frac{\frac{\partial z}{z}}{\frac{\partial x}{x}} = \frac{\partial z}{\partial x} \cdot \frac{x}{z}$$

$$\varepsilon_{zy} = \frac{\frac{\partial z}{z}}{\frac{\partial y}{y}} = \frac{\partial z}{\partial y} \cdot \frac{y}{z}$$

when all variables are positive,

$$\varepsilon_{zx} = \frac{\partial lnz}{\partial lnx}$$

$$\varepsilon_{zy} = \frac{\partial lnz}{\partial lny}$$

Note $\frac{\partial lnz}{\partial lnx} = \frac{\frac{\partial z}{z}}{\frac{\partial x}{x}} = \frac{\partial z}{\partial x} \cdot \frac{x}{z} \ \therefore \frac{dlnx}{dx} = \frac{1}{x}$

For n variables, $f(x_1, x_2, ..., x_n)$:

$$\varepsilon_{zi} = \frac{\partial lnz}{\partial lnx_i} = \frac{\partial f(x_1,x_2,...,x_n)}{\partial lnx_i} \cdot \frac{x_i}{f(x_1,...,x_n)}$$

Elasticities of demand

Given demand function $Q_1 = \alpha - \beta P_1 + \gamma P_2 + \mu Y \quad or \quad Q_1 = f(P_1, P_2, Y)$

, where P_2 = price of substitute (goods 2) and Y = income

Own price elasticities of demand $\varepsilon_d = \frac{\partial Q_1}{\partial P_1} \cdot \frac{P_1}{Q_1} = (-\beta) \cdot \frac{P_1}{Q_1}$

Income elasticities of demand $\varepsilon_Y = \frac{\partial Q_1}{\partial Y} \cdot \frac{Y}{Q_1} = \mu \cdot \frac{Y}{Q_1}$

Cross-price elasticities of demand $\varepsilon_c = \frac{\partial Q_1}{\partial P_2} \cdot \frac{P_2}{Q_1} = (\gamma) \cdot \frac{P_2}{Q_1}$

ex. $Q_1 = 100 - P_1 + 0.75P_2 - 0.25P_3 + 0.0075Y$

At Y = 10,000 , $P_1 = 10, P_2 = 20, P_3 = 40$ and $Q_1 = 100$ Find price elasticity, income elasticity and cross-price elasticity of demand.

$\varepsilon_d =$

$\varepsilon_Y =$

$\varepsilon_{12} =$

$\varepsilon_{13} =$

Output Elasticities

Given linearly homogeneous Cobb-Douglas production function

$Q = F(K, L) = AK^{\alpha}L^{\beta}$ where $A, \alpha, \beta \neq 0$

Output elasticity of capital $= \varepsilon_{QK} = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q} = \alpha$

Output elasticity of labor $= \varepsilon_{QL} = \frac{\partial Q}{\partial L} \cdot \frac{L}{Q} = \beta$

ex. $Q = 6K^{\frac{1}{2}}L^{\frac{1}{2}}$

$\varepsilon_{QK} \;=$

$\varepsilon_{QL} \;=$

Application : Production Function

Given a production function $Q \;=\; 36KL - 2K^2 - 3L^2$

$$MP_K \;=$$

$$MP_L \;=$$

If MR at L = 3 and K = 2 is \$5 Find $MRP_{L=3}$ and $MRP_{K=2}$

$MRP_{L=3} \;=\; MR \cdot MP_L \;=$

$MRP_{K=2} \;=\; MR \cdot MP_K \;=$

Application : Multipliers in Macroeconomic Model

Consider a Keynesian-cross model

$$
\begin{aligned}
Y &= C + I_0 + G \\
C &= C_0 + c(Y - T) \qquad C_0>0,\ 0<c<1 \\
T &= T_0 + tY \qquad T_0>0,\ 0<t<1 \\
Y^* &= \tfrac{C_0 - cT_0 + I_0 + G_0}{1-c-ct}
\end{aligned}
$$

Conduct comparative static derivative ;

$$\tfrac{\partial Y^*}{\partial G_0} =$$

$$\tfrac{\partial Y^*}{\partial T_0} =$$

$$\tfrac{\partial Y^*}{\partial t} =$$

$$=$$

$$=$$

$$\tfrac{\partial Y^*}{\partial c} =$$

$$=$$

$$=$$

$$=$$

$$\tfrac{\partial Y^*}{\partial I_0} = \tfrac{\partial Y^*}{\partial G_0} = \tfrac{\partial Y^*}{\partial C_0} = \tfrac{1}{1-c+ct} \quad \Delta\text{autonomous expenditure}$$

