EE422 Mathematical Economics 2

Continuous Time: Second-Order Differential Equations and Systems in the Plane

Read Chapter 6 Sydaeter et al.

Only 6.4 – 6.9

topics

- 6.1 introduction
- 6.2 Linear Differential equations, second-order
- 6.3 Constant Coefficients
- 6.4 Stability for linear equations
- 6.5 Simultaneous equations in the plane
- 6.6 Equilibrium points for linear systems
- 6.7 Phase plane analysis
- 6.8 Stability for nonlinear systems
- 6.9 Saddle Points

6.1 Introduction

Second-order dif equation is written as

$$\ddot{x} = F(t, x, \dot{x}) \;\ldots\ldots\ldots\ldots(1)$$

where $x(t)$ is the unknown function.

A solution of (1) is a twice-diffentiable function that satisfy (1).

6.1 Introduction: simplest case

$Ex1. \ddot{x} = k \quad where\, k \text{ is a cosntant.}$

$\text{Solution: } \ddot{x} = \dfrac{d\dot{x}}{dt}.\ \text{Direct integration}$

$\Rightarrow \dot{x} = \int k\, dt = kt + A.$

$\text{Integrating once more,we get}$

$$x = \frac{1}{2}kt^2 + At + B.$$

6.1 Introduction:
when x or t is missing

- These two special cases are

$$(2a)\,\ddot{x} = F(t,\dot{x})$$

$$(2b)\,\ddot{x} = F(x,\dot{x})$$

$In\,(2a), x\,is\,\text{missing}.\,Let\,u \equiv \dot{x}.\,Then\,(2a)\,becomes$

$\dot{u} = F(t,u)\,which\,is\,first-order\,equation.$

$If\,we\,can\,find\,u(t), then\,\text{integrating}\,u(t)\,will\,give$

$us\,the\,general\,solution\,of\,(2a).$

6.1 Introduction: when x or t is missing

$$Ex1.\, \ddot{x} = F(t, \dot{x}) = t + \dot{x}.$$

$$Let\, u \equiv \dot{x}\, yields\, \dot{u} = t + u.$$

$$Solution\, is\, u(t) = Ae^{t} - t - 1.$$

$$Hence\, \dot{x} = Ae^{t} - t - 1. \text{Integrating}\, yields$$

$$x(t) = \int (Ae^{t} - t - 1)\, dt = Ae^{t} - 0.5t^{2} - t + B.$$

6.1 Introduction: when x or t is missing

$$(2b)\,\ddot{x} = F(x, \dot{x})$$

t is not here, eq is called autonomous.

(2b) cannot be solved explicitly, except in special cases.

$Ex.\ \ \ddot{x} = -x\dot{x}^3\ \ see\ p.222$

6.1 Introduction:

$$\ddot{x} = F(t, x, \dot{x}) \ldots\ldots\ldots\ldots (1)$$

$Similarly, (1)\, is\, even\, harder\, to\, solve\, for\, x(t)$

$when\, both\, \dot{x}\, and\, t\, are\, there.$

$Generally, we\, use\, numerical\;\; solution\;\, for\, a$

$given\, initial\, condition.$

$Some\, case\, we\, can\;\, find\, values\, of\;\, A\, and\, B\, by$

$some\, initial\, conditions. For\, example,$

$$x(t_0) = x_0 \quad and \quad \dot{x}(t_0) = a.$$

6.1 Introduction:

$$Ex.\ddot{x} = \dot{x} + t \ and\, x(0) = 1 \quad and \quad \dot{x}(0) = 2.$$

$$Solution: x = Ae^{t} - 0.5t^{2} - t + B.$$

$$For\, x(0) = 1, \quad 1 = A + B$$

$$For\, \dot{x}(0) = 2, \quad 2 = A - 1$$

$$Thus, A = 3, \;\; B = -2$$

$$The\, unique\, solution\, is\, x = 3e^{t} - .5t^{2} - t - 2.$$

6.2 Linear Differential Equation

The general 2nd-order linear dif eq is

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t) \ldots\ldots\ldots\ldots\ldots(1)$$

There is no explicit solution of (1) in the general case or non-homogenous case.

When RHS=0, we have:

Homogenous: $\quad let\; f(t) = 0$

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0 \ldots\ldots\ldots\ldots(2)$$

6.2 Linear Differential Equation

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0 \ldots\ldots\ldots\ldots(2)$$

$First, we\ claim\ that\ if\ u_1 = u_1(t)\ and\ u_2 = u_2(t)$

$both\ satisfy\ (2), then\ x = Au_1 + Bu_2\ also\ satisfies (2).$

$Check : \dot{x} = A\dot{u}_1 + B\dot{u}_2\ and\ \ddot{x} = A\ddot{u}_1 + B\ddot{u}_2$

$Substitute\ these\ in\ (2), we\ get$

$$A[\underbrace{\ddot{u}_1 + a(t)\dot{u}_1 + b(t)u_1}_{0}] + B[\underbrace{\ddot{u}_2 + a(t)\dot{u}_2 + b(t)u_2}_{0}]$$

$by\ assumption\ that\ u_1\ and\ u_2\ solve\ (2). Thus, we\ prove\ that$

$x = Au_1 + Bu_2\ satisfies\ (2).$

The sum of two or more solutions of homogenous dif eq. is also a solution.

6.2 Linear Differential Equation

Theorem 6.2.1

(a) Homogenous dif eq. $\ddot{x} + a(t)\dot{x} + b(t)x = 0$

has the general solution

$$x = Au_1(t) + Bu_2(t) \quad \text{where}$$

$u_1(t)$ *and* $u_2(t)$ are two solutions that are

not proportional. (multiples of each other)

(b) The nonhomogenous dif eq has the general

solution $x = Au_1(t) + Bu_2(t) + u^*(t)$

where $u^*(t)$ is any particular solution of the non

homogenous equation.

6.2 Linear Differential Equation

Ex. find the general solutions of $\ddot{x} - x = t$

First look at homogenous part. We need some functions $x(t)$ that do not change when differentiated twice.

How about $x = e^t \Rightarrow \dot{x} = e^t, \ddot{x} = e^t$,

$and\, x = e^{-t} \Rightarrow \dot{x} = -e^{-t}, \ddot{x} = e^{-t}$ not proportional.

The general solution of the homogenous part is

$x = Ae^t + Be^{-t}$.

6.2 Linear Differential Equation

For non-homogenous, the general solution is a complementary function + a particular function.

By trial and error we find that

$u^*(t) = \text{-}t$ where $u^*(t)$ is any particular solution.

Thus, the general solution is

$$x = Ae^{t} + Be^{-t} - t$$

We have no general method to find two solutions of homogenous part, except when $a(t)$ and $b(t)$ are both constants.

6.3 Linear Differential Equation: Homogenous with constant coefficients.

Consider the homogenous eq.

$$\ddot{x} + a\dot{x} + bx = 0 \ldots\ldots\ldots\ldots\ldots(1)$$

By Thm 6.2.1 we need to find $u_1(t)$ *and* $u_2(t)$

solutions that are not proportional.

Consider $x = e^{rt}$ this function has propeties

$$\dot{x} = re^{rt} = rx \quad and \quad \ddot{x} = r^2 e^{rt} = r^2 x.$$

So $x, \dot{x}, and\, \ddot{x}$ are constant multiples of each other.

6.3 Linear Differential Equation: Homogenous with constant coef.

Idea: we try to adjust r *so that* $x = e^{rt}$ *satisfy* (1).

This requires $r^2 x + arx + bx = 0$

$$\Rightarrow x(r^2 + ar + b) = 0.$$

We need $r^2 + ar + b = 0$, *Characteristic eq. of* (1).

Two characteristic roots are

$$r_1 = -\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^2 - b},\quad r_2 = -\tfrac{1}{2}a - \sqrt{\tfrac{1}{4}a^2 - b}.$$

The roots are real iff $\boxed{\tfrac{1}{4}a^2 - b \geq 0}$.

$$[Note: Ax^2 + Bx + C \Rightarrow x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}]$$

Theorem 6.3.1: Homogenous

Theorem 6.3.1

The general solution of $\ddot{x} + a\dot{x} + bx = 0$ is

(1) If $\frac{1}{4}a^2 - b > 0$, $\boxed{x = Ae^{r_1 t} + Be^{r_2 t}.}$

(2) If $\frac{1}{4}a^2 - b = 0$, $\boxed{x = (A + Bt)e^{rt}}$, where $r = -.5a$

(3) If $\frac{1}{4}a^2 - b < 0$, $\boxed{x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)}$,

where $r = \alpha \pm i\beta$ and $\alpha = -.5a$, $\beta = \sqrt{b - \frac{1}{4}a^2}$.

(1) Two distinct real roots; (2) one real double root;

(3) complex roots or no real roots

Theorem 6.3.1: Homogenous

In case (2), the general solution is

$x = Ae^{rt} + Bg(t), where\, g(t)$ is some solution that is not a constant multiple of e^{rt}.

We can see that a possible $g(t)\, is\ te^{rt}$.

Hence the general solution is $(A + Bt)e^{rt}$.

Theorem 6.3.1: Homogenous

Note that in (3): $x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

α *and* $\beta > 0$ and are real.

This is derived from $r_1 = \alpha + i\beta$ *and* $r_2 = \alpha - i\beta$.

Thus, $x = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}$

$$= e^{\alpha t}(Ae^{i\beta t} + Be^{-i\beta t})$$

where $i \equiv \sqrt{-1}$ *or* $i^2 = -1$.

In practice, we prefer to work with real functions than complex exponentials. We can apply Euler's formula

Next, use $e^{i\beta t} = \cos\beta t + i\sin\beta t,$

and $e^{-i\beta t} = \cos\beta t - i\sin\beta t.$

Theorem 6.3.1: Homogenous

$$x = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t} = e^{\alpha t}(Ae^{i\beta t} + Be^{-i\beta t})$$

$$= e^{\alpha t}(A\cos\beta t + Ai\sin\beta t + B\cos\beta t - Bi\sin\beta t)$$

$$= e^{\alpha t}(A'\cos\beta t + B'\sin\beta t)$$

$$where\,A' = A + B\,and\,B' = i(A - B).$$

$$These\,A'\,and\,B'\,are\,real\,numbers\,when\,A\,and\,B$$

$$are\,complex\,conjugates.$$

$$Another\;\;form\,is\,x = Ce^{\alpha t}\cos(\beta t + D).$$

$$Ex1.\, \ddot{x} - 3x = 0$$

$$x = e^{rt}, \ddot{x} = r^2 x \Rightarrow r^2 x - 3x = 0$$

$$\Rightarrow x(r^2 - 3) = 0$$

$$Characteristic\, eq.: r^2 - 3 = 0\, has\, 2\, real\, roots$$

$$r_1 = -\sqrt{3}\, and\, r_2 = \sqrt{3}.$$

$$The\, general\, solution\, is$$

$$x = Ae^{-\sqrt{3}t} + Be^{\sqrt{3}t}.$$

$$Ex2.\ \ddot{x} - 4\dot{x} + 4x = 0$$

$$\Rightarrow x(r^2 - 4r + 4) = 0$$

$$\Rightarrow r^2 - 4r + 4 = (r-2)^2\ has\ r = 2.$$

The general solution is $x = (A + Bt)e^{2t}$.

$$Ex3.\ \ddot{x} - 6\dot{x} + 13x = 0$$

$$\Rightarrow r^2 - 6r + 13 = 0\, has\, no\ real\, roots.$$

$$check\, \frac{1}{4}(-6)^2 - 13 < 0.$$

$$\alpha = -a/2 = -(-6)/2 = 3;$$

$$\beta = \sqrt{13 - .25(-6)^2} = 2$$

$$The\, general\, solution\, is$$

$$x = e^{3t}(A\cos 2t + B\sin 2t).$$

6.3 Non-Homogenous Equation

Consider the nonhomogenous eq.

$$\ddot{x} + a\dot{x} + bx = f(t)\ldots\ldots\ldots\ldots(4)$$

By thm 6.2.1, the general solution is

$$x(t) = Au_1(t) + Bu_2(t) + u^*(t).$$

How do we find $u^*(t)$? The method of undetermined coefficients works in many cases.

6.3 Non-Homogenous Equation

$\text{Consider 3 cases of } f(t)\, when\, b \neq 0$

$(\text{note that } if\, b = 0, \text{ we have a case of missing } x\, (case\, 2a)).$

$\boxed{\text{Case 1. } f(t) = A \text{ (constant)}}$

$\text{The particular sol of } \ddot{x} + a\dot{x} + bx = A$

$$is \quad u^{*}(t) = A \,/\, b.$$

$If\, u^* = c, then\, \dot{u}^* = \ddot{u}^* = 0, so\, bc = A\ or\, c = A \,/\, b.$

6.3 Non-Homogenous Equation

$\boxed{\text{Case 2. } f(t) \text{ is polynomial.}}$

$\text{Suppose } f(t)\, is \text{ degree } n.\text{ The particular sol of}$

$$\ddot{x} + a\dot{x} + bx = f(t) \quad ....(4)$$

$$is \quad u^{*}(t) = A_{n}t^{n} + A_{n-1}t^{n-1} + ... + A_{1}t + A_{0}.$$

$\text{We find all } A's \text{ by requiring } u^{*} \text{ satisfying } (4).$

6.3 Non-Homogenous Equation

$$Ex.\ddot{x} - 4\dot{x} + 4x = \underset{f(t)}{t^2 + 2}\ldots\ldots(*)$$

$f(t)$ is polynomial of degree 2.

$$Let\, u^* = At^2 + Bt + C, next$$

try to adjust A, B, C to give a solution

$$(i)\,\dot{u}^* = 2At + B; (ii)\,\ddot{u}^* = 2A$$

Sub $(i), (ii)\, and\, u$ $*$ into $(*)$ yields

$$2A - 4(2At + B) + 4(At^2 + Bt + C) = t^2 + 2$$

6.3 Non-Homogenous Equation

$$\text{rearrange LHS to get similar to RHS}$$

$$\underset{1}{4A}\,t^2 + \underbrace{(4B-8A)}_{0}\,t + \underbrace{(2A-4B+4C)}_{2} = t^2 + 2$$

$$\Rightarrow 4A = 1,\ or\ A = 1\,/\,4.$$

$$\Rightarrow 4B - 8A = 0\ \ or\ \ B = 2A = 1\,/\,2.$$

$$\Rightarrow (2\frac{1}{4} - 4\frac{1}{2} + 4C) = 2\ \ or\, C = 7\,/\,8.$$

$$So, u^* = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{7}{8}.$$

6.3 Non-Homogenous Equation

$$Ex.(you\,try)\;\; \ddot{x}+4\dot{x}-2x=\underbrace{2t^2-2t+6}_{f(t)}......(*)$$

$f(t)$ is polynomial of degree 2.

For Homogenous part, the auxiliary equation is

$r^r+4r-2=0$. We get two distinct real roots:

$r_1=-2-\sqrt{6}\quad and\, r_2=-2+\sqrt{6}$.

The general solution $is\, x=c_1e^{-2-\sqrt{6}}+c_2e^{-2+\sqrt{6}}$

For non-homogenous part, $Let\, u^*=At^2+Bt+C$.

