

ENDOGENOUS GROWTH MODELS

EE 462 Development Macroeconomics

Topics

- Transitional Dynamics in Solow Model
- The AK Model
- The Lucas Model
- Romer's Model

Transitional Dynamics (Solow Model)

- Recall from Solow model, capital accumulation can be written as:

$$\Delta k = sy - (n+d)k$$

- Rewrite in terms of the rate of change in k :

$$\Delta k/k = sy/k - (n+d)$$

- Given the production function: $y = f(k) = k^\alpha$. Then,

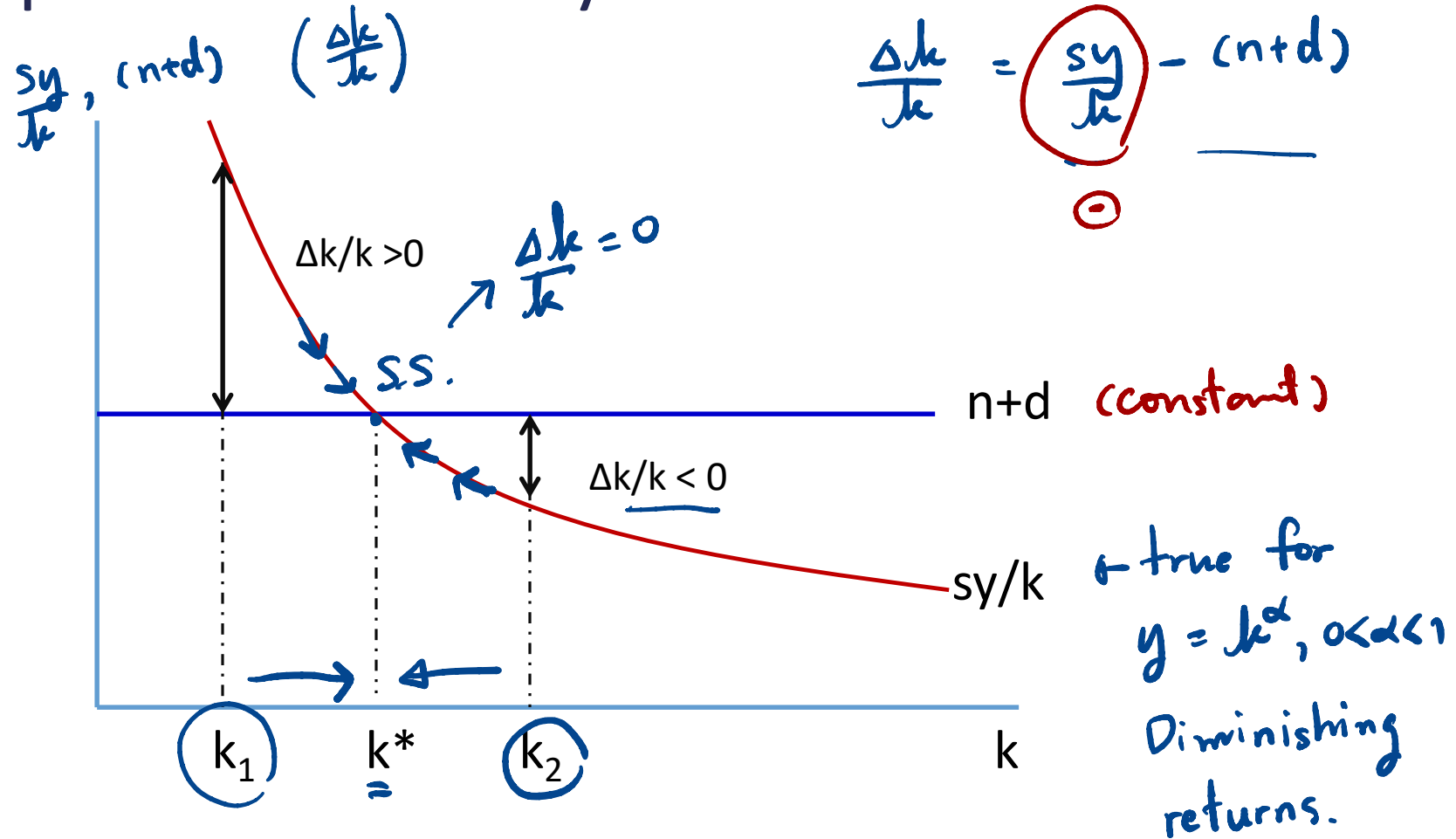


$$\Delta k/k = sk^{\alpha-1} - (n+d)$$

$(0 < \alpha < 1)$
At s.s., $\frac{\Delta k}{k} = 0$

- Countries with high levels of k ($k > k_{ss}$) will have negative growth rates of k .
- Countries with low levels of k ($k < k_{ss}$) will have a positive growth rates of k .