1.2 Second-order partial derivatives

Consider $z = f(x, y)$ which gives rise to:

$f_x \equiv \frac{\partial z}{\partial x}$ and $f_y \equiv \frac{\partial z}{\partial y}$

"rate of change of z after x (y) changes"

We can find rate of change of f_x with respect to z, given fixed y by a second-order partial derivative:

- with respect to x

$f_{xx} \equiv \frac{\partial}{\partial x} f_x \quad or \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

- with respect to y

$f_{yy} \equiv \frac{\partial}{\partial y} f_y \quad or \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

We can also find cross partial derivatives

$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

Young's Theorem : let $y = f(x_1, ..., x_n)$ is twice differentiable, then

$\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial x_i}; i =\neq j$

$\therefore f_{xy} = f_{yx}$

$\underline{ex.}\quad Q = aK^b L^{1-b} = bK^{\frac{1}{4}} L^{\frac{3}{4}}$

$MP_K = \frac{\partial Q}{\partial K} =$

$MP_L = \frac{\partial Q}{\partial L} =$

$$
\begin{aligned}
\frac{\partial MP_K}{\partial K} &= \frac{\partial}{\partial K}\left(\frac{\partial Q}{\partial K}\right) = \\
\frac{\partial MP_L}{\partial L} &= \frac{\partial}{\partial L}\left(\frac{\partial Q}{\partial L}\right) = \\
Q_{KL} &= \frac{\partial}{\partial K}\left(\frac{\partial Q}{\partial L}\right) = \\
Q_{LK} &= \frac{\partial}{\partial L}\left(\frac{\partial Q}{\partial K}\right) = \\
Q_{KL} &= Q_{LK}
\end{aligned}
$$

Hessian Matrix: The second-order partial derivatives can be written in a matrix form

$$\nabla^2 f(x) = H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

for $f(x_1, x_2)$ $\quad \nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

<u>ex.</u> $Q = aK^{\frac{1}{4}}L^{\frac{3}{4}} = Q(K, L)$

$$H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix} =$$

<u>ex.</u> $z = x^2 \cdot e^{-y}$ Find f_x, f_y, and Hessian matrix

1^{st} order:

$\frac{\partial z}{\partial x} =$

$\frac{\partial z}{\partial y} =$

$$
\begin{aligned}
2^{nd} \text{ order} : \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) &= \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}(\frac{\partial z}{\partial x}) &= \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}(\frac{\partial z}{\partial x}) &= \\
\frac{\partial^2 z}{\partial x^2} &= \\
\frac{\partial^2 z}{\partial y^2} &=
\end{aligned}
$$

$H =$

Application: Agricultural Production Function

Consider an agricultural production function.

$$Q = F(K, L, T) = AK^{\alpha} L^{\beta} T^{\gamma}, \text{ where } A>0, \alpha>0, \beta>0, \gamma>0$$

$$
\begin{aligned}
\text{where K} &= \text{capital} \\
\text{L} &= \text{labor} \\
\text{T} &= \text{land}
\end{aligned}
$$

1^{st} - order

$$
\begin{aligned}
MP_K &= \\
MP_L &= \\
MP_T &=
\end{aligned}
$$

2^{nd} - order

$$\frac{\partial MP_K}{\partial K} =$$

$$\frac{\partial MP_K}{\partial L} =$$

$$\frac{\partial MP_K}{\partial T} =$$

$$\frac{\partial MP_L}{\partial K} =$$

$$\frac{\partial MP_L}{\partial L} =$$

$$\frac{\partial MP_L}{\partial T} =$$

$$\frac{\partial MP_T}{\partial K} =$$

$$\frac{\partial MP_T}{\partial L} =$$

$$\frac{\partial MP_T}{\partial T} =$$

$H =$

2 Differentials and Total Differentials

Recall definition of derivative;

$$\begin{aligned}
\frac{dy}{dx} &= f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
\Rightarrow dy &= f'(x)dx \\
\text{where } dy &= \text{differential of y tells approx change in y} \\
dx &= \text{differential of x tells approx change in x}
\end{aligned}$$