6.3 Non-Homogenous Equation

Next try to adjust A, B, C to give a solution

$$(i)\,\dot{u}^* = 2At + B; and\ (ii)\,\ddot{u}^* = 2A$$

Sub $(i), (ii)$ and $u*$ into $(*)$ yields

$$2A + 8At + 4B - 2At^2 - 2Bt - 2C = 2t^2 - 3t + 6$$

rearrange LHS to fit RHS

$$-2At^2 + (8A - 2B)t + (2A + 4B - 2C) = 2t^2 - 3t + 6$$

Thus the particular solution is

$$u = -t^2 - \tfrac{5}{2}t - 9$$

6.3 Non-Homogenous Equation

Case 3. $f(t) = pe^{qt}$, where p and q are constants.

$\ddot{x} + a\dot{x} + bx = pe^{qt}$ has the particular solution

$$u^* = \frac{p}{q^2 + aq + b} e^{qt}.$$

In all 3 cases, notice that u^* has a same form $as\ f(t)$.

6.3 Non-Homogenous Equation

Euler's Differential Equation

$$t^2\ddot{x} + at^1\dot{x} + bx = 0, \ (t > 0)$$

Trick: make it to a new independent variable s

where $t = e^s$ *or* $s = \ln t$. *Then*

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{ds}\frac{1}{t}$$

$$\ddot{x} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{1}{t}\right) = -\frac{1}{t^2}\frac{dx}{ds} + \frac{1}{t}\frac{d}{dt}\left(\frac{dx}{ds}\right)$$

6.3 Non-Homogenous Equation

$$= -\frac{1}{t^2}\frac{dx}{ds} + \frac{1}{t}\frac{d}{ds}\left(\frac{dx}{ds}\right)\frac{ds}{dt}$$

$$= -\frac{1}{t^2}\frac{dx}{ds} + \frac{1}{t}\frac{d^2x}{ds^2}\frac{1}{t}. \quad Use\, these\, we\, get$$

$$-\frac{dx}{ds} + \frac{d^2x}{ds^2} + at\frac{dx}{ds}\frac{1}{t} + bx = 0, or$$

$$\boxed{\frac{d^2x}{ds^2} + (a-1)\frac{dx}{ds} + bx = 0}$$

this is an ordinary 2nd eq. with constant coef.

6.3 Non-Homogenous Equation

Ex. solve $\quad t^2\ddot{x} + t\dot{x} - x = 0.\ [a = 1, b = -1]$

$$\Rightarrow \frac{d^2x}{ds^2} + (1-1)\frac{dx}{ds} + (-1)x = 0 \ \ where\, s = \ln t.$$

Now, 2^{nd} diff. eq. with constant coef. : $a = 0, b = -1.$

charecteristic eq : $\quad r^2 + ar + b = 0$

$$\Rightarrow \quad r^2 + 0.r + (-1) = 0.$$

check roots : $\quad \frac{1}{4}a^2 - b = 0 - (-1) > 0.$

So, we have a case with 2 different real roots.

6.3 Non-Homogenous Equation

$$x(s) = Ae^{r_1 s} + Be^{r_2 s}; use\, formula$$

$$r_1 = -\frac{1}{2}0 + \sqrt{\tfrac{1}{4}0^2 - (-1)} = 1;$$

$$r_2 = -1.$$

$$\text{Thus}\;\; x(s) = Ae^{1s} + Be^{-1s}.$$

$$\text{Since}\;\; s \equiv \ln t\;\, or\;\, e^{s} = t, so$$

$$x(t) = At + Bt^{-1}.$$

6.3 Non-Homogenous Equation

$$ex. \quad t^2\ddot{x} + 3t\dot{x} + \frac{3}{4}x = 0$$

$\Rightarrow$ show that this is an Euler equation

whose the general solution is

$$x(t) = At^{-1/2} + Bt^{-3/2} + 4.$$

6.4 Stability for Linear Equations

- A solution of the system may be interpreted as the resulting path of a particle that is initially placed at position X(0) or initial conditions.
- If X(0) is just a critical points, then the particle remains stationary.
- Now, we look at the behavior of the solution when X(0) is chosen close to a critical point of the system.
- Now, suppose X_1 is critical points, we are interested in answering these questions:

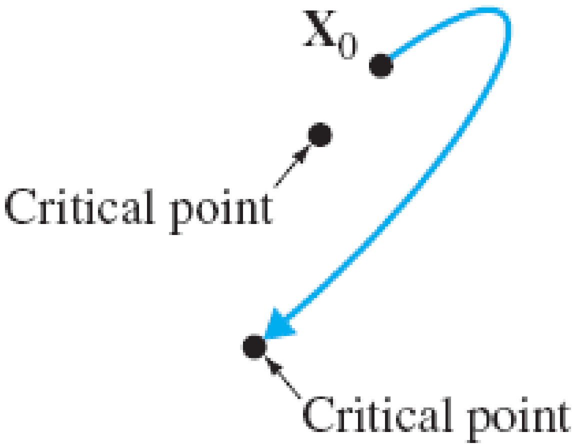
6.4 Stability for Linear Equations

- If X(0) is placed near $X_{1,}$ or we think this as one or more initial conditions are changed, obviously, the solution will change. We want to know
- (a) will the particle return to the critical point? More precisely does lim $X(t)=X_1$?
- (b) If the particle does not return to the critical point, does it remain close to the critical points or move away from it?

6.4 Stability for Linear Equations

(a) Locally stable

(b) Locally stable



(c) Unstable

6.4 Stability for Linear Equations

- Will small changes in the initial conditions have any effect on the long-run behavior of the solution, or will the effect die out as *t* goes to infinity (meaning that the particle eventually goes to the critical points) ?
- Two concepts: Stability requires only that small perturbations of the equilibrium yield a solution that remains close to the equilibrium, whereas asymptotic stability requires that the solution eventually returns to the equilibrium (not just only stay close) as *t* goes to infinity.

6.4 Stability for Linear Equations

- If their effects die out, we have the system that is asymptotically stable.
- If it leads to big changes in the behavior of the solution in the long run, then the system is unstable.
- If a neighborhood of an equilibrium point is extended to the whole domain, we call it globally asymptotically stable. That is, all solution paths approach equilibrium, regardless of initial conditions. Ex. $\dot{x} = ax \, with \, a < 0$.
- Obviously, asymptotic stability is a stronger property than stability.

6.4 Stability for Linear Equations

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t) \ldots\ldots\ldots\ldots(1)$$

The solution *of* (1) *is* $x = Au_1(t) + Bu_2(t) + u^*(t)$,

where $Au_1(t) + Bu_2(t)$ is the general solution of

homogenous *and* $u^*(t)$ is a particular solution.

Eq.(1) is called globally asym. stable if *every* general

solutions of homogenous equation tends to 0

as $t \to \infty$ for all values of A *and* B.

Then the effect of the initial conditions dies out eventually.

6.4 Stability for Linear Equations with constant coefficients

$$\ddot{x} + a\dot{x} + bx = f(t)\, is\, globally\, asym.\, stable$$

$$iff \quad a > 0 \quad and \quad b > 0. \quad ....(2)$$

$$or \;\; iff\, both\, roots\, of\, the\, characteristic$$

$$equation \quad r^2 + ar + b = 0 \quad have$$

$$negative\, real\;\; parts \;\; ....(3).$$

6.4 Stability for Linear Eqs with constant coefficients

$$If \;\; r^2 + ar + b = 0 \;\;\; has\, real\, roots,$$

$$then \;\; r_1 + r_2 = -a \quad ...(*)$$

$$and \qquad r_1 r_2 = b \qquad ...(**).$$

$$To\, claim\, that\, both\, roots\, are\, negative, this$$

$$needs \; (i) \quad a \;\; is \;\; positive \;\; ..from(*),$$

$$and \quad (ii) \;\; b \;\; is \;\; positive \quad ..from(**).$$

$$Conversely, if \;\; a, b \;\; are\, both\, positive, then$$

$$we\, can\, show\, that\, , from\, (*), either\, r_1 \;\; or \;\; r_2$$

$$must\, be\, negative. \; So, if\, one\, is\, positive,$$

$$from\, (**), both\, must\, be\, negative.$$

6.4 Stability for Linear Equations with constant coefficients

$\mathrm{Look\,at\,all\,3\,cases\,from\,Theorem\,6.3.1}$

$$(1)\ x = Ae^{r_1 t} + Be^{r_2 t}.$$

$$(2)\ x = (A + Bt)e^{rt}, \quad where\, r = -.5a.$$

$$(3)\ x = e^{\alpha t}(A\cos\beta t + B\sin\beta t), where\,\alpha = -.5a$$

$$and\ \beta = \tfrac{1}{2}\sqrt{4b - a^2}.$$

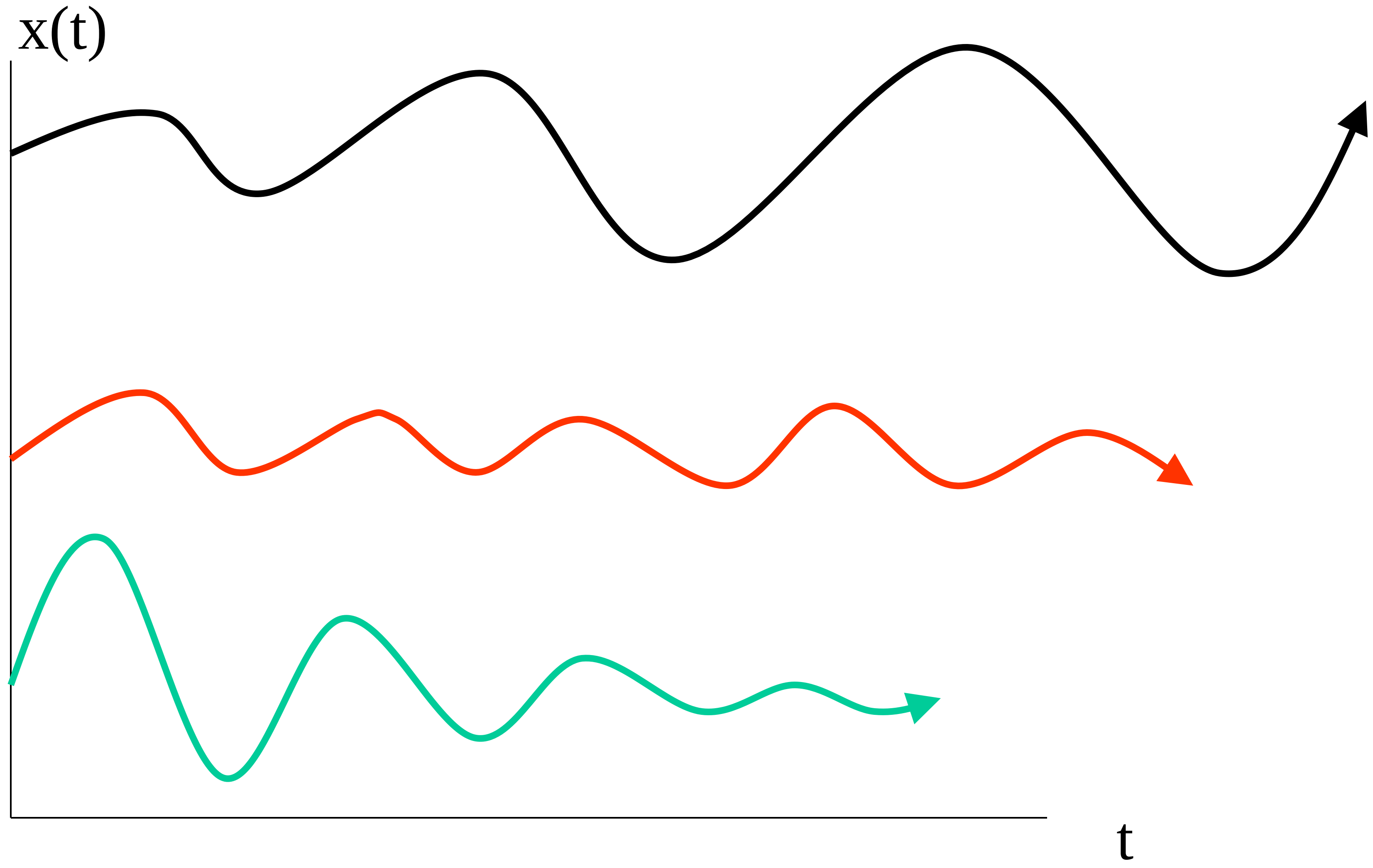
6.4 Stability for Linear Equations with constant coefficients

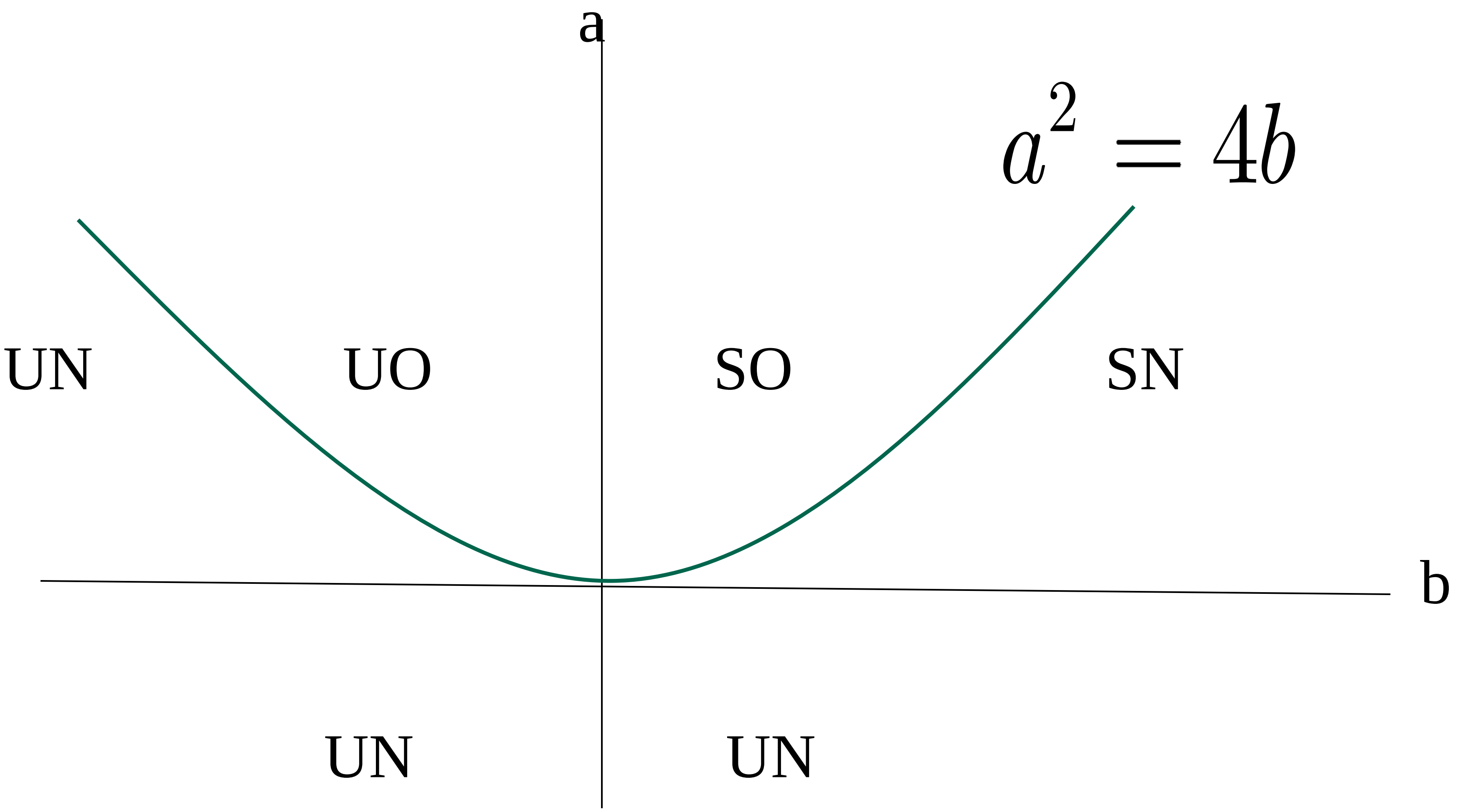
$\text{Case 3 depends on the value of } \alpha.$

$If\ \alpha > 0, \text{ we have explosive fluctuation.}$

$If\ \alpha = 0, \text{ we have uniform fluctuation.}$

$If\ \alpha < 0, \text{ we have damped fluctuation or convergence time path.}$


x(t)
t



S=stable, U=unstable, O=oscillatory, N=non-oscillatory

6.5 Simultaneous Eqs in the plane

Many economic models involve several unknown functions that satisfy a number of simultaneous diff equations :

$$\dot{x} = f(t, x, y)$$
$$\dot{y} = g(t, x, y).$$

In economics, $x(t)$ *and* $y(t)$ are state variables.

