Assignment 5

1. Let $A = \{1, 2, 3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A. Define a relation r from $\mathcal{P}(A)$ to $\mathbb{Z}$ as

$$r = \{(x, y) \in \mathcal{P}(A) \times \mathbb{Z} | y = \text{ the number of elements in } x\}.$$

   (a) List all the elements in r.
   (b) Is r a function? If so, find the domain, co-domain, and range of r.

2. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define a function $f : X \to Y$ by $f = \{(1, a), (2, a), (3, c)\}$.

   (a) Find the domain of f, co-domain of f,and range of f.
   (b) What is the inverse image of a?
   (c) What is $f(2)$?
   (d) Draw the arrow diagram of f.

3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (2x + 1, \tfrac{1-y}{2}) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

   (a) Is H one-to-one? Prove or give a counterexample.
   (b) Is H onto? Prove or give a counterexample.
   (c) Is H bijective? If so, find H^{-1}, the inverse function of H.

4. Let $f : (0, \infty) \to \mathbb{R}$ defined by

$$f = \begin{cases} |x|, & x \in (0, 1] \\ x^2 & x \in (1, 3] \\ 6 + |x|, & x \in (3, \infty). \end{cases}$$

   (a) Is f one-to-one? Prove or give a counterexample.
   (b) Is f onto? Prove or give a counterexample.
   (c) Is f bijective? If so, find the inverse function of f.

5. Find the largest sets D and S such that the function $f : D \to S$ defined by $f(x) = \frac{2}{x-1}$ is an onto function.

6. Let $f : \mathbb{R} - \{1\} \to S$ be a *bijective* function defined by $f(x) = \frac{x+1}{x-1}$.

   (a) Determine the set S.
   (b) Determine the inverse function f^{-1}.
   (c) Compute $f \circ f$ and $f \circ f^{-1}$.

7. Graph the function $f : \mathbb{R} \to \mathbb{Z}^+ \cup \{0\}$, $f(x) = \lfloor x \rfloor - \lceil x \rceil$ on the closed interval $[-4, 4]$.

8. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f : X \to Z$ is onto, must both f and g be onto? Prove or give a counterexample.

9. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}$, $\quad g(x) = x^3 + 1$.

   (b) $f(x) = x^5$, $\quad g(x) = x^{1/5}$.

10. Let f and g be functions defined by $f(x) = \sqrt{2-x}, x \geq 2$ and

$$g = \begin{cases} x^2 - 1, & x \in (-\infty, -1] \\ \frac{1}{x}, & x \in (-1, \infty). \end{cases}$$

   Find $g \circ f$ and its domain.