Graph: Transitional Dynamics in Solow Model



Endogenous Growth Theory

- The *endogenous growth theory* or *new growth theory* focuses on understanding the economic forces underlying technological progress.
- By endogenous growth, it means persistent income growth that is determined by the system governing the production process rather than forces outside that system.
- The main differences from neoclassical models:
 - Discard the assumption of diminishing marginal returns to capital
 - Allow increasing returns to scale in aggregate production
 - Try to explain the divergent long-term growth patterns among countries
 - Often focus on the role of externalities and human capital

$$Y = AK^{\alpha}L^{\beta}$$

$\alpha + \beta > 1$

The AK Model

Before
 $y = AK^\alpha L^\beta$, $\alpha + \beta = 1$.

- Assume a production function without diminishing returns:

$$Y = AK$$

where $A > 0$ (A is a constant.)

- Output per capita can be written as:

$$y = Ak$$

$$\ln y = \ln A + \ln k \rightarrow \frac{1}{y} dy = \frac{d(\ln A)}{\ln A} + \frac{dk}{k}$$

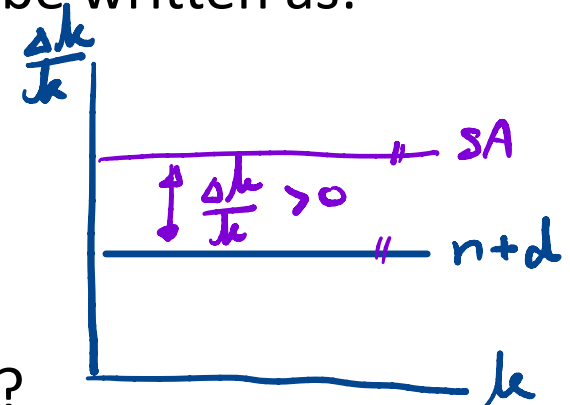
where $y = Y/L$ and $k = K/L$

s, n, d are constants.

- The accumulation of capital per worker can be written as:

$$\rightarrow \Delta k/k = sA - (n+d) > 0$$

$$\rightarrow \Delta k = sAk - (n+d)k$$



- There is no convergence in the long run.

i.e. There's no steady state.

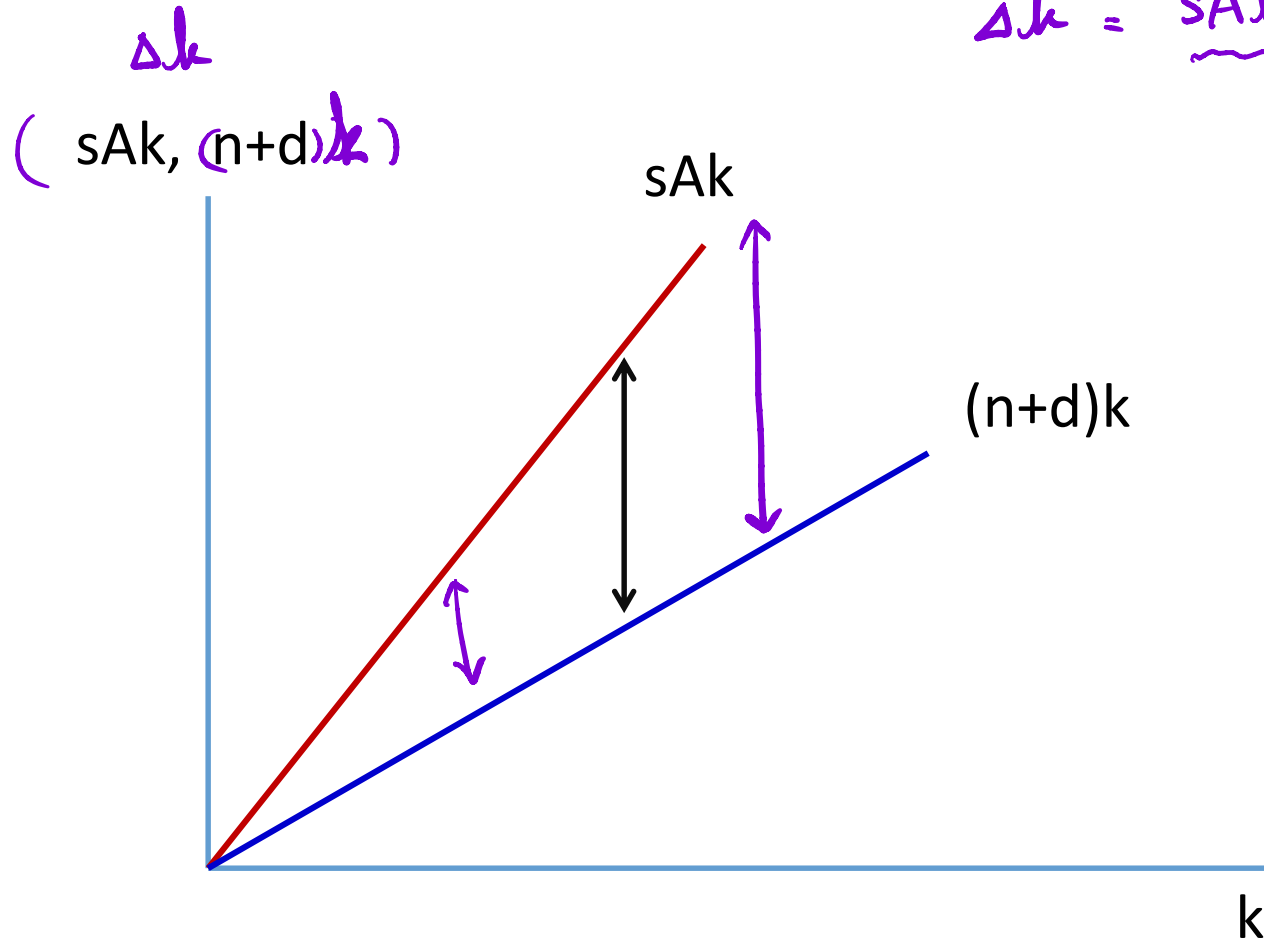
- What's the growth rate in per capita output?