Usually, $(x(t_0), y(t_0))$ is known and the future path is then uniquely determined.

6.5 Simultaneous Eqs in the plane

However, rate of change of each variable depends also on other variable, thus x and y are interacted.

Systems of this type are very complicated.

A solution of (1) is a pair of differentiable functions $(x(t), y(t))$ satisfying (1) and (2).

There is no general method. However, we can find explicit solutions in some special cases.

6.5 Simultaneous Eqs in the plane

We will discuss some special cases:

(a) *E*limination

(b) Recursive systems

(c) Linear systems with constant coefficients.

(p.238)

6.5 Simultaneous Eqs in the plane

(a) Elimination (p.237):

Idea is to reduce the system into only one unknown.

Transform the first equation

$$\dot{x} = f(t, x, y) \text{ to } y = h(t, x, \dot{x}).$$

Differentiate this wrt t and substitute y *and* $\dot{y}$ into the second equation, hence we express all in terms of x.

Then, solve for x(t) and then we can find y(t).

6.5 Simultaneous Eqs in the plane

Ex.1. Find the solution of the system

$$\dot{x} = 2x + e^{t}y - e^{t}$$

$$\dot{y} = 4e^{-t}x + y$$

Solution:

Solving the first eq. $y = \dot{x}e^{-t} - 2xe^{-t} + 1$

Diff wrt t : $\dot{y} = \ddot{x}e^{-t} - \dot{x}e^{-t} - 2\dot{x}e^{-t} + 2xe^{-t}$

Insert this and y into the second equation gives

$$\ddot{x} - 4\dot{x} = e^{t}.$$

Use the formula in p.230 section 6.3

6.5 Simultaneous Eqs in the plane

(b) Recursive (p.237):

When the system takes the special form:

$$\dot{x} = f(t, x, y)\, and\, \dot{y} = g(t, y).$$

In this case, y is independent.

First, solve $\dot{y}\, and\, get\, y(t)$.

Second, substitute y into $\dot{x}$ and solve for $x(t)$.

6.5 Simultaneous Eqs in the plane

Consider linear System with constant Coeff.

$$\dot{x} = a_{11}x + a_{12}y + b_1(t)$$

$$\dot{y} = a_{21}x + a_{22}y + b_2(t)$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

We will solve for

$$x = \phi(t; A, B) \quad and \quad y = \varphi(t : A, B)$$

$A\,and\,B$ can be specified by initial conditions

for each variable, say $x(t_0) = x_0\, and\, y(t_0) = y_0$.

See the formula for the solution of this system in text.

6.5 Simultaneous Eqs in the plane

Suppose $b_t(t)$ and $b_2(t) = 0,$ the linear system with constant coeff. reduces to the linear homo system (solution based on Eigenvalues p. 239)

$$\dot{x} = a_{11}x + a_{12}y$$

$$\dot{y} = a_{21}x + a_{22}y$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad ....(7)$$

Let $x = v_1 e^{\lambda t}$ *and* $y = v_2 e^{\lambda t}$ solution of (7) for an appropriate choice of numbers v_1, v_2.

6.5 Simultaneous Eqs in the plane

Inserting $\dot{x} = v_1 \lambda e^{\lambda t}$ and $\dot{y} = v_2 \lambda e^{\lambda t}$ in (7) yields

$$\begin{pmatrix} v_1 \lambda e^{\lambda t} \\ v_2 \lambda e^{\lambda t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{pmatrix}$$

Get rid of $e^{\lambda t}$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{X} = \lambda \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{X}$$

where X is Eigenvector of matrix A $with$ λ Eigenvalue.

$$\boxed{AX - \lambda X = 0 \Rightarrow (A - \lambda I)X = 0.}$$

6.5 Simultaneous Eqs in the plane

$$For\ X \neq 0, we\ need \left|A - \lambda I\right| = 0, or$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - \underbrace{(a_{11} + a_{22})}_{trA}\lambda + \left|A\right| = 0.$$

So, if A has different real eigenvalues, $\lambda_1 and\ \lambda_2, then$

A has 2 eigenvectors that are linearly independent.

Thus, the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + Be^{\lambda_2 t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

$$Ex.\ \dot{x} = 0 + 2y$$

$$\dot{y} = x + y$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$The\, characteristic\, of\ A\, is\ \begin{vmatrix} 0 - \lambda & 2 \\ 1 & 1 - \lambda \end{vmatrix}$$

$$= \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2). Thus, eigenvalues$$

$$are\, \lambda_1 = -1\, and\ \lambda_2 = 2. The\, corresponding$$

$$eigenvectors\, are\ \begin{pmatrix} -2 \\ 1 \end{pmatrix} and \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

When $\lambda_1 = -1$, the corresponding eigenvectors

$$is\ AX = -1X;\ \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix};$$

$$\begin{pmatrix} 2v_2 \\ v_1 + v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}; or \boxed{v_1 = -2v_2}; or \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

When $\lambda_2 = 2$, the corresponding eigenvector

$$is\ AX = 2X;\ \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; or$$

$$\boxed{2v_2 = 2v_1};\ or \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$The\ general\ solution\ is$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + Be^{\lambda_2 t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= Ae^{-1t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + Be^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2Ae^{-1t} + Be^{2t} \\ Ae^{-1t} + Be^{2t} \end{pmatrix}.$$

EXAMPLE 1 Distinct Eigenvalues

Solve

$$\begin{aligned}\frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= 2x + y.\end{aligned} \tag{4}$$

SOLUTION We first find the eigenvalues and eigenvectors of the matrix of coefficients.

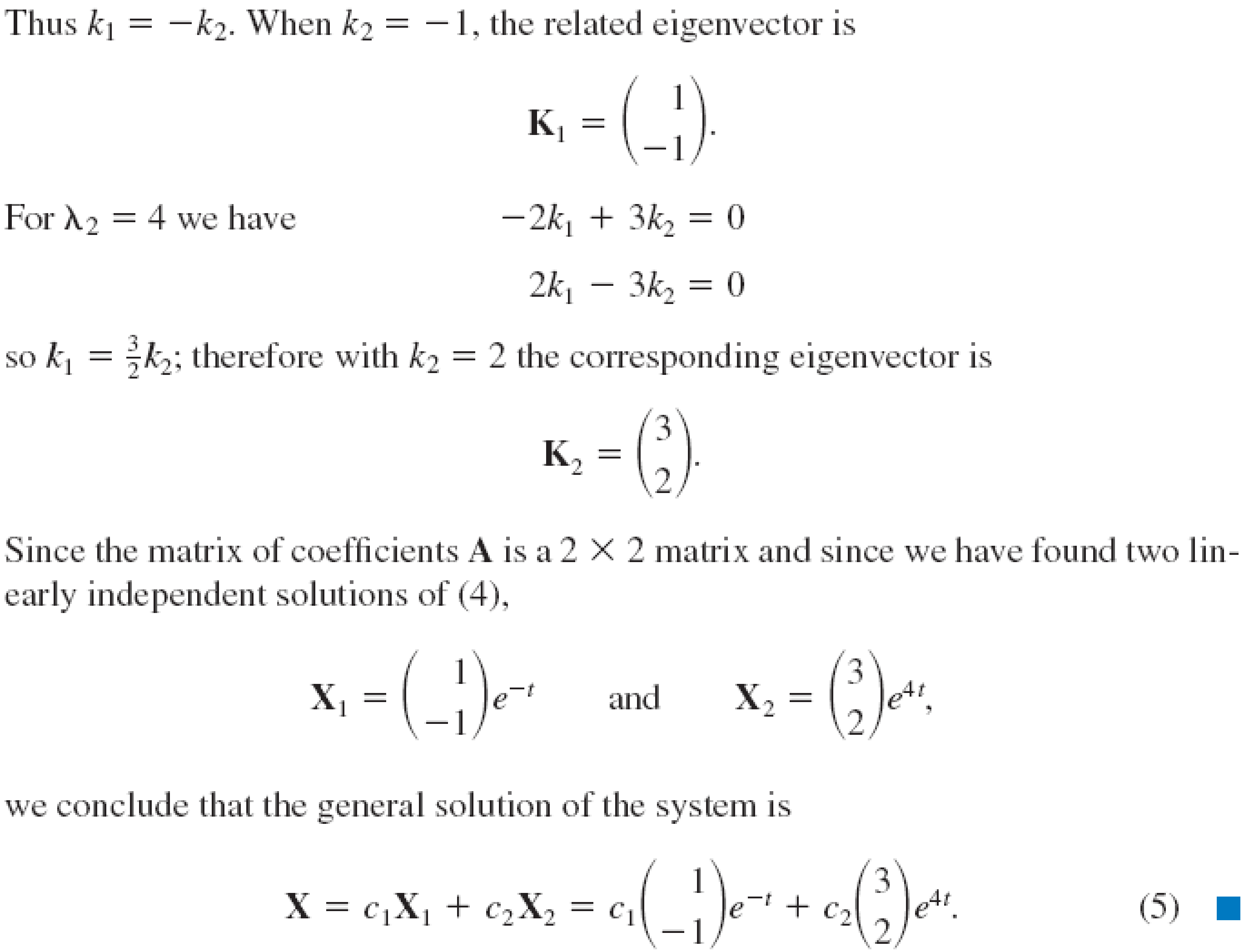
From the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$$

we see that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

Now for $\lambda_1 = -1$, (3) is equivalent to

$$\begin{aligned} 3k_1 + 3k_2 &= 0 \\ 2k_1 + 2k_2 &= 0. \end{aligned}$$



Thus $k_1 = -k_2$. When $k_2 = -1$, the related eigenvector is

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 4$ we have

$$\begin{aligned} -2k_1 + 3k_2 &= 0 \\ 2k_1 - 3k_2 &= 0 \end{aligned}$$

so $k_1 = \frac{3}{2}k_2$; therefore with $k_2 = 2$ the corresponding eigenvector is

$$\mathbf{K}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

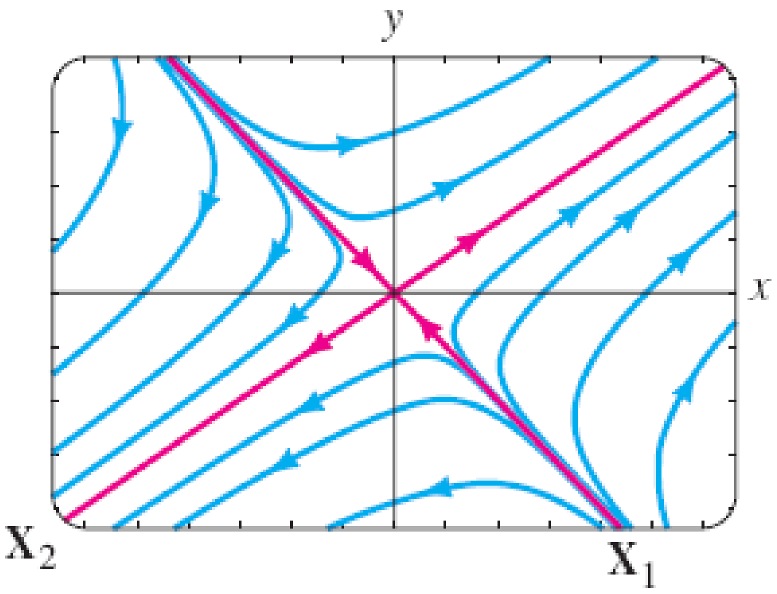
Since the matrix of coefficients $\mathbf{A}$ is a 2×2 matrix and since we have found two linearly independent solutions of (4),

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t},$$

we conclude that the general solution of the system is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}. \qquad (5)$$

■



This shows a typical phase portrait of all 2x2 homogenous linear system with real eigenvalues of opposite signs. (saddle)

If both eigenvalues are positive, the system moves away from the origin.(Repeller)

If both eigenvalues are negative, the system Moves toward the origin. (Attractor)

$\mathrm{Ex\,2.\;\;Let\,add\,some\,constants}$

$$\dot{x} = 2y + 6$$

$$\dot{y} = x + y - 3$$

$\mathrm{Trick\,is\,to\,transform\,this\,into\,homogenou\,system}$

$\mathrm{by\,introducing\,new\,variables\,as\,a\,deviation\,from}$

$\mathrm{equilibrium.}\,When\,\dot{x} = \dot{y} = 0, (x, y) = (6, -3).$

$\mathrm{Let\,call}\;\; z = x - 6 \;\; and \;\; w = y + 3$

$and \quad \dot{z} = \dot{x} \;\;, \dot{w} = \dot{y} \;\; \mathrm{so\,the\,problem\,becomes}$

$$\begin{aligned} \dot{z} &= 2(w-3) + 6 = 2w \\ \dot{w} &= (z+6) + (w-3) - 3 = z + w \end{aligned} \Leftrightarrow \begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

$$this\ \ is\, the\, same\, as\, before, thus$$

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} -2Ae^{-1t} + Be^{2t} \\ Ae^{-1t} + Be^{2t} \end{pmatrix}.$$

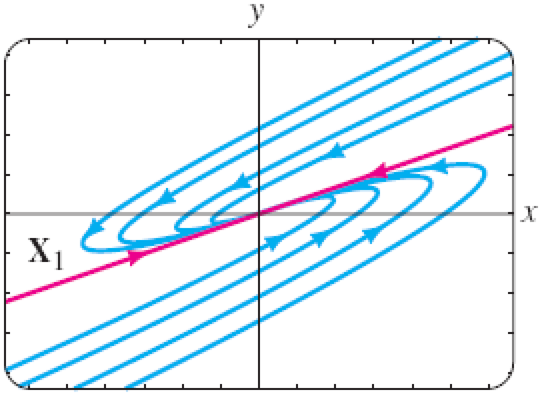
$$Thus, x = z + 6 = ...$$

$$and \quad y = w - 3 = ...$$

Ex. When A give repeated roots, try this

$$A=\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}, find\,\lambda = -3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = see\,formula(4)\,in\,p.238.$$



$$\mathbf{X} = c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{-3t} + c_2\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix}te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}e^{-3t}\right].$$

This shows a typical picture of phase diagram of all 2x2 homogenous linear systems that have two repeated negative eigenvalues.