The process to find "differential of y" from a function y = f(x) is called "differentiation." So,

$$\begin{aligned}
\frac{dy}{dx}: &\quad \text{derivative} \leftrightarrow \text{differentiation w.r.t. x.} \\
dy: &\quad \text{differential of y} \leftrightarrow \text{differentiation} \\
&\quad \text{product} \qquad \text{process}
\end{aligned}$$

Total Differentials

Total Differentials is sum of the approximated changes from all parameters

$$\begin{aligned}
\text{For } y &= f(x_1, x_2) \\
dy &= f_1 dx_1 + f_2 dx_2 \qquad \text{where } f_1 = \frac{\partial f}{\partial x_1} \\
& \qquad\qquad\qquad\qquad\quad f_2 = \frac{\partial f}{\partial x_2}
\end{aligned}$$

<u>ex.</u>

$$\begin{aligned}
y &= 2x_1 + 3x_2 \\
dy &= 2dx_1 + 3dx_2 \approx \Delta y = 2\Delta x_1 + 3\Delta x_2
\end{aligned}$$

Utility function with n variables :

$$
\begin{aligned}
U &= U(x_1, ..., x_n) \\
dU &= \tfrac{\partial U}{\partial x_1} dx_1 + \tfrac{\partial U}{\partial x_2} dx_2 + ... + \tfrac{\partial U}{\partial x_n} dx_n \\
dU &= U_1 dx_1 + ... + U_n dx_n \ = \sum_{i=1}^{n} U_i dx_i
\end{aligned}
$$

Consider n = 2, $U(x_1, x_2)$

$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2$

Holding U constant $\Rightarrow dU = 0$

$0 = MU_1 dX_1 + MU_2 dx_2$

$-MU_2 dx_2 = MU_1 dX_1 \Rightarrow \frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2}$ = MRS = Slope of IC

<u>ex.</u> Given the utility function, $U = x^{\alpha} y^{\beta}$, find MRS

$$
\begin{aligned}
dU &= \\
0 &= \\
\frac{dy}{dx} &=
\end{aligned}
$$

Production function

$$
\begin{aligned}
Q &= F(K, L) \\
dQ &= \frac{\partial F}{\partial K} \cdot dK + \frac{\partial F}{\partial L} \cdot dL \\
dQ &= \underbrace{F_K}_{MP_K} dK + \underbrace{F_L}_{MP_L} dL \\
\text{Holding products constant} \Rightarrow dQ &= 0 \\
0 &= MP_K dK + MP_L dL \\
\frac{dK}{dL} &= -\frac{MP_L}{MP_K} = \text{MRTS} = \text{Slope of isoquant}
\end{aligned}
$$

<u>ex.</u> $Q = 60K^{0.25}L^{0.75}$, find MRTS

$$
\frac{dK}{dL} = -\frac{MP_L}{MP_K} =
$$

3 Total Derivatives

Consider the function that independent variables are related one another.

ex. y = f(x,w) where x = g(w), or y = f(g(w),w). Find $\frac{dy}{dw}$

1. By chain rule: $\frac{dy}{dw} = \frac{\partial f}{\partial x}\frac{dx}{dw} + \frac{\partial f}{\partial w} = \overbrace{f_x \frac{dx}{dw}}^{indirect} + \overbrace{f_w}^{direct}$

2. Total differential: $dy = \frac{\partial y}{\partial x} \cdot dx + \frac{\partial y}{\partial w} \cdot dw$

$$\frac{dy}{dw} = \frac{\partial y}{\partial x} \cdot \frac{dx}{dw} + \frac{\partial y}{\partial w} \cdot \frac{dw}{dw} = \overbrace{f_x \frac{dx}{dw}}^{indirect} + \overbrace{f_w}^{direct}$$

$\frac{dy}{dw}$ = total derivative of y w.r.t. w

$\frac{\partial y}{\partial w}$ = partial derivative of y w.r.t. w

ex. Find $\frac{dy}{dw}$, given
$$\begin{aligned} y &= f(x,w) = 3x - w^2 \\ x &= g(w) = 2w^2 + w + 4, \end{aligned}$$