$$g_y = \frac{\Delta y}{y} \approx \frac{dy}{y} = \frac{dk}{k} \approx \frac{\Delta k}{k} = sA - (n+d)$$

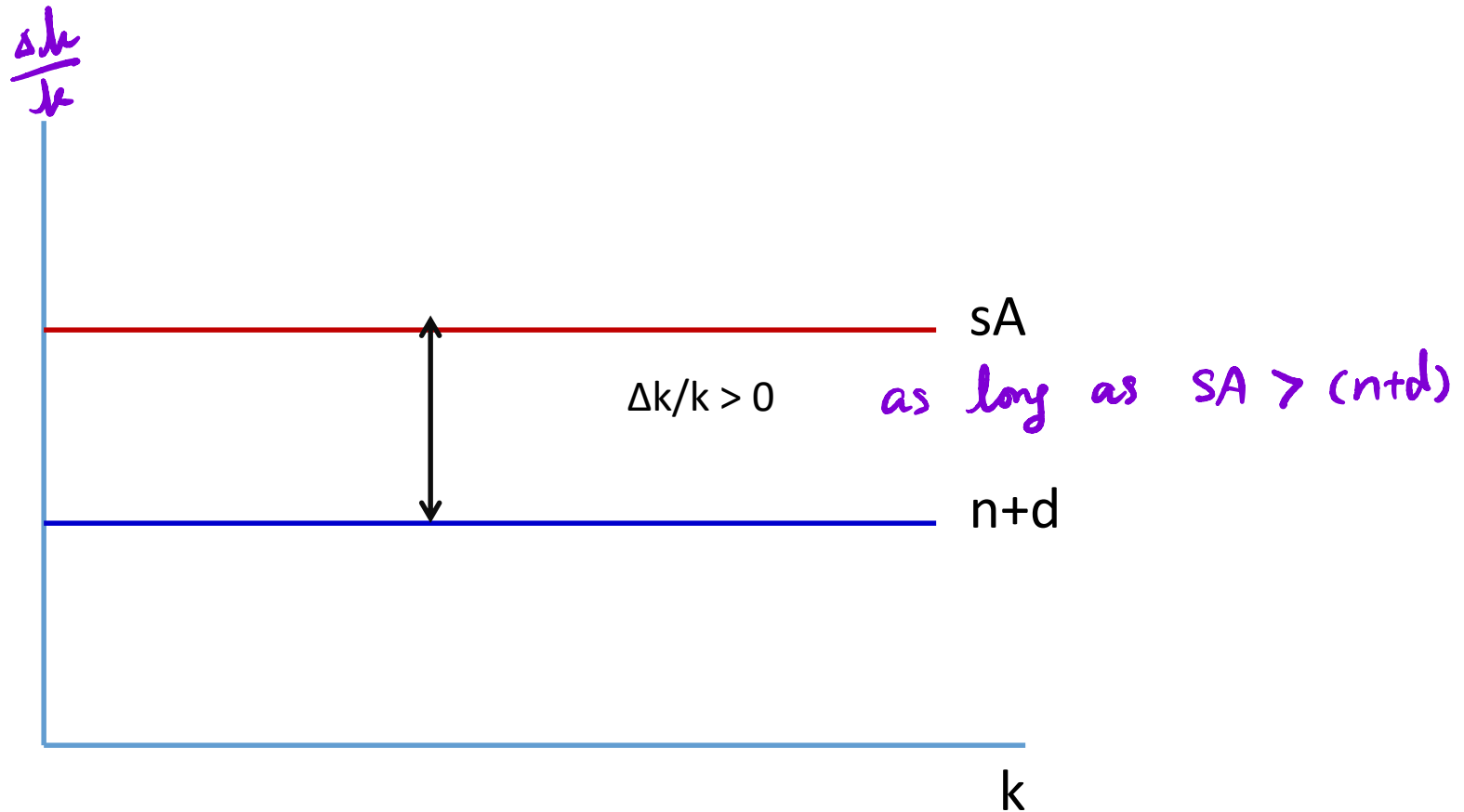
Graph: The AK Model

$$y = Ak$$

$$\Delta k = \underbrace{sAk} - \underbrace{(n+d)k}$$



Transitional Dynamics: AK Model



The Lucas (1988) Model

- similar to Solow Model with technological progress.

- Lucas (JME, 1988) defines human capital as the skill embodied in workers.
- The production function is given by:

$$Y = AK^\alpha(uhN)^{1-\alpha} = AK^\alpha(L)^{1-\alpha}$$

where N = total workers

h = stock of human capital (of each worker)

u = proportion of total labor spent on producing output

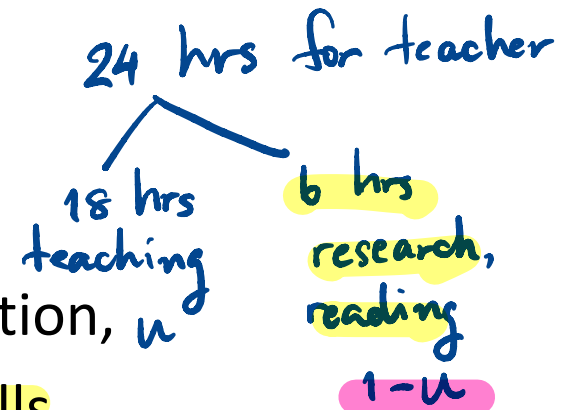
$L = uhN$ = effective worker

- Human capital grows at a constant rate:

$$\Delta h/h = a(1-u)$$

Where a reflects quality or efficiency of education, u

$(1-u)$ reflects time spent accumulating skills.



Lucas Model (cont'd)