6.6 Equilibrium Points for Linear Systems

Consider Linear system with constant coefficients

$$\dot{x} = a_{11}x + a_{12}y + b_1$$

$$\dot{y} = a_{21}x + a_{22}y + b_2$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{...........}(1)$$

The equilibrium points are determined by putting $\dot{x} = \dot{y} = 0 \; in \, (1).$

6.6 Equilibrium Points for Linear Systems

$$a_{11}x + a_{12}y + b_1 = 0$$

$$a_{21}x + a_{22}y + b_2 = 0$$

$$\Leftrightarrow a_{11}x + a_{12}y = -b_1$$

$$a_{21}x + a_{22}y = -b_2 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$From\,(2), we\,can\,use\,Cramer's\,rule$$

$$x^* = \frac{a_{12}b_2 - a_{22}b_1}{|A|}, y^* = \frac{a_{21}b_1 - a_{11}b_2}{|A|}$$

6.6 Equilibrium Points for Linear Systems

$$\text{Ex. Find the equilibrium point for}$$

$$\dot{x} = -2x + y + 2$$

$$\dot{y} = -2y + 8$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$First, \left|A\right| = 4 - 0 = 4 > 0 \, and \, (x^*, y^*) = (3, 4)$$

$$\text{Also you can solve for the general solution}$$

$$x(t) = Ae^{-2t} + Bte^{-2t} + 3, \; and$$

$$y(t) = Be^{-2t} + 4.$$

6.6 Equilibrium Points for Linear Systems: Theorem 6.6.1

The equilibrium point (x^*, y^*) for

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

is globally stable iff

$$tr(A) = a_{11} + a_{22} < 0, and$$

$$\det(A) > 0.$$

Equivalently, iff both eigenvalues of A have negtive real parts.

Some nice properties

(1) $\boxed{tr(A) = \lambda_1 + \lambda_2}$

so if both have negative real parts,

then $tr(A) < 0$.

(2) $\boxed{\det\left|A\right| = \lambda_1\lambda_2}$

$$check : Suppose\, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, we\, know\, that$$

$$trace(A) = a + d,\ and\ \ |A| = ad - bc.$$

$$Now\, find\, eigenvalue\, of\, this\, matrix\, A$$

$$|A - \lambda I| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - (a + d)\lambda + |A|.$$

$$Show\, that\, (a)\ \lambda_1 + \lambda_2 = a + d$$

$$and\, (b)\ \lambda_1 \lambda_2 = ad - bc.$$

$$\text{Ex. Check the stability at } (0,0) \text{ for}$$

$$\dot{x} = y \ and \ \ \dot{y} = -2x - y \ \Leftrightarrow$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$Solution.$

$$tr(A) = a_{11} + a_{22} = 0 - 1 < 0, and$$

$$\det\left|A\right| = 0(-1) + 2 = 2 > 0$$

$$\text{By thm 6.6.1 the equil. point } (0,0) \text{ is stable.}$$

Alternative Behaviors around Equil.Points

Consider linear system

$$\underset{\dot{\mathbf{x}}}{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{A} \underset{\mathbf{x}}{\begin{pmatrix} x \\ y \end{pmatrix}} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

If $\mathbf{A}$ is 2x2 real matrix, then there exists $\mathbf{T}$ such that

$$\mathbf{AT} = \mathbf{TB} \;\; or \;\; \mathbf{T}^{-1}\mathbf{AT} = \mathbf{B}$$

where $\mathbf{T}$ is eigenvector matrix, and

$\mathbf{B}$ eigenvalue diagonal matrix.

Alternative Behaviors around Equil.Points

Then we can define $\mathbf{X} = \mathbf{TY}$. For $\dot{\mathbf{X}} = \mathbf{T}\dot{\mathbf{Y}}$, so

$$\dot{\mathbf{X}} = \mathbf{AX} \Rightarrow \mathbf{T}\dot{\mathbf{Y}} = \mathbf{ATY} \quad or$$

$$\dot{\mathbf{Y}} = \mathbf{T}^{-1}\mathbf{ATY} = \mathbf{BY}.$$

So, we can see the qualitative properties of solution of the system from Y which is only a linear tranformation of X.

Ex. $\dot{x} = 0 + 2y$

$$\dot{y} = x + y$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \ We\, have$$

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

where T is eigenvector matrix, and

B eigenvalue diagonal matrix.

Verify $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{B}$

Then we can define $\dot{\mathbf{Y}} = \mathbf{BY}$.

Alternative Behaviors around Equil.Points

B could take the following form:

(a) a sink: both neg real, all solution curves converges.

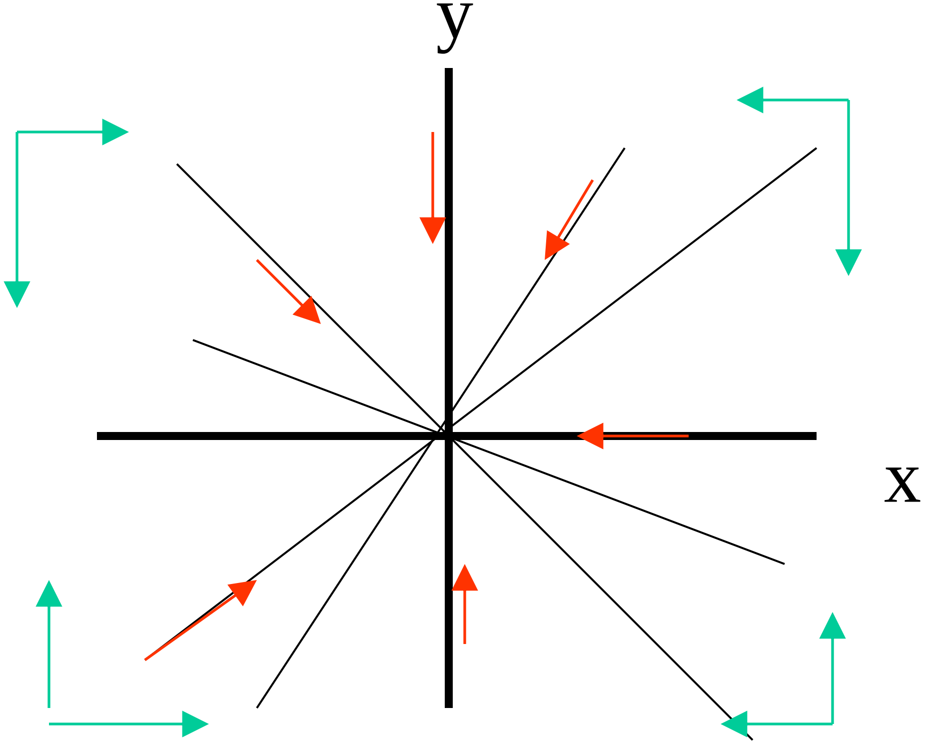
(b) a source: both pos real, diverges from the equil point.

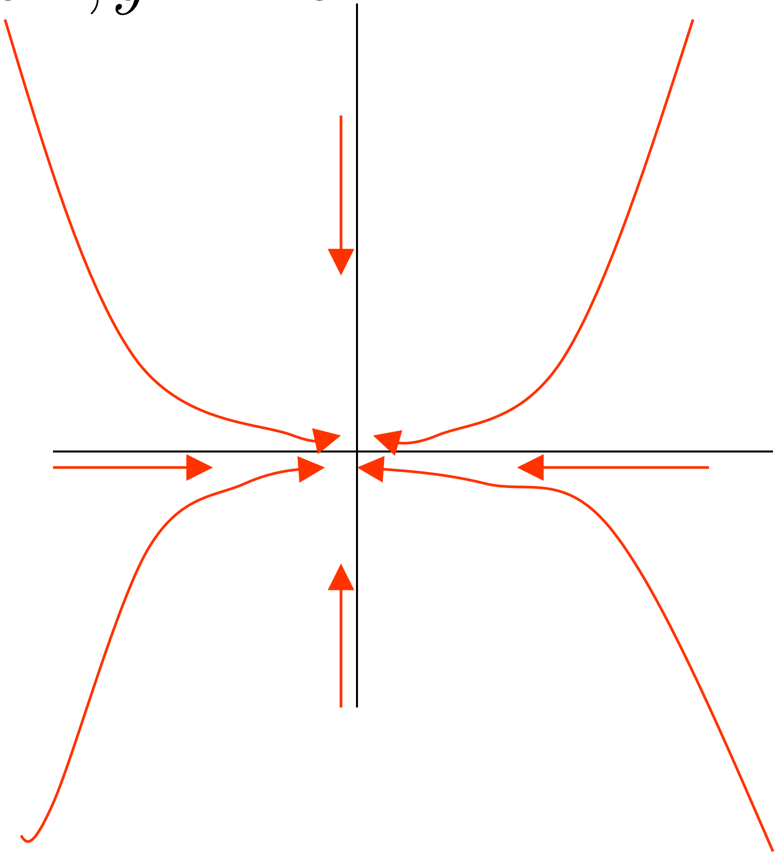
(c) a saddle: real with opposite signs.

(d) a centre: both imaginary, all solution curves are periodic with the same period. The curves are ellipses, or circles.

$$(a)\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \; if\, both\, are\, negative\, real\, parts, a\, SINK:$$

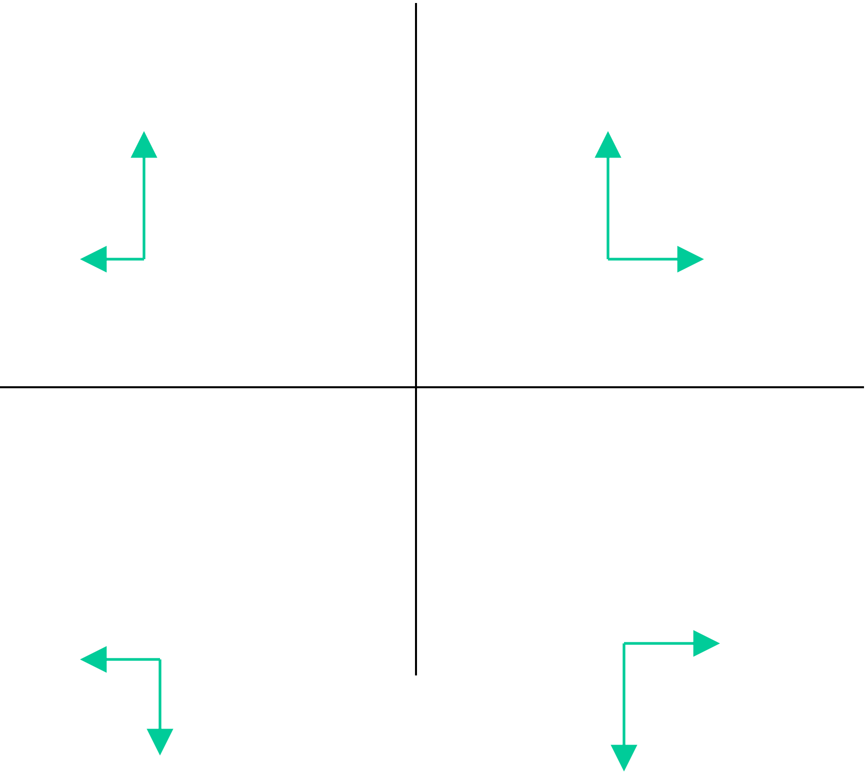
$$Ex.\, \dot{x} = -x, \dot{y} = -2y \Rightarrow x = Ae^{-t}, y = Be^{-2t}$$

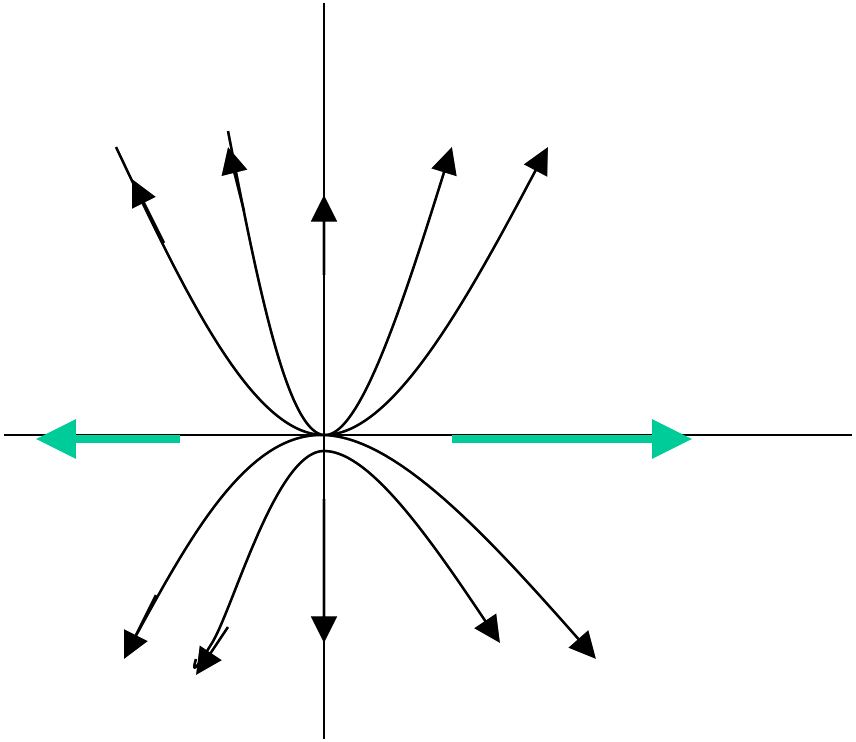




$$(b)\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\ if\ both\ are\ pos\ real,\ a\ SOURCE$$