1. By chain rule: $\frac{dy}{dw} =$

$=$

$=$

2. Total differential: $dy =$

$=$

$\frac{dy}{dw} =$

ex. Q =Q(K,L,t) where K =K(t) and L =L(t)

$$\frac{dQ}{dt} = Q_K\frac{dK}{dt} + Q_L\frac{dL}{dt} + Q_t\frac{dt}{dt} \qquad \underline{\text{Note}}\ \dot{x} = \frac{dx}{dt}\text{growth rate due to time}$$

$$\dot{Q} = Q_K \cdot \dot{K} + Q_L \cdot \dot{L} + Q_t$$

ex. $y = f(x_1, x_2, u, v)$ where $x_1 = g(u, v)$ and $x_2 = h(u, v)$

$$\frac{dy}{du}|_{v=\overline{v}} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial u}|_{v=\overline{v}} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial u}|_{v=\overline{v}} + \frac{\partial y}{\partial u}$$

$$\frac{dy}{dv}|_{u=\overline{u}} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial v}|_{u=\overline{u}} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial v}|_{u=\overline{u}} + \frac{\partial y}{\partial v}$$

ex. Utility function U =U(C,l), where C = consumption, l = leisure = 24 - n, and C = f(n). Given $\frac{\partial U}{\partial C} > 0$ and $\frac{\partial U}{\partial l} > 0, U = U(f(n),\ l(n)) = U(f(n), 24 - n)$. Find $\frac{dU}{dn}$

$$\frac{dU}{dn} =$$

$\frac{dU}{dn}$>0 if $MU_C \cdot f'(n) > MU_l$

$\frac{dU}{dn}$<0 if $MU_C \cdot f'(n) < MU_l$

4 Implicit Function

Consider the following function:

$$\left.\begin{cases} y = f(x) = 3x^4 \rightarrow \text{explicit function} \\ F(y,x) = y - 3x^4 \rightarrow \text{implicit function} \end{cases}\right\} \text{same func in different form}$$

A given function of the form $F(y, x_1, x_2, ..., x_n) = 0$ defines an implicit function of the form $f(x_1, x_2, ..., x_n)$ if

1. $F(y, x_1, x_2, ..., x_n)$ has continuous partial derivatives: $\frac{\partial F}{\partial y}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, ..., \frac{\partial F}{\partial x_n}$ need to be continuous, and

2. $\frac{dF}{dy} \neq 0$

<u>ex.</u> Given $F(y, x, w) = y^3x^2 + w^2 + yxw - 3 = 0$, does F(y,x,w) defines an implicit function $y = f(x, w)$?

Check 2 conditions:

1. continuous partial derivatives $\rightarrow \frac{\partial F}{\partial y}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial w}$

$$\left.\begin{aligned} \frac{\partial F}{\partial y} &= \text{............................} \\ \frac{\partial F}{\partial x} &= \text{............................} \\ \frac{\partial F}{\partial w} &= \text{............................} \end{aligned}\right\} \text{all are continuous?}$$

2. $\frac{dF}{dy} \neq 0$

$\frac{dF}{dy} =$ .... if evaluate at $(y, x, w) = (1, 1, 1)$

$\frac{dy}{dx} =$ ....

$\therefore$ implicit function y = f(x,w) is defined by F(y,x,w)

Derivatives of implicit functions

$$\begin{aligned} F &= F(y, x_1, x_2, ..., x_n) \Rightarrow y = f(x_1, x_2, ..., x_n) \\ dF &= d(0) = 0 \end{aligned}$$

Or

$$\begin{aligned} dF &= F_y dy + F_1 dx_1 + ... + F_n dx_n = 0 \\ \text{(From } dy &= f_1 dx_1 + ... + f_n dx_n) \\ dF &= F_y f_1 dx_1 + F_1 dx_1 + F_y f_2 dx_2 + F_2 dx_2 + ... + F_y f_n dx_n + F_n dx_n \\ dF &= (F_y f_1 + F_1)dx_1 + ... + (F_y f_n + F_n)dx_n = 0 \end{aligned}$$

Holds if $F_y f_i \; + F_i = 0, \forall i = 1, ..., n$

$$\therefore f_i = \frac{-F_i}{F_y} \quad \text{or} \quad \frac{\partial f}{\partial x_i} = \frac{-\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}}$$