$$\frac{\Delta k}{k} = \frac{sY}{k} - (n+d) \rightarrow \Delta k = sY - (n+d)k$$

- Physical capital accumulation is defined as usual: $\Delta K = sY - dK$.
- Rewrite the production function in terms of per effective worker:

$$y_e = Ak_e^\alpha$$

$$(A > 0)$$

$$k_e \equiv \frac{K}{L} = \frac{K}{uhN}$$

- The growth rate of physical capital per effective worker is:

$$\frac{\Delta k_e}{k_e} = \frac{\Delta K}{K} - \frac{\Delta u}{u} - \frac{\Delta h}{h} - \frac{\Delta N}{N} = \theta$$

$$\frac{\Delta k_e}{k_e} = sAk_e^{\alpha-1} - d - a(1-u) - n$$



$$\Delta k_e = sy_e - [d + a(1-u) + n]k_e$$

- At the equilibrium in the long run, (s.s.)

- Capital per effective worker is: $k_e^* = [s/(d + a(1-u) + n)]^{1/(1-\alpha)}$
- Output per effective worker is: $y_e^* = A[s/(d + a(1-u) + n)]^{\alpha/(1-\alpha)}$

- Note: This simplified version of Lucas model is similar to the Solow Model with augmented labor technology.

Lucas Model (cont'd)

(S.S.)