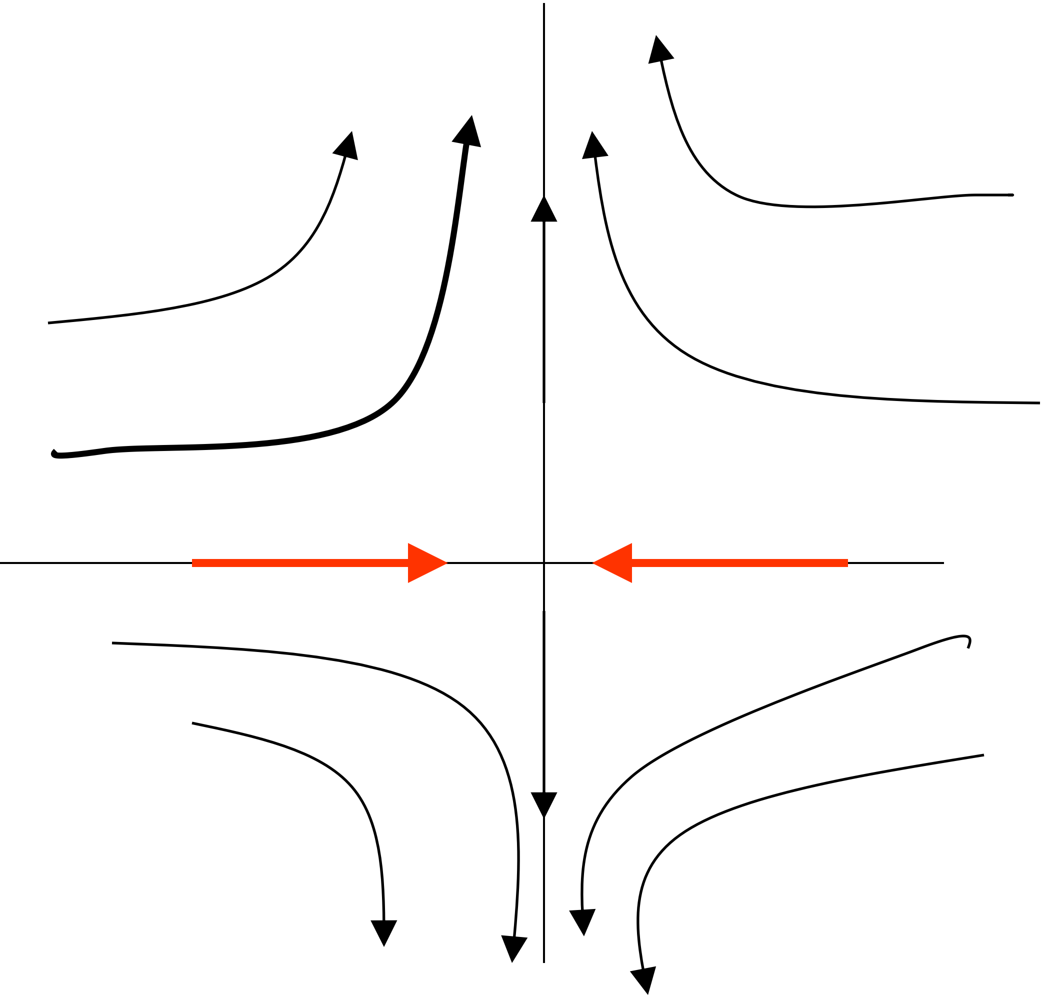
$$ex.\ \dot{x} = x, \dot{y} = 2y \Rightarrow x = Ae^{t}, y = Be^{2t}$$

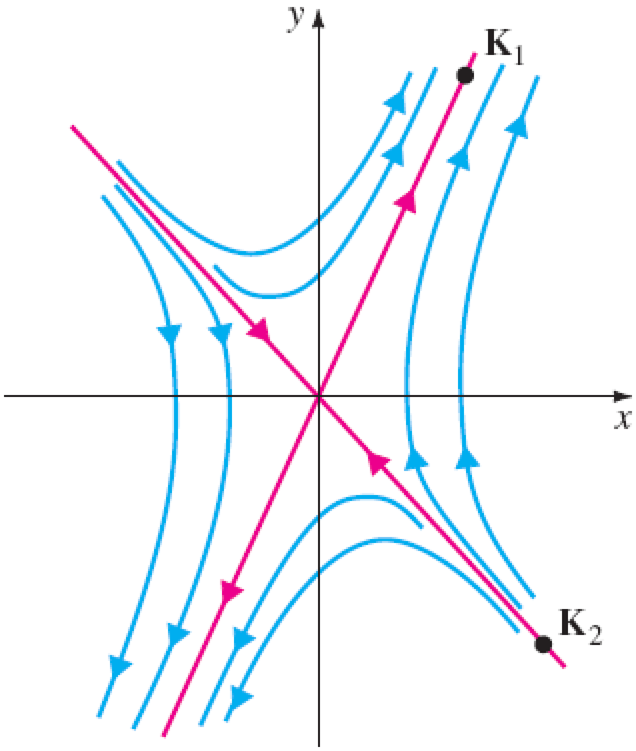




(c) *if both are real with opposite sign, a SADDLE.*

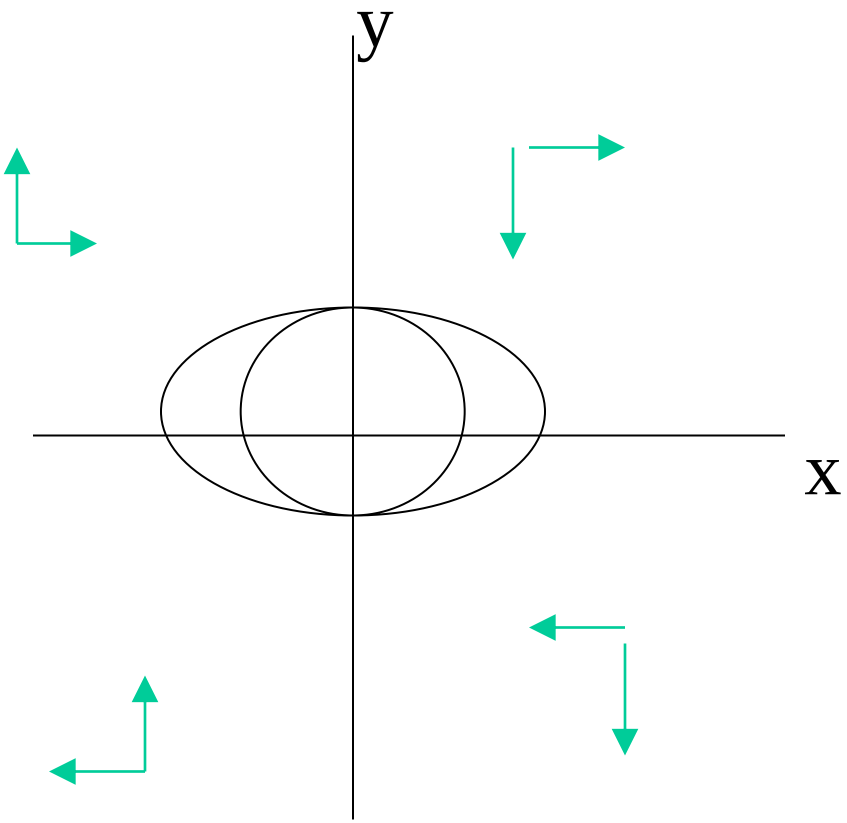
ex. $\dot{x} = -x, \dot{y} = y$, *a stable branch is* $x-$*axis.*

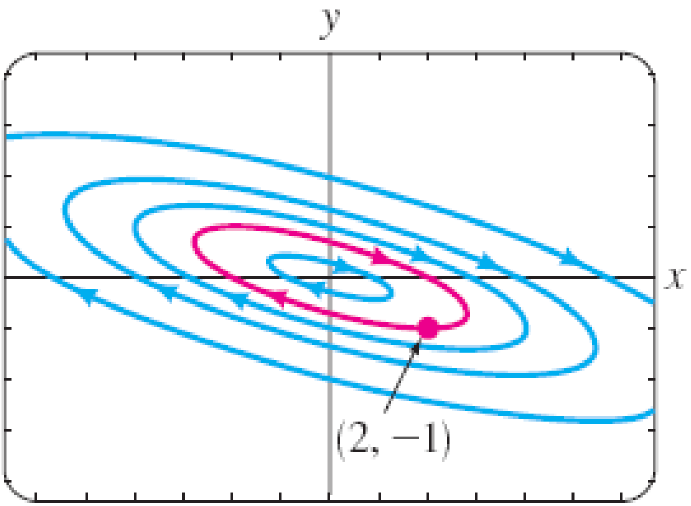




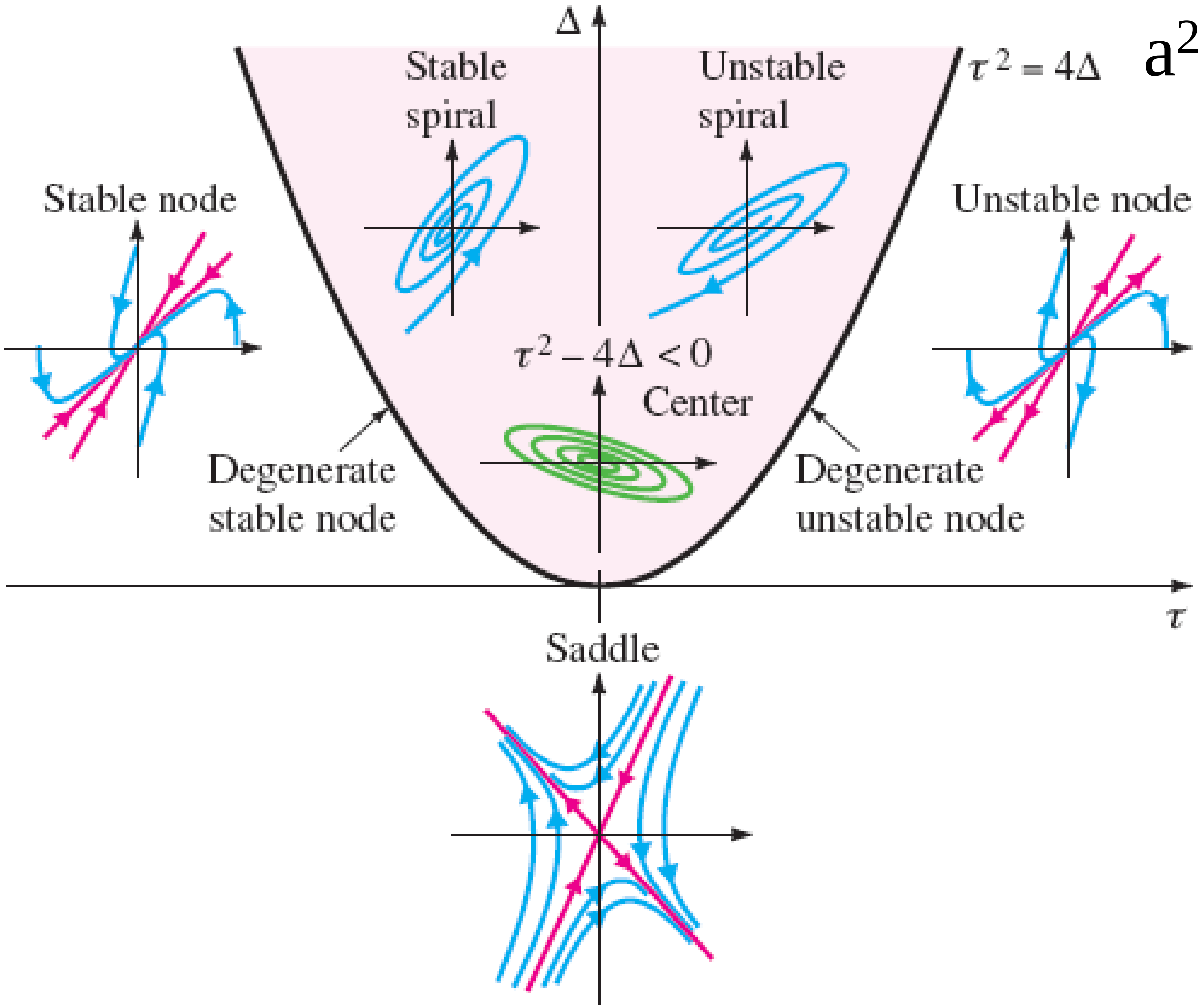
$$(d)\, if\, both\, are\, purely\, imaginary, a\, CENTER.$$

$$ex. \dot{x} = 4y, \dot{y} = -x => r = \pm 2i.$$





b


Δ
Stable spiral
Unstable spiral
τ 2 = 4Δ
Stable node
Unstable node
τ 2 – 4Δ < 0
Center
Degenerate stable node
Degenerate unstable node
τ
Saddle

$a^2=4b$

a

6.7 Phase Plane Analysis

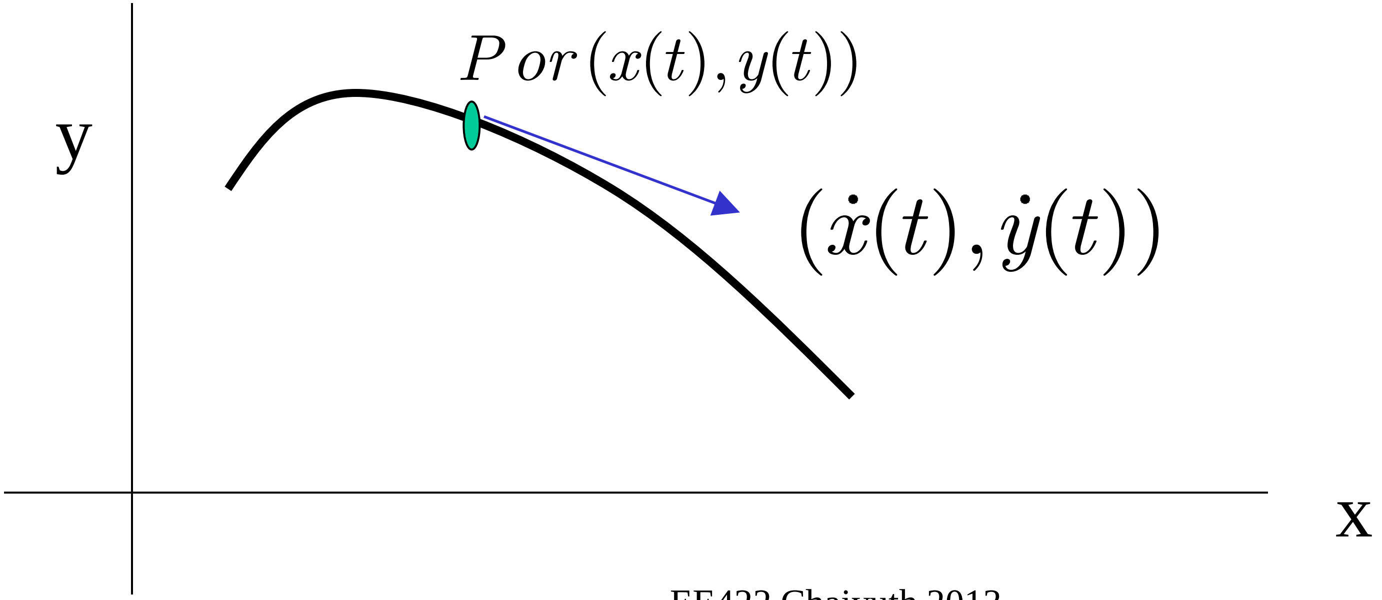
- When the explicit solutions are unavailable, we can find geometric arguments to shed light on the structure of the solutions of autonomous systems of differential equations.
- From topic 6.7, the autonomous system

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

- Solution (x(t), y(t)) describes a curve or path in x-y plane.
- Since xdot is given by f(.) and ydot by g(.), both not related to t. So, (xdot, ydot) is uniquely determined at the point (x(t), y(t)).

