Recall:

Statement:

In Mathematics we are constantly dealing with statements. By a statement we mean a sentence that is true or false, but not both.

Truth Values:

Every statement has a truth value, namely true (denoted by T) or false (denoted by F). We often use P or Q to denote statements, or perhaps $P_1, P_2, ..., P_n$ if there are several statements involved.

Compound statement:

A compound statement is a statement that can be expressed in terms of one or more (simpler) statements.

The Negation of a Statement:

The negation of a statement P is the statement "Not P" and is denoted by "$\sim$P".

The Disjunction of Statements:

The disjunction of the statements P and Q is the statement "P or Q" and is denoted by $P \vee Q$.

The conjunction of Statements:

The conjunction of the statements P and Q is the statement "P and Q" and is denoted by $P \wedge Q$.

The Implication:

For statements P and Q, the implication is the statement "If P, then Q" and is denoted by $P \rightarrow Q$.

Section 1.5 Quantified Statements and Their Negations

- **Symbols**

$\sim$ negation (not)

$\vee$ disjunction (or)

$\wedge$ conjunction (and)

$\rightarrow$ implication

$\leftrightarrow$ equivalence

$\forall$ universal quantifier (for every, for all)

$\exists$ existential quantifier (there exists, for some)

Each of the phrases "for every, "for each", and "for all" is referred to as a **universal quantifier** and is commonly expressed by the symbol $\forall$. For example, "For every real number *x*, its square $x^2 \geq 0$." can be written in symbols as

$$\forall x \in R, x^2 \geq 0.$$

In fact, if we define the open sentence $p(x)$ by $p(x): x^2 \geq 0$, then we can write statement $\forall x \in R, x^2 \geq 0$ as $\forall x \in R, p(x)$. We are saying then that for every element x in the set of real numbers (the universal set in this case), $p(x)$ holds.

Suppose, however, that we were to let $q(x): x^2 \leq 0$. The statement $\forall x \in R, q(x)$ (that is, for every real number *x*, its square $x^2 \leq 0$) is false. Of course this means that its negation is true. If it were not the case that for every real number *x*, its square $x^2 \leq 0$, then there must exist some real number *x* such that $x^2 > 0$. The phase "there exists" or "there is" is called an **existential quantifier** and is denoted in symbols by $\exists$. Hence this statement can be written as

$$\exists x \in R, x^2 > 0 \quad \text{or} \quad \exists x \in R, \sim q(x)$$

Therefore, if we are considering some universal set *S* and if $p(x)$ is an open sentence concerning some element $x \in S$, then

$$\sim (\forall x \in S, p(x)) \text{ is logical equivalent to } \exists x \in S, \sim p(x).$$

Example 1: Find the truth value of the following statements.

1) $\forall x \in N[x-1=3] \vee \exists x \in N[x+2=0]$

2) $\exists x[x^2=-1] \rightarrow \exists x[x-1>2]$

3) $\forall x[x \neq 0 \rightarrow x^2 \neq 0] \wedge \exists x \in I[2x=1]$

4) $\exists x \in N[x+1=0] \leftrightarrow \exists x[x^2<0]$

Theorem: For all statements *p*, *q*, and *r*, the following statements are tautologies.

1. Law of Identity:

$$p \leftrightarrow p$$

2. Law of Double Negation:

$$p \leftrightarrow \sim(\sim p)$$

3. Law of Excluded Middle:

$$p \vee \sim p$$

4. Law of Contradiction:

$$\sim(p \wedge \sim p)$$

5. Idempotent Laws:

$$(p \wedge p) \leftrightarrow p$$

$$(p \vee p) \leftrightarrow p$$

6. Law of Addition:

$$p \rightarrow (p \vee q)$$

7. Law of Equivalence:

$$[p \leftrightarrow q] \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$$

8. Law of Contraposition:

$$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$$

9. Law of (Hypothetical) Syllogism:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow [p \rightarrow r]$$

10. Commutative Laws:

$$(p \wedge q) \leftrightarrow (q \wedge p)$$

$$(p \vee q) \leftrightarrow (q \vee p)$$

11. Associative Laws:

$$[p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r] \leftrightarrow [p \wedge q \wedge r]$$

$$[p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r] \leftrightarrow [p \vee q \vee r]$$

12. Distributive Laws:

$$\left[p \wedge (q \vee r)\right] \leftrightarrow \left[(p \wedge q) \vee (p \wedge r)\right]$$

$$\left[p \vee (q \wedge r)\right] \leftrightarrow \left[(p \vee q) \wedge (p \vee r)\right]$$

13. Absorption Laws:

$$\left[p \wedge (p \vee q)\right] \leftrightarrow p$$

$$\left[p \vee (p \wedge q)\right] \leftrightarrow p$$

14. De Morgan's Laws:

$$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$$

$$\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$$

15. Law of Implication:

$$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$$

$$(p \vee q) \leftrightarrow (\sim p \rightarrow q)$$

16. Law of Negation for Implication:

$$\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$$

Example 2: Show that $\left[\sim (\sim p \vee q) \rightarrow p\right]$ is tautology.

Example 3: Show that $(p \to q) \wedge p \wedge \sim q$ is contradiction.

Example 4: Find the negation of the following statements.

1) $(\sim r \to t) \wedge (s \vee t)$
2) $\sim (q \vee \sim p) \to (r \wedge \sim t)$
3) $(\sim x \to y) \vee (\sim y \wedge \sim z)$