- In the long run, there is no change in physical capital per effective worker. Thus, the growth rate of physical capital per worker can be written as:

but there's a change in physical capital per worker

$$\frac{\Delta k_e}{k_e} = 0 \Rightarrow 0 = \frac{\Delta K}{K} - \frac{\Delta N}{N} - \frac{\Delta h}{h}$$

$$\frac{dK}{K} - \frac{dN}{N} = \frac{dk_e}{k_e} \approx \frac{\Delta k_e}{k_e}$$

$$\frac{\Delta k}{k} = \frac{\Delta K}{K} - \frac{\Delta N}{N} = \frac{\Delta h}{h} = a(1 - u)$$

$$\Rightarrow \boxed{\frac{\Delta k}{k} = a(1 - u)}$$

$$k_e \equiv \frac{K}{L} = \frac{K}{u h N}$$

$$\ln(k_e) = \ln(K) - \ln(u) - \ln(h) - \ln(N)$$

$$\frac{dk_e}{k_e} = \frac{dK}{K} - \frac{du}{u} - \frac{dh}{h} - \frac{dN}{N}$$

$$\frac{dk}{k} = \frac{dh}{h}$$

- Main driver for growth is the growth rate of human capital.
 - Policies that promote knowledge accumulate will result in higher growth rate of output.
- $1-u$ can be thought of as the production of knowledge.

a = quality

Romer Model

$$y = \underline{A} k^\alpha \quad \text{where} \quad y \equiv \frac{Y}{L}, \quad k \equiv \frac{K}{L}$$

- The Romer model endogenizes technological progress by introducing the search for new ideas by researchers interested in profiting from their inventions.
- This model is aimed to explain mainly why and how the advanced countries exhibit sustained growth.
- A firm's R&D raises its profits, but also create a positive externality on other firm (lead to increasing returns overall).
- Thus, technological progress is driven by profit-maximizing firms or investors' own interests.

Romer Model: Basic Elements (1)

- Let's start with a model of firms that invest in R&D.
- The production function is given by:

$$Y = K^\alpha (AL_Y)^{1-\alpha}, \quad (0 < \alpha < 1) \quad \text{-- (1)}$$

where A = a given level (or stock) of knowledge

L_Y = labor used in the production sector

L = total labor: $L = L_Y + L_A$ (L_A = labor used in R&D)

- The production function exhibits CRTS w.r.t. K and L , but exhibits IRTS w.r.t. all three inputs (K , L , and A).

increasing returns to scale

- Capital accumulations as people forego consumption at rate s_K .

$$\Delta K = s_K Y - dK \quad \text{-- (2)}$$

Romer Model: Basic Elements (2)

- Labor (equivalent to population) grows at a constant rate n :

$$\frac{\Delta L}{L} = n \quad \text{-- (3)}$$

Before A is constant in Solow model.

- Romer assumes that the number of new ideas (ΔA) at any given point in time is can be derived from:

$$\Delta A = \bar{\delta} L_A, \quad \text{-- (4)}$$

where L_A is the number of people attempting to discover new ideas, and $\bar{\delta}$ is the rate at which new ideas are discovered.

- $\bar{\delta}$ could a function of A: $\bar{\delta} = \delta A^\phi$ -- (5)
 - $\phi > 0 \rightarrow$ Productivity of research increases with the stock of knowledge
 - $\phi < 0 \rightarrow$ the “fishing out” case
 - $\phi = 0 \rightarrow$ Productivity of research is independent of the stock of knowledge

Romer Model: Basic Elements (3)

- Also, it is possible that the average productivity of research depends on the number people searching for new ideas.
- The general **production function for ideas** can be written as:

$$\Delta A = \delta A^\phi L_A^\lambda \quad -- (6)$$

where δ = a constant

ϕ = **knowledge spillover** ($\phi = 0$: no new ideas),

λ = **originality of idea** (i.e. $\lambda = 1$: no duplication) $\lambda = 0$ "copy cat"

$$g_A \equiv \frac{\Delta A}{A} = ?$$

- Suppose further that $L_A/L = s_R \rightarrow L_Y/L = 1 - s_R \quad -- (7)$

Growth rate of ideas (A)

$$\Delta A = \delta A^\phi L_A^\lambda$$

ϕ = spillover
 λ = originality of idea
= externality

$$Q: g_A \equiv \frac{\Delta A}{A} = ?$$

$$g_A = \frac{\Delta A}{A} = \frac{\delta A^\phi L_A^\lambda}{A} = \delta A^{\phi-1} L_A^\lambda$$

$$\ln(g_A) = \ln(\delta) + (\phi-1) \ln(A) + \lambda \ln(L_A)$$

Totally differentiate:

$$\frac{d(g_A)}{g_A} = \frac{d\delta}{\delta} + (\phi-1) \cdot \frac{dA}{A} + \lambda \cdot \frac{d(L_A)}{L_A}$$

$$\approx \frac{\Delta A}{A} = g_A$$

$$\left(\frac{dL_A}{L_A} = \frac{dL}{L} = n \right)$$

$$(1-\phi) g_A = \lambda \cdot n \Rightarrow$$

$$g_A = \frac{\lambda n}{1-\phi}$$

Growth in Romer Model

- Along the balanced growth path, one can show that

$$g_y = g_k = g_A \quad \text{-- (8)}$$

- Question: What determine the growth rate of A (i.e. growth rate of ideas)?
- Define the *growth rate of ideas* as $g_A = \Delta A / A$.
- Recall that $\Delta A = \delta A^\phi L_A^\lambda$. Hence,