We can portrait the dynamics of the system by drawing a family of vectors, A Vector Field: We ask as t increases, how the system moves from the point P=(x(t), y(t)). Suppose f(P) >0, g(P)<0. This means that at P, x will increase and y will decrease. The direction is given by the tangent vector to the path at P, with speed given by the length of the vector.



Following the vector field and see if we can find a globally asymtotically stable point, all paths from this system tend to the equi. point.

A point (a,b) where f(a,b)=g(a,b)=0 is called an equilibrium point since $\dot{x}=\dot{y}=0$.

Next,we will show how to use the phase-plane diagram: partition regions where we know the direction of each variables, and see if certain equilibrium point is stable.

Drawing Phase Diagram:

First draw $\dot{x} = f(x,y) = 0, and\ \dot{y} = g(x,y) = 0$

$On\ \dot{x} = 0, draw \uparrow if\ \dot{y} > 0, \downarrow if\ \dot{y} < 0.$

(since x is not moved, we canfind areas where

y rises, and y falls)

$On\ \dot{y} = 0, draw \longrightarrow if\ \dot{x} > 0, \downarrow if\ \dot{x} < 0.$

(since y is not moved, we find areas where

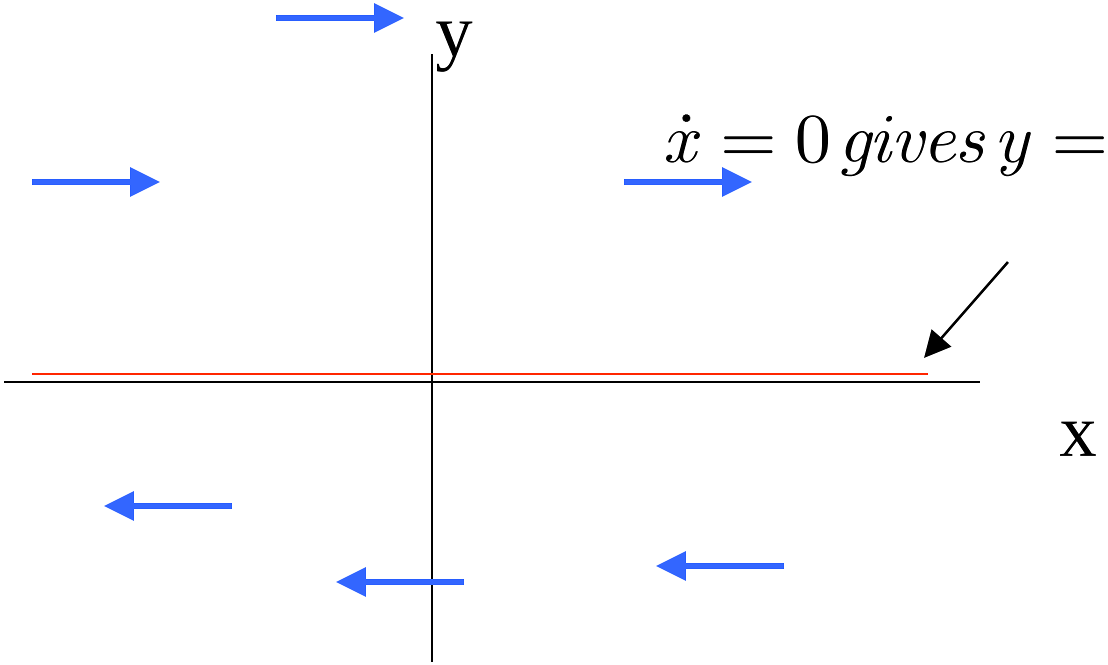
x rises, and x *falls*).

From $\dot{x}, \dot{y} = 0$, we form 4 areas.

$Ex.\, \dot{x} = y,\ \dot{y} = -2x - y;$

$In\, areas\, (I),(II), y > 0, so\, \dot{x} > 0; In\, areas (III),(IV),$

$y < 0, so\, \dot{x} < 0;$

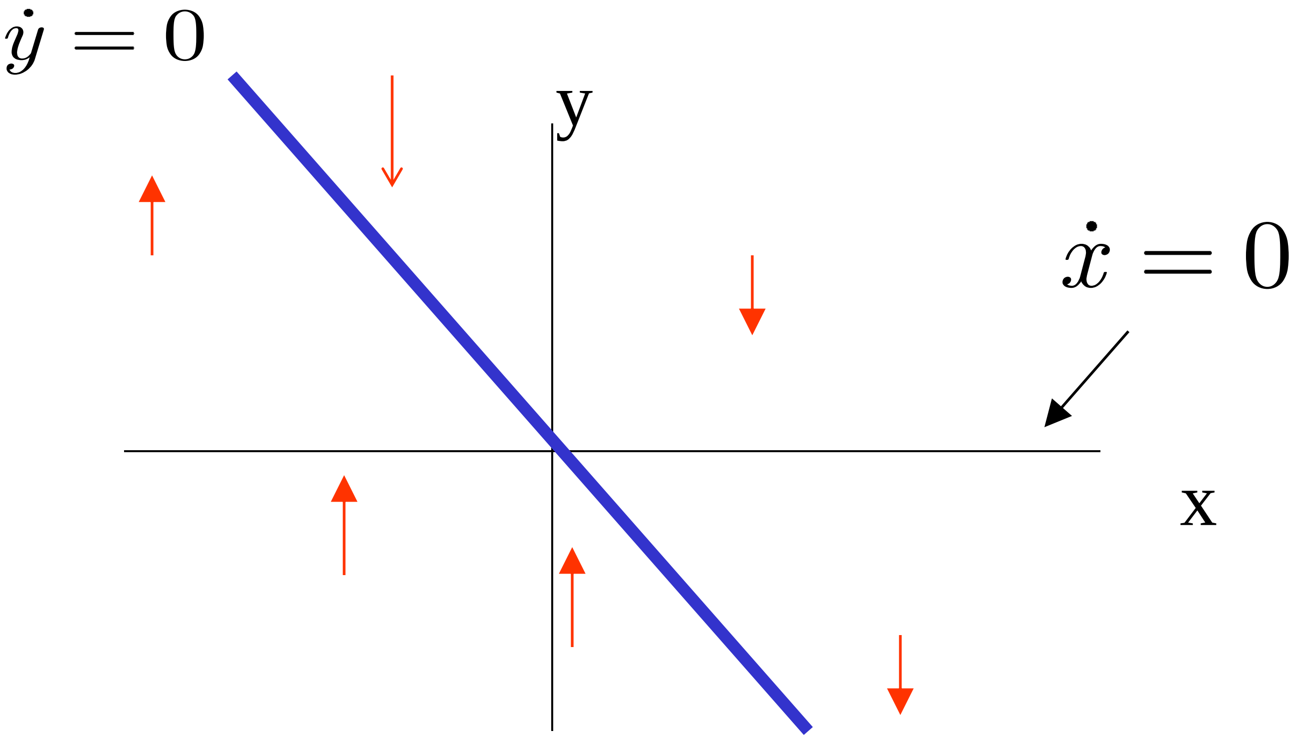


$\dot{x} = 0\, gives\, y = 0\, or\, x - axis.$

$$Ex.\,\dot{x} = y,\ \dot{y} = -2x - y;$$

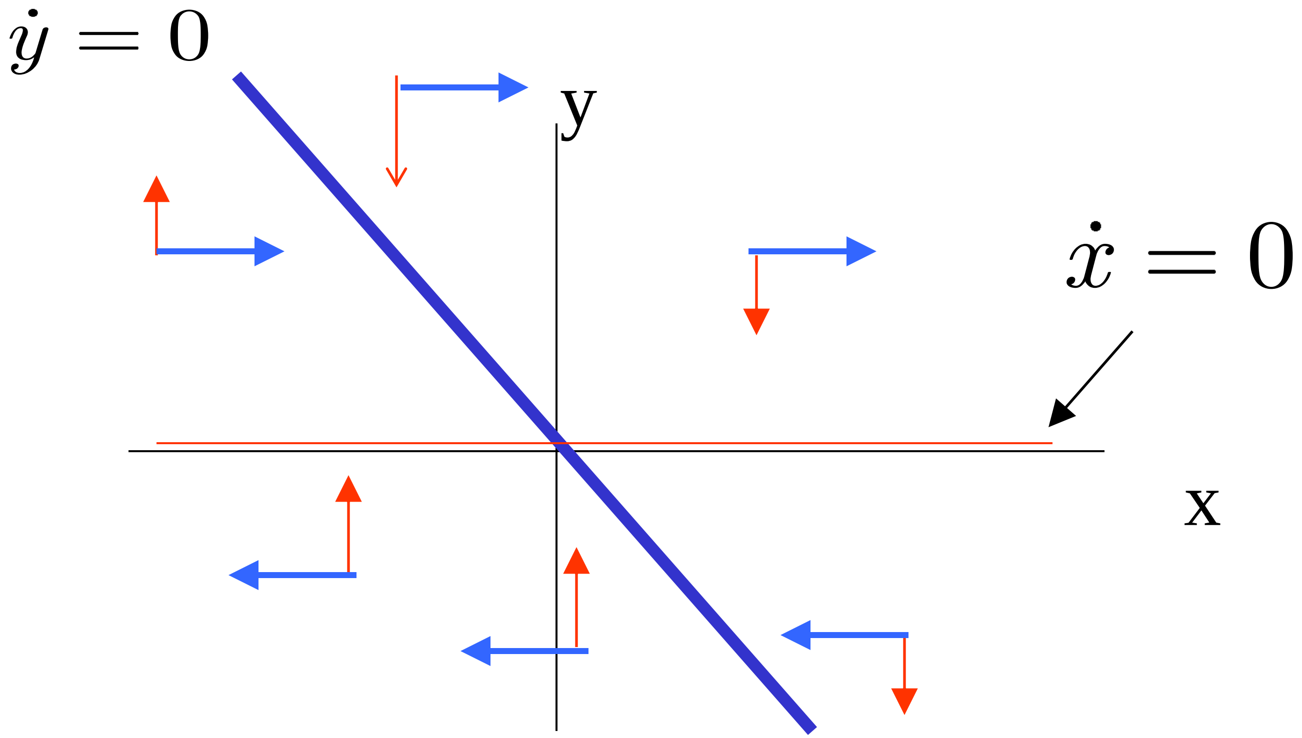
$$When\ \dot{y} = 0, we\ get\ y = -2x. Right\ area\ of\ \dot{y} = 0\ or$$