<u>ex.</u> Find $\frac{dy}{dx}$ for implicit function $y - 3x^4 = 0$

$\frac{dy}{dx} = -\frac{F_x}{F_y} =$ .... by implicit function

$\frac{dy}{dx} =$ .... by normal differentiation

<u>ex.2</u> Find $\frac{\partial y}{\partial x}$ for any implicit function that may be defined by

$F(y, x, w) = y^3 x^2 + w^3 + yxw - 3 = 0$

$\frac{\partial y}{\partial w} = -\frac{F_w}{F_y} =$

$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} =$

<u>ex.3</u> Given $F(Q,K,L)=0$ and $Q=f(K,L)$, find MP_K, MP_L

① $\frac{\partial Q}{\partial K} = -\frac{F_K}{F_Q} = MP_K$

$\frac{\partial Q}{\partial L} = -\frac{F_L}{F_Q} = MP_L$

$\frac{MP_L}{MP_K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}}|_{Q=\overline{Q}} = \frac{\partial K}{\partial L} =?$

② $Q = f(K,L) = 0$

$dQ = f_K dK + f_L dL = 0$

$\frac{dK}{dL} = -\frac{f_L}{f_K} = -\frac{MP_L}{MP_K}$

<u>ex.4</u> If $Q(K,L) = 16K^{0.25}L^{0.75} = 2,144$, use implicit function rule to find the slope

$\frac{dK}{dL} - MRTS$

$F(Q,K,L) =$

$\frac{\partial K}{\partial L} = -\frac{F_L}{F_K} =$

Application: Partial Market Equilibrium

Suppose now Q^d is a function of P as well as Y

$$\begin{aligned} Q^d &= Q^s \\ Q^d &= D(P,Y) \\ Q^s &= S(P) \end{aligned}$$

where D_p<0, S_p>0 ,and D_Y>0

$$\begin{aligned} \text{Endogenous variable} &= Q, P \\ \text{Exogenous variable} &= Y \end{aligned}$$

$$\begin{aligned} \text{Equilibrium condition} &: \quad Q^d = Q^s \text{ or } D(P,Y) - S(P) = 0 \\ \text{Equilibrium price} &: \quad P^* = P^*(Y) \\ \text{Equilibrium identity} &: \quad D(P^*,Y) - S(P^*) \equiv 0 \end{aligned}$$

where $F(P^*,Y) = D(P^*,Y) - S(P^*)$

Find $\frac{dP^*}{dY}, \frac{dQ^*}{dY}$

Method I.

1) $\frac{dP^*}{dY_0} = -\frac{F_Y}{F_{P^*}}$ by implicit function theorem

$F_{Y_0} = \frac{\partial D}{\partial Y_0} - 0$ (no Y in S)

$F_P = \frac{\partial D}{\partial P} - \frac{\partial S}{\partial P}$

$\therefore \frac{dP^*}{dY_0} = \frac{-\frac{\partial D}{\partial Y_0}}{\frac{\partial D}{\partial P} - \frac{\partial S}{\partial P}}$>0

2) $\frac{dQ^*}{dY_0} = \frac{d}{dY_0}(S(P^*(Y_0))) = \frac{dS}{dP}\frac{dP}{dY} > 0$

$\frac{dQ^*}{dY_0} = \frac{d}{dY_0}(D(P^*(Y_0), Y_0)) = \frac{dD}{dP} \cdot \frac{dP}{dY} + \frac{dY_0}{dY_0} = D_P \cdot \frac{dP}{dY} + 1$

Method II.

Use total derivative for $\frac{dP^*}{dY}$ from $F = D(P^*, Y) - S(P^*) = 0, P^* = P(Y)$

$$\frac{dF}{dY} = \underbrace{(\frac{\partial D}{\partial P^*} \cdot \frac{dP^*}{dY})}_{\text{indirect effect of } Y \text{ on D}} + \underbrace{\frac{\partial D}{\partial Y}}_{\text{direct effect of } Y \text{ on D}} - \underbrace{\frac{\partial S}{\partial P^*} \cdot \frac{dP^*}{dY}}_{\text{indirect effect of } Y \text{ on S}}$$

$$\Rightarrow (\tfrac{\partial D}{\partial P^*} - \tfrac{\partial S}{\partial P^*})\tfrac{dP^*}{dY} = -\tfrac{\partial D}{\partial Y}$$

$$\Rightarrow \frac{dP^*}{dY} = \frac{-\frac{\partial D}{\partial Y}}{\frac{\partial D}{\partial P^*} - \frac{\partial S}{\partial P^*}}$$

We can find $\frac{dP^*}{dY}$ with the same process we have in I.