$$\frac{\Delta A}{A} = \delta A^{\phi-1} L_A^\lambda. \quad \text{-- (9)}$$

- Along a balanced growth path, $\Delta A / A \equiv g_A$ is constant. Taking logs and derivatives of both sides of equation (9) gives:

$$g_A = \frac{\lambda n}{1-\phi}$$

$$\left. \begin{array}{l} \lambda \uparrow \text{ (more original ideas) } \\ \phi \uparrow \text{ (more spillover) } \end{array} \right\} \Rightarrow g_A \uparrow$$

-- (10)
→ R & D subsidy

(incentives).
Patents

Implications (if $\lambda = 1$ and $\phi = 0$)

- Population growth is the *only source of technological progress* and any long-run growth in per capita income.
 - A higher population growth in the Romer model means a *higher long-run economic growth rate*.
- Government cannot change the growth rate of per capita income by *any policy except a policy to increase the fertility rate*.
- For instance, a policy of increasing the R&D share of labor s_R cannot change the long-run growth rate of the economy. Why?
 - They only increase the growth rate during the transition period (but not in the long run).

Comparative Statics: A Permanent Increase in R&D Share (Optional)

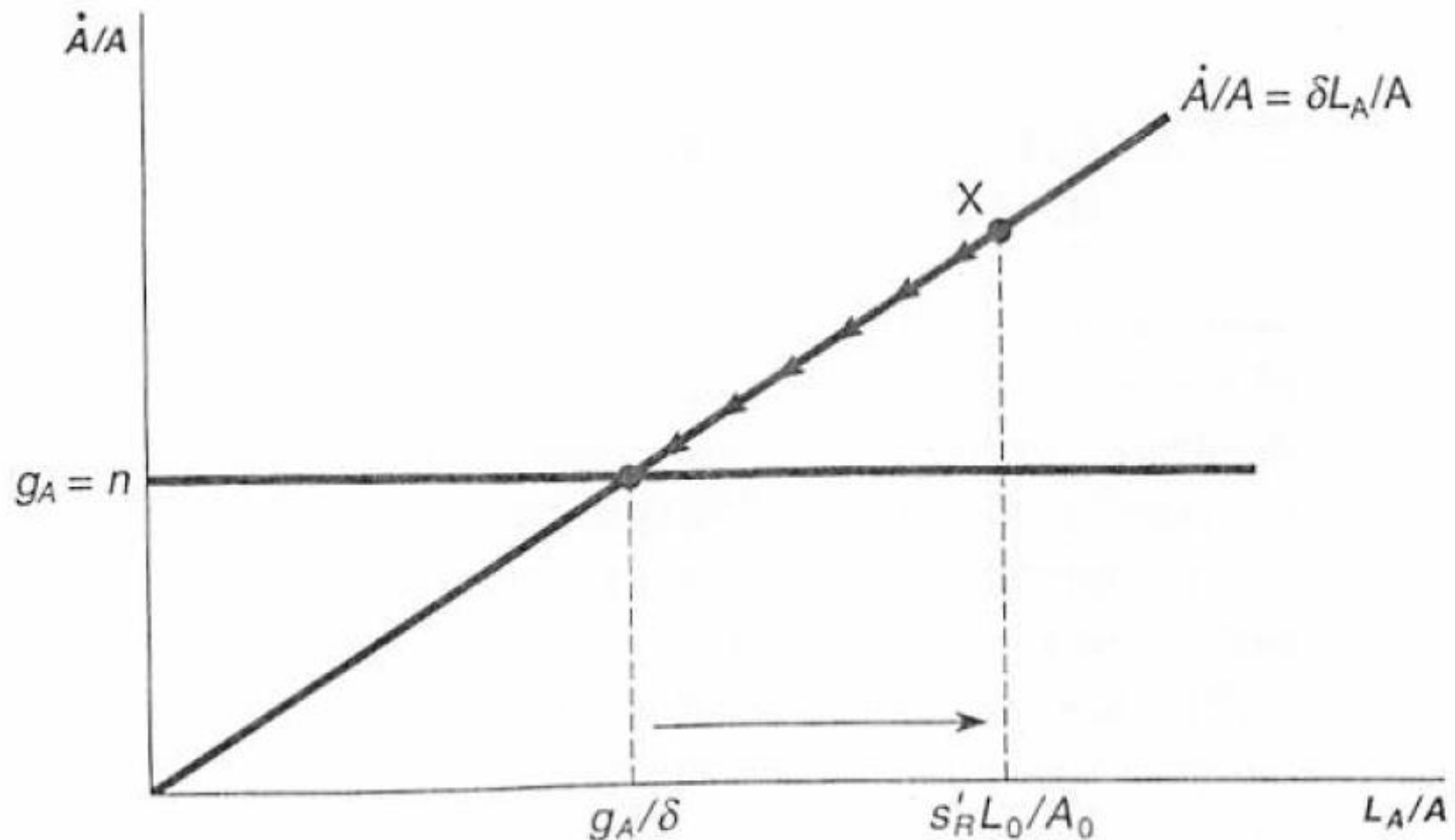
- For simplicity, assume $\lambda = 1$ and $\phi = 0$. Equation (9) can be rewritten as:

$$\frac{\Delta A}{A} = \delta \frac{S_R L}{A}, \quad \text{-- (11)}$$

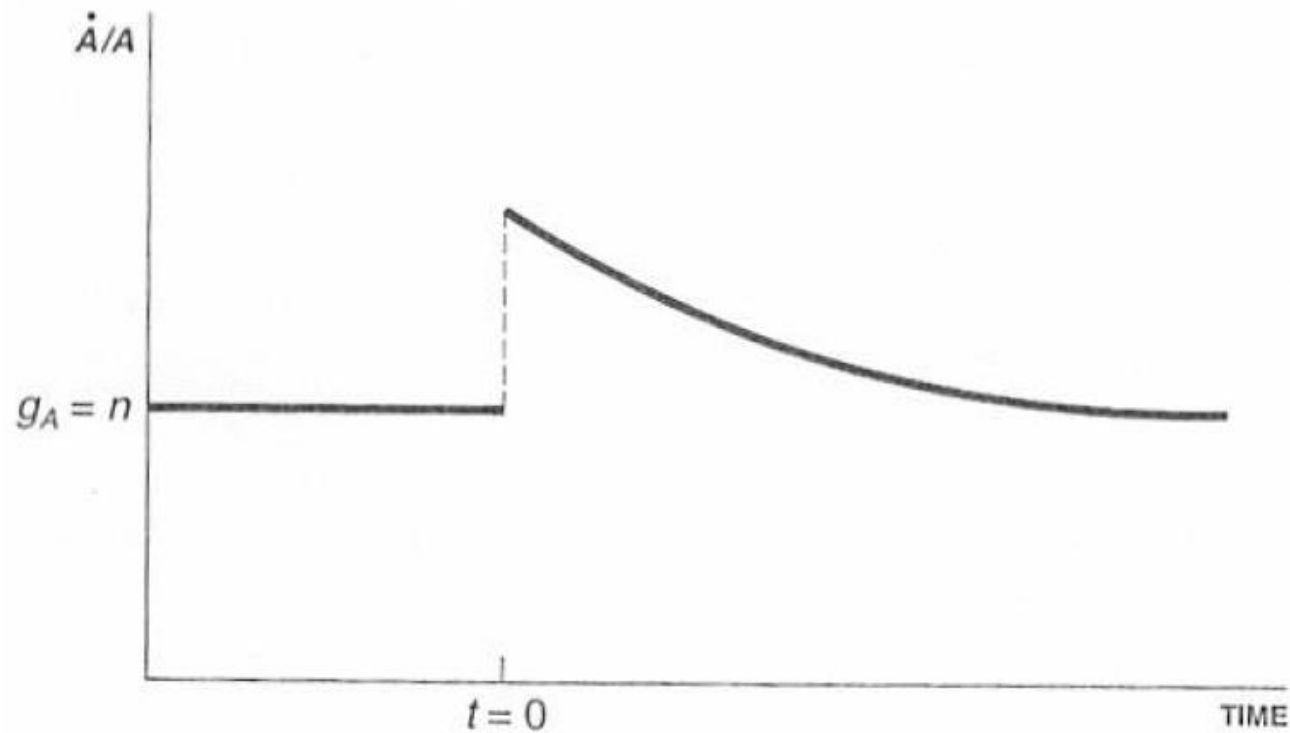
where S_R is the share of labor engaged in R&D.

- Question: What happens to the per capita output (y) if S_R increases?
 - Need to know the impact of an increase in R&D share on technological progress.
 - Use a Solow framework to derive output per effective labor and output per worker along a balanced growth path.