$$the\ linear\ line\ is\ \dot{y} < 0.$$



$$Ex.\, \dot{x} = y,\ \dot{y} = -2x - y;$$

$$Connect\, all\, arrows\, together.$$



$Ex2. Growth\ Model$

$$\dot{K} = aK - bK^2 - C;$$

$$\dot{C} = w(a - 2bK)C.$$

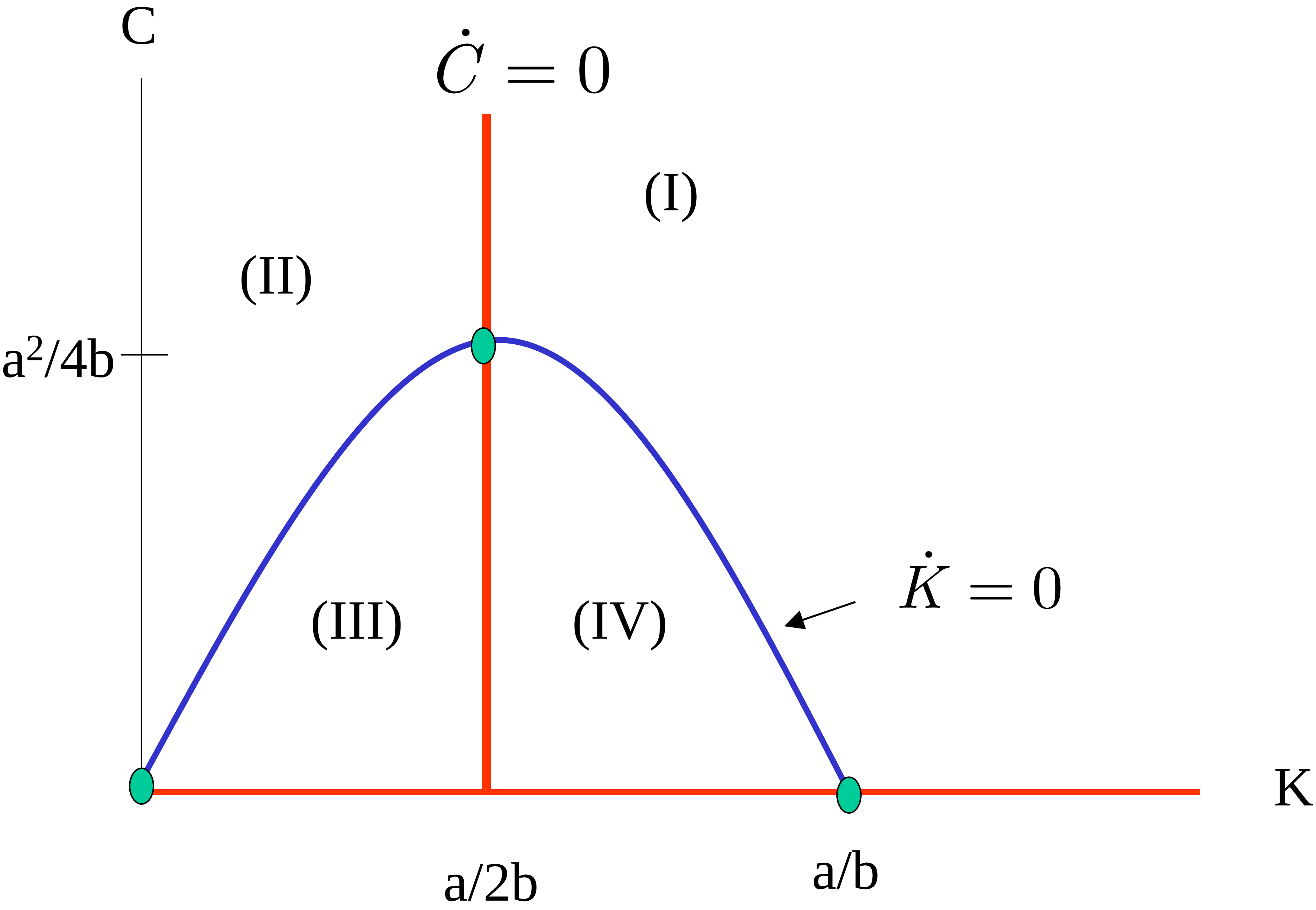
$Assume\ K \geq 0,\ C \geq 0.$

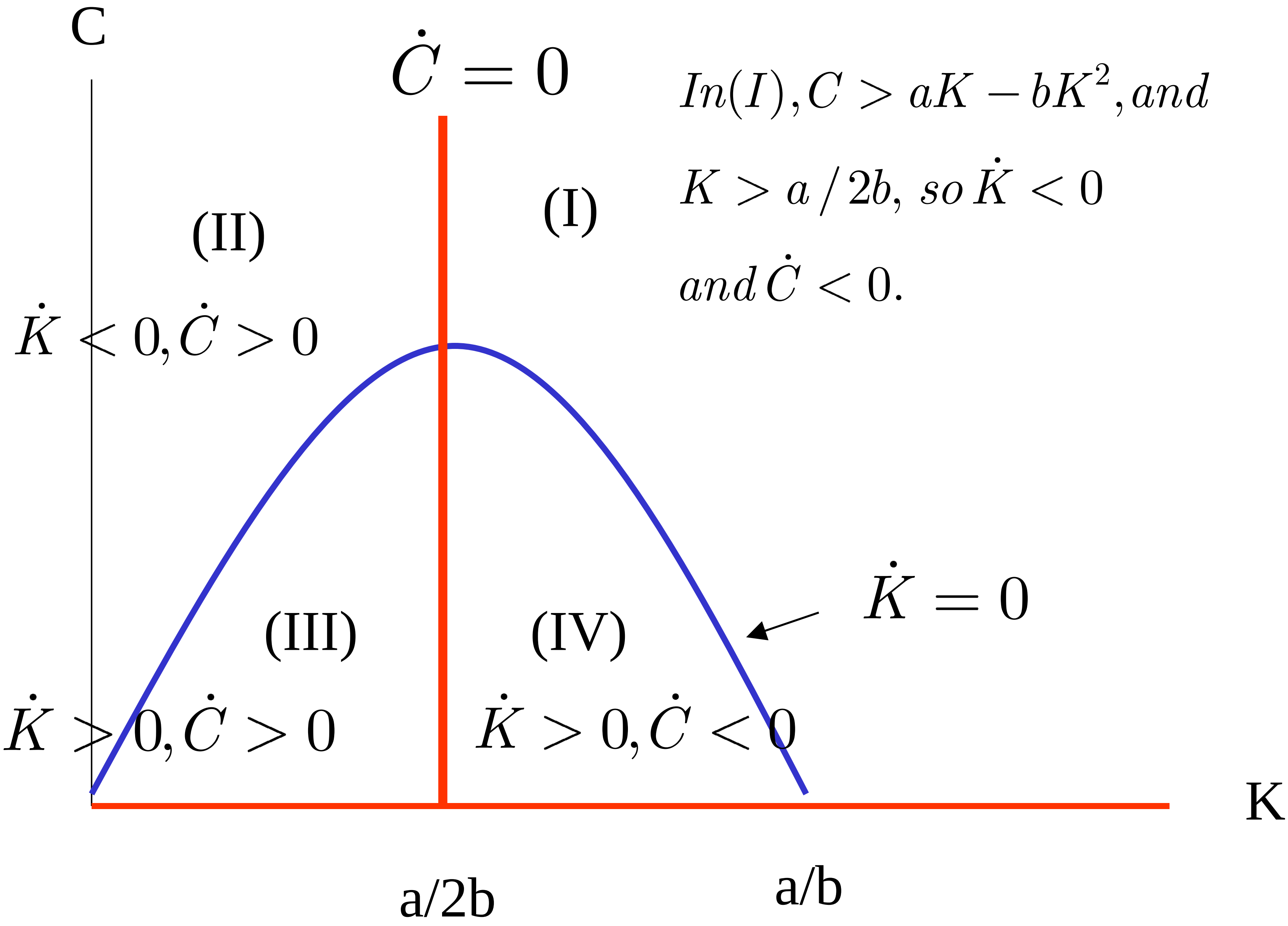
$First, draw\ the\ line\ \dot{K} = 0\ (the\ blue\ line\ where$

$C = aK - bK^2), and\ the\ lines\ \dot{C} = 0(two\ reds\ \ lines$

$C = 0, and\ K = a/2b.)$

Three quilibrium points where two lines intersect.

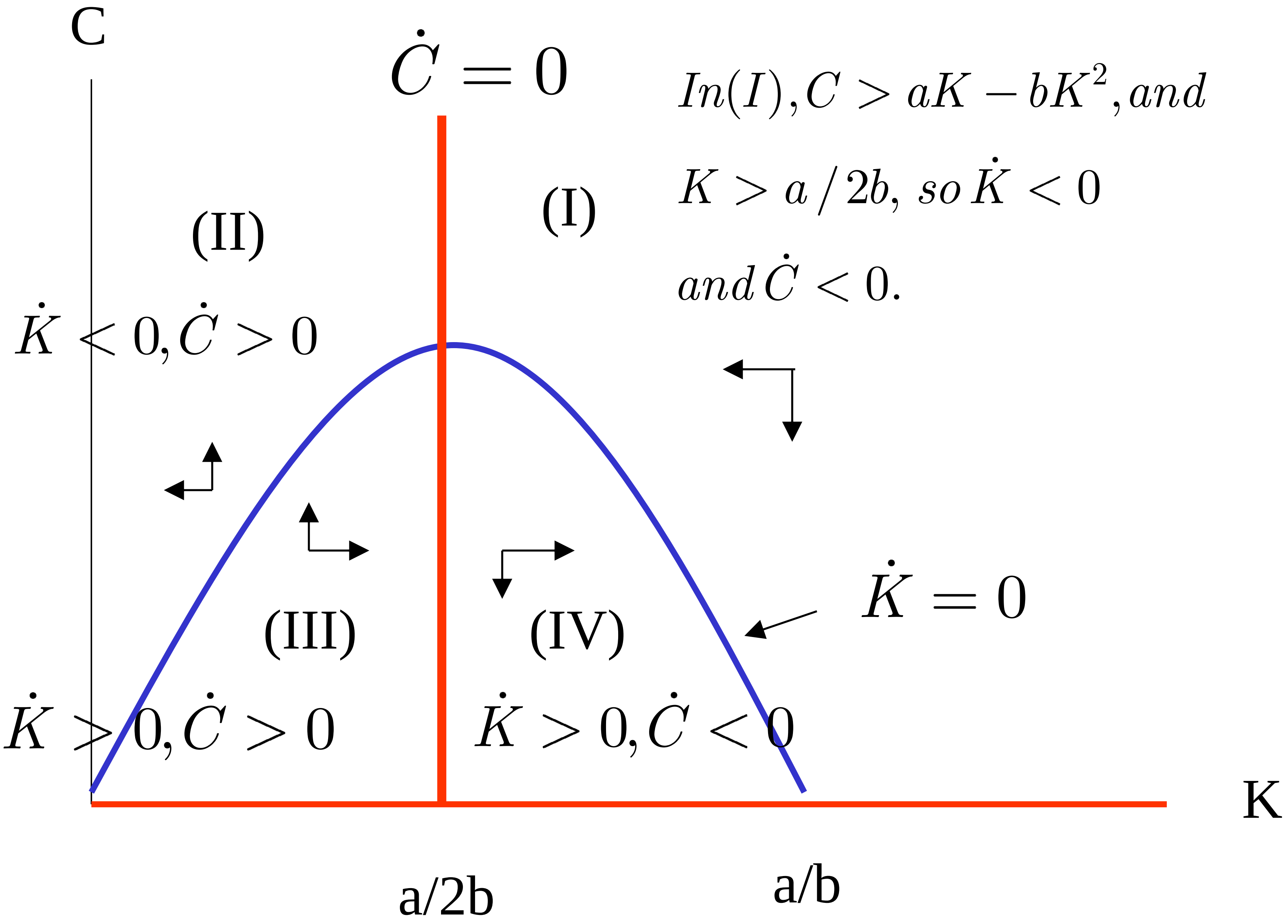

C
$\dot{C} = 0$
(I)
(II)
$a^2/4b$
(III)
(IV)
$\dot{K} = 0$
K
a/2b
a/b



$In(I), C > aK - bK^2, and$

$K > a/2b,\ so\ \dot{K} < 0$

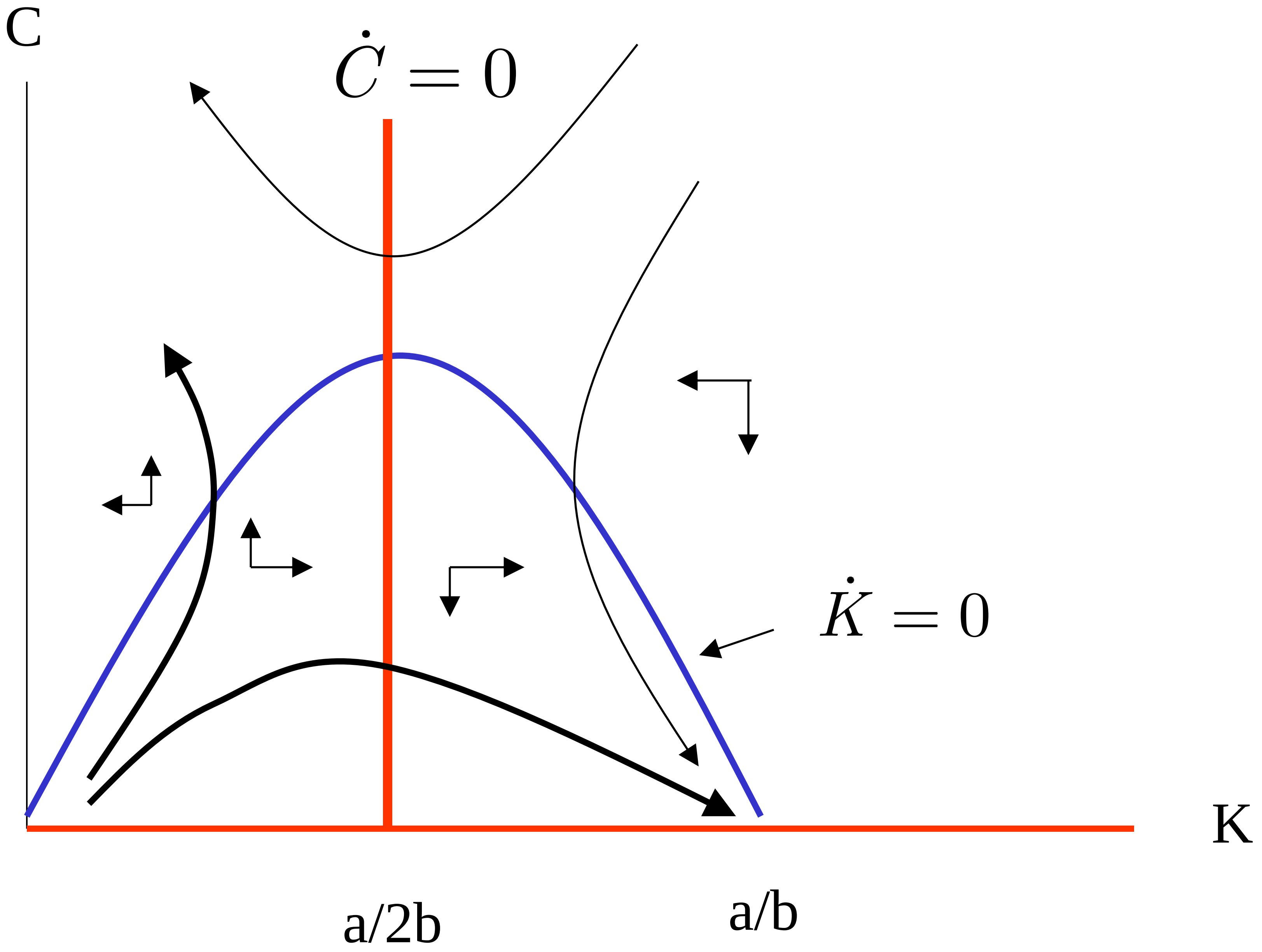
$and\ \dot{C} < 0.$



$In(I), C > aK - bK^2, and$

$K > a/2b,\ so\ \dot{K} < 0$

$and\ \dot{C} < 0.$


C
$\dot{C} = 0$
$\dot{K} = 0$
K
a/2b
a/b

$$ex.\,\dot{x} = -x + 2y - 1$$

$$\dot{y} = -3x - 8y + 25$$

$$We\,can\,find\,equilibrim\,at\,(3,2)$$

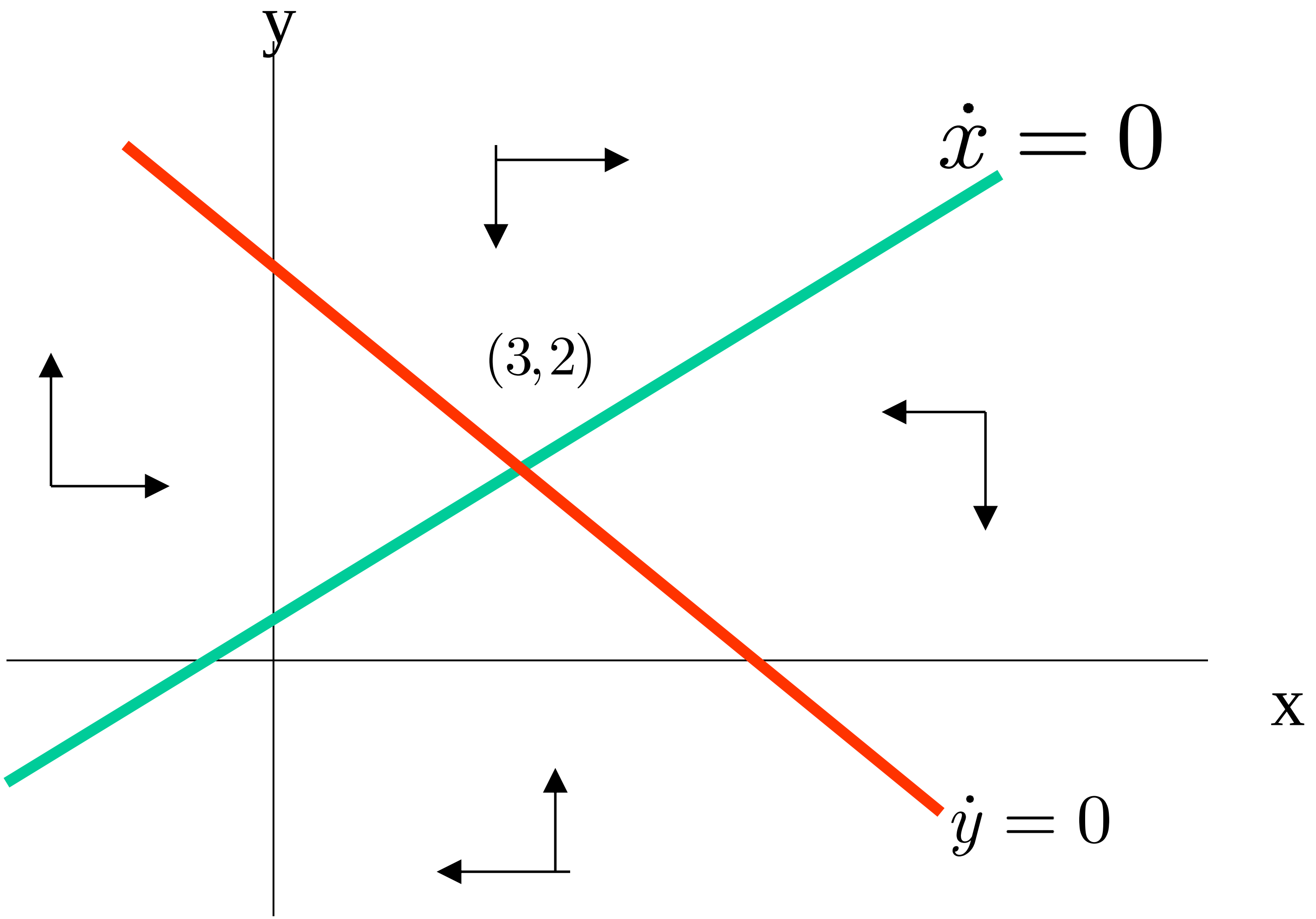
$$A = \begin{bmatrix} -1 & 2 \\ -3 & -8 \end{bmatrix}; |A - \lambda I| = (-1-\lambda)(-8-\lambda) + 6 = 0$$

$$\lambda_1 = -2,\; and\,the\,coressponding\,eigenvector$$

$$are\,AX = -2X\,yield - x + 2y = -2x\,or x = -2y$$

$$and\,\lambda_2 = -7, yield\,y = -3x. Thus\,the\,solution$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + Be^{-7t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$


y
$\dot{x} = 0$
$(3, 2)$
x
$\dot{y} = 0$

6.8 Stability for Nonlinear System

We discuss stability of the autonomous system.

An equi. point (a, b) is locally asym. stable if any path starting near (a, b) tends to (a, b) $as\, t \to \infty$.

An equi. point (a, b) is globally asym.stable if any solution converge to (a, b) $as\, t \to \infty$.

To see if (a, b) is locally asym. stable,we examine how solutions behave in a neighborhood of (a, b) by approximating $f\; and\; g$ about (a, b).

6.8 Stability for Nonlinear System

$$f(x,y) \simeq \underset{=0}{f(a,b)} + f_1'(a,b)(x-a) + f_2'(a,b)(y-b)$$

$$g(x,y) \simeq \underset{=0}{g(a,b)} + g_1'(a,b)(x-a) + g_2'(a,b)(y-b)$$

$$becomes\,(u\,\mathrm{s}\,e\,f(a,b) = 0\,to\,get\,b_1; g(a,b) = 0\,to\,get\,b_2)$$

$$\dot{x} = f(x,y) = f_1'(a,b)x + f_2'(a,b)y + b_1$$

$$\dot{y} = g(x,y) = g_1'(a,b)x + g_2'(a,b)y + b_2$$

$$around\,(a,b), this\,sytem\,behaves\,likes\,the\,linear$$

6.8 Stability for Nonlinear System

$$\dot{x} = f(x, y) = f_1'(a, b)x + f_2'(a, b)y + b_1$$
$$\dot{y} = g(x, y) = g_1'(a, b)x + g_2'(a, b)y + b_2$$

From theorem 6.1, this linear system is globally asym. stable iff the eigenvalues of the matrix A both have negative real parts.

Equivalenlty, iff A has negative trace $(\lambda_1 + \lambda_2)$ and positive determinant $(\lambda_1 \lambda_2)$.

Theorem 6.8.1 Lyapunov Stability

The system is locally asy. stable at (a, b) iff the eigenvalues of matrix A have both neg. real parts. Equivalently iff $trA < 0, and \left|A\right| > 0$, where the Jacobian matrix

$$A = \begin{bmatrix} f_1'(a,b) & f_2'(a,b) \\ g_1'(a,b) & g_2'(a,b) \end{bmatrix}.$$

Limiting behavior of nonlinear sytem near equil. point is similar to that of the linearlized system.

6.8 Stability for Nonlinear System

$$Ex.\,\dot{x} = -3x - 2y + 8x^2 + y^3$$

$$\dot{y} = 3x + y - 3x^2y^2 + y^4$$

$$check\, if\, (0,0)\, equil.\, is\,\, locally\, asym\, stable.$$

$$find\, f_1^{'}(0,0) = -3, \quad f_2^{'}(0,0) = -2$$

$$g_1^{'}(0,0) = 3, \qquad g_2^{'}(0,0) = 1.$$

$$trA = -3 + 1 = -2 < 0$$

$$\det A = -3 - (-6) = 3 > 0$$

6.8 Stability for Nonlinear System

$Ex.(population\, and\, growth)$

$$\dot{K} = sK^{1-\alpha} - \delta K = K(sK^{\alpha-1} - \delta)$$

$$\dot{P} = K^{\beta} - \gamma P \quad where\, \beta > 1$$

$where\, P\, is\, stock\, of\;\, pollution, decay with\, \gamma.\; Check$

$$(1)\, A = \begin{bmatrix} \delta(\alpha-1) & 0 \\ \beta\left(\dfrac{\delta}{s}\right)^{\frac{\beta-1}{\alpha-1}} & -\gamma \end{bmatrix} (2) trA < 0, |A| > 0$$

$$(3)\lambda_1 = \delta(\alpha-1),\; \lambda_2 = -\gamma.$$

6.8 Stability for Nonlinear System

We skip the theorem for globally stability of an autonomous system of differential equations in the plane. See Olech’s theorem in p. 251 in which we require to check for the whole domain in R2, not just only at the equilibrium point.

6.9 Saddle Points

Many dynamic economic models have equilibria that are not asymptotically stable.

In some cases a different type of behavior near equilibrium is seen.

Two paths approach the equilibrium points from opposite direction as t goes to infinity, while all other paths move away from the equilibrium point.

This is called “Local Saddle Point Equilibrium”

6.9 Saddle Points

Thm 6.9.1 Local Saddle Point

Suppose f *and* g *are* $c1$ functions, (a, b) be equilibrium of the system $\dot{x} = f(x, y)$ *and* $\dot{y} = g(x, y)$

Let A be Jacobian matrix

$$A = \begin{bmatrix} f_1'(a,b) & f_2'(a,b) \\ g_1'(a,b) & g_2'(a,b) \end{bmatrix}.$$

If eigenvalues of A are nonzero real with opposite signs, or, equivalently det A<0.

Such equilibrium is saddle point.

6.9 Saddle Points

Thm 6.9.1 Local Saddle Point

For any starting point t_0, there exist exactly two solution paths $(x_1(t), y_1(t))$ *and* $(x_2(t), y_2(t))$ that converge toward (a, b) from opposite directions in the phase-plane.

6.9 Saddle Points

Note

$\text{The characteristic equation of } A \text{ is}$

$$r^2 - tr(A) + \det(A) = 0.$$

$\text{The quadratic equation } r^2 + ar + b = 0$

$$\text{has two roots, } r_{1,2} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b.}$$

$\text{If } b < 0, \text{both roots are real, also } r_1 r_2 = b < 0,$

$\text{so the roots are of opposite signs.}$

$If\ r_1\ and\ r_2 \text{ are real and opposite signs,}$

$\text{then } b = r_1 r_2 < 0.$

6.9 Saddle Points

$\mathrm{Ex.\,Growth\,Model\,(see\,section\,6.7)}$

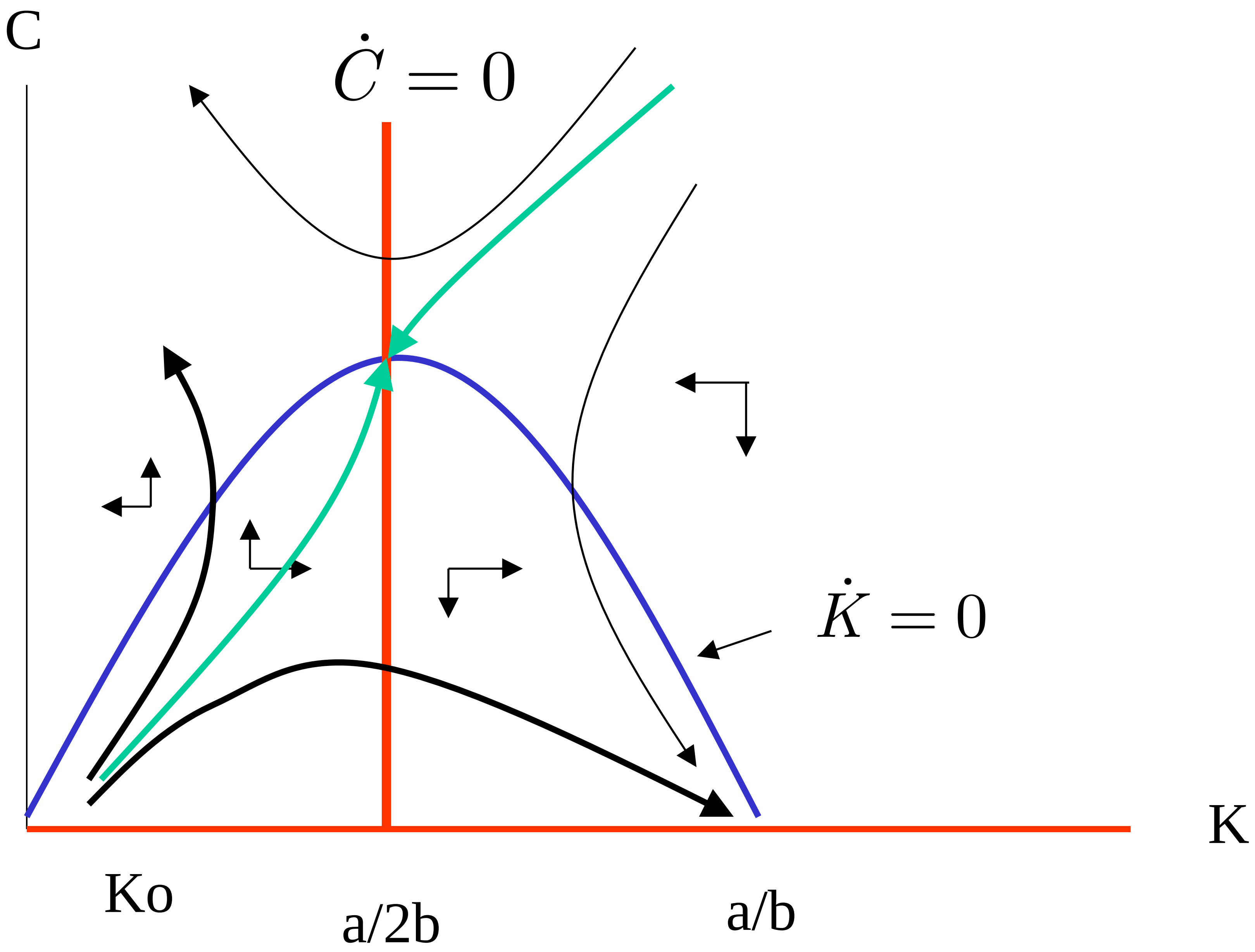
$$\dot{K} = aK - bK^2 - C$$

$$and \quad \dot{C} = w(a - 2bK)C$$

$\mathrm{One\,equil.\,point}\, is\,(K^*, c^*) = (a / 2b, a^2 / 4b)$

$\mathrm{Check\ \ if\,det\,A\,at\ \ this\,point} < 0.$

$\mathrm{If\,so,\,such\,equilibrium\,is\,saddle\,point.}$


C
$\dot{C} = 0$
$\dot{K} = 0$
K
Ko
a/2b
a/b

ex. Check the equil property of the following system

$$\dot{x} = 2y$$

$$\dot{y} = 3x - y$$

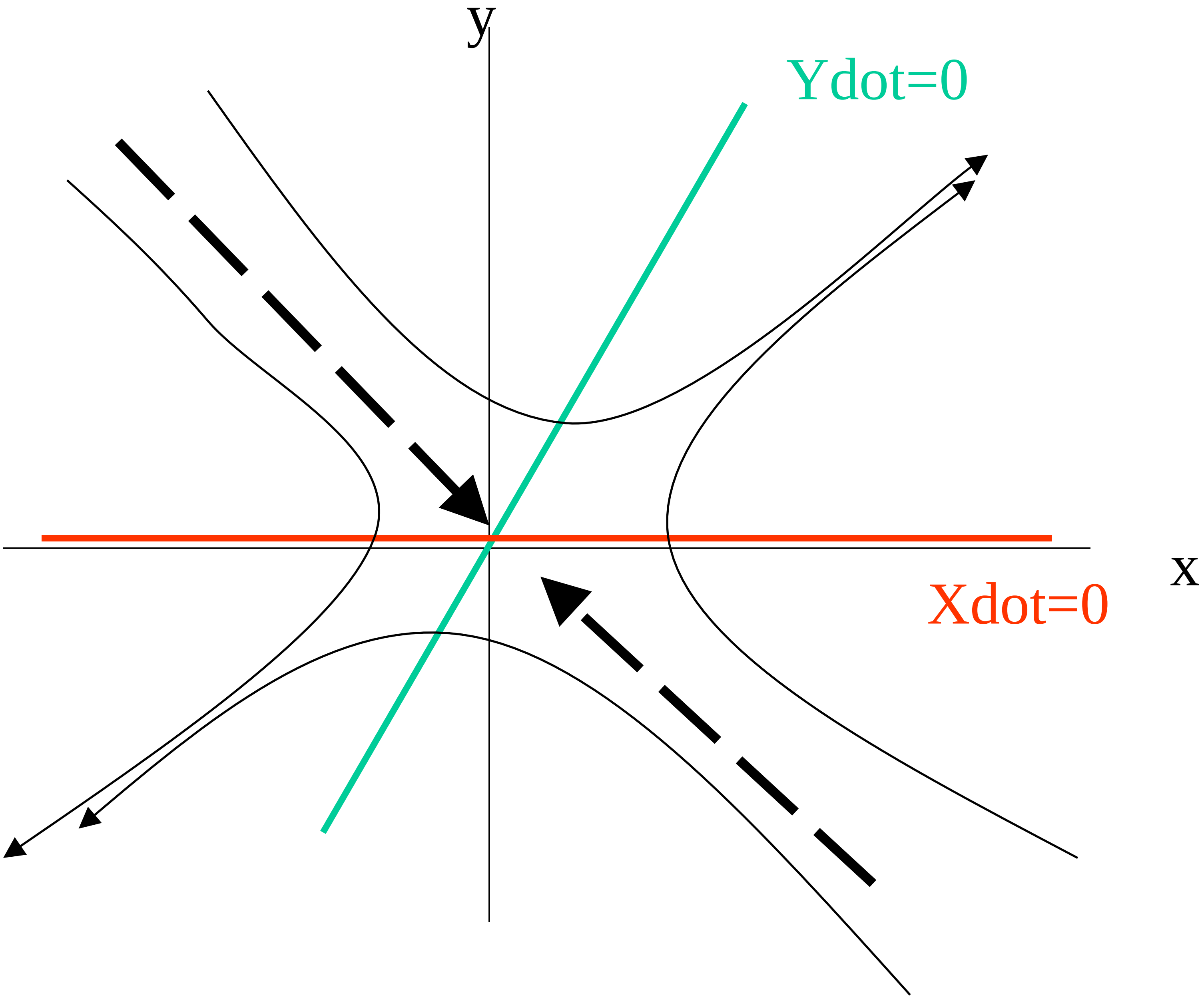
We can find equilibrim at $(0,0)$

$$A = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix};\ \det(A) = -6$$

$$|A - \lambda I| = (0 - \lambda)(-1 - \lambda) - 6 = 0$$

$$\lambda^2 + \lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda + 3)$$

For $\lambda_2 = -3$, *yield associated eigenvector* $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$


y
Ydot=0
x
Xdot=0