Differential Equations

Differential equations are often used by economists. As its name suggested, it is an equation that includes the **derivatives** of the **unknown**.

Difference equation ➔ function of discrete time
Differential equation ➔ function of continuous time

In this lecture, the main objective is to learn to solve ***first-order and second-order ordinary differential equations***. The unknown and the derivative define the type of differential equation. An *ordinary differential equation* is one for which the **unknown** is a function of only **one** variable, for example, $\dot{x} = f(t, x)$ or $\ddot{x} = f(t, x, \dot{x})$, where $x = x(t)$ is the unknown function. If an equation contains only the **first-order derivatives** of the unknown function, it is called a first-order differential equation. On the other hand, if an equation contains the **second-order derivatives** of the unknown function, it is called a second-order differential equation.

Differential equation	General form	Example
First-order ordinary	$\dot{x} = f(t, x)$	$\dot{x} = 7x + 2t - 1$
Second-order ordinary	$\ddot{x} = f(t, x, \dot{x})$	$\ddot{x} = 3\dot{x} + 2x + t + 4$

If the unknown is a function of more than one variable, the equation becomes a *partial differential equation*. In this class, we will only study ordinary differential equation, so from here, all differential equations in the lecture will mean **ordinary differential equations**.

1. Separate Differential Equations

For a differential equation $\dot{x} = f(t, x)$, if $f(t, x)$ can be written as a product $g(t)h(x)$ of two functions, one of