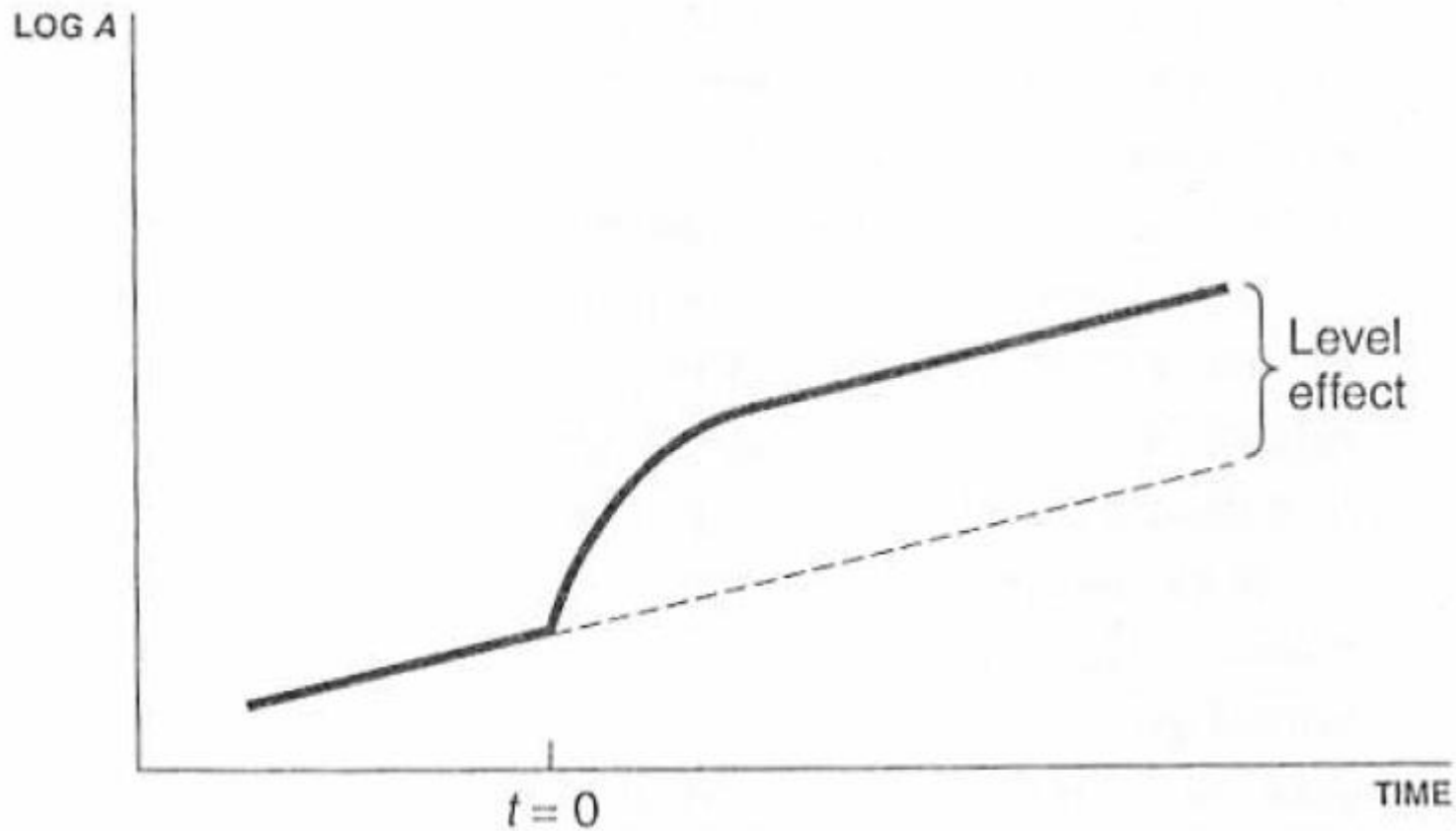
Technological Progress: An Increase in The R&D Share (Optional)



Technological Progress Over Time (Optional)



The Level of Technology Over Time(Optional)



Steady-State Output Per Capita (Optional)

- Based on the Solow framework, the ratio y/A along a balanced growth path is constant and is given by:

$$\left(\frac{y}{A}\right)^* = \left(\frac{s_K}{n+g_A+d}\right)^{\alpha/(1-\alpha)} (1 - s_R) \quad -- (12)$$

- Also, along a balanced growth path, from equation (11), the level of A can be solved in terms of the labor force:

$$A = \delta \frac{s_R L}{g_A} \quad -- (13)$$

- Based on the above information, we can derive:

$$y^* = \left(\frac{s_K}{n+g_A+d}\right)^{\alpha/(1-\alpha)} (1 - s_R) \frac{\delta s_R L}{g_A} \quad -- (14)$$

- s_R affects y^* both negatively (more researchers=fewer workers producing output) and positively (more researchers = more ideas).