Method III.

$$Q^d : dQ^* = D_P dP + D_Y dY \qquad \Rightarrow \frac{dQ^*}{dY_0} - D_P \frac{dP}{dY_0} = D_Y \frac{dY}{dY}$$

$$Q^s : dQ^* = S_P dP \qquad \Rightarrow \frac{dQ^*}{dY} - S_P \frac{dP}{dY} = 0$$

$$\begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \begin{bmatrix} \frac{dQ^*}{dY} \\ \frac{dP^*}{dY} \end{bmatrix} = \begin{bmatrix} D_Y \\ 0 \end{bmatrix}$$

By Cramer's rule

$$\frac{dQ^*}{dY} = \frac{\begin{vmatrix} D_Y & -D_P \\ 0 & -S_P \end{vmatrix}}{\begin{vmatrix} 1 & -D_P \\ 1 & -S_P \end{vmatrix}} = \frac{-D_Y S_P}{D_P - S_P} = S_P \cdot \frac{dP^*}{dY}$$

$$\frac{dP^*}{dY} = \frac{\begin{vmatrix} 1 & D_Y \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -D_P \\ 1 & -S_P \end{vmatrix}} = \frac{-D_Y}{D_P - S_P}$$

Interpretation

$$\text{Normally,}\quad \left.\begin{array}{ll} S_P > 0 & \text{slope of supply} \\ D_Y > 0 & \text{normal goods} \\ D_P < 0 & \text{law of demand} \end{array}\right\} \frac{dQ^*}{dY} > 0\ ,\ \frac{dP^*}{dY} > 0$$

$$\text{But if}\quad \left.\begin{array}{ll} S_P > 0 & \\ D_Y < 0 & \text{inferior goods} \\ D_P < 0 & \end{array}\right\} \frac{dQ^*}{dY} < 0\ ,\ \frac{dP^*}{dY} < 0$$

Consider giffen goods

$$\left.\begin{array}{l} S_P > 0 \\ D_Y > 0 \\ D_P > 0 \end{array}\right\} \frac{dQ}{dY} = \frac{-D_Y S_P}{D_P - S_P}$$

Case 1, if $S_P > D_P$

$$\frac{dQ^*}{dY} = \frac{-D_Y S_P}{D_P - S_P} = \frac{\ominus}{\ominus} = \oplus$$

Case 2, if $S_P < D_P$

$$\frac{dQ^*}{dY} = \frac{-D_{Y_0} S_P}{D_P - S_P} = \frac{\ominus}{\oplus} = \ominus$$

Example

$$\begin{array}{rclcrcl} Q^d & = & D(P, Y) & \quad & \text{endo} & = & P, Q \\ Q^s & = & S(P, T) & & \text{exo} & = & Y, T \\ P^* & = & P^*(Y, T) & & & & \\ Q^* & = & Q^*(Y, T) & & & & \end{array}$$

Solve for $\frac{dP^*}{dT}$ (suppose $T \not\leftrightarrow Y$)

Application: IS-LM model

$$\begin{array}{rrcl} \text{IS:} & Y & = & C(Y) + I(r) + G_0 & \text{endo} & = & Y, r \\ \text{LM:} & L(Y,r) & = & M_0 & \text{exo} & = & M_0, G_0 \end{array}$$

Solve for $\frac{dr^*}{dM_0}$

1) Take total differential

IS: ...... = ...

LM: .... = ...

2) Move endogenous variables to left-hand side to form the matrix.

.... = ...

.... = ...

3) Divide all with dM_0

.... = ... = 0

.... = ... = 1

4) Write matrix form

$$\begin{bmatrix} \dots\dots\dots & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots \end{bmatrix} \begin{bmatrix} \frac{dY^*}{dM_0} \\ \frac{dr^*}{dM_0} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

5) Solve by Cramer's rule

$\frac{dr^*}{dM_0} =$

How about $\frac{dr^*}{dG_0}, \frac{dY^*}{dM_0}$, and $\frac{dY^*}{dG_0}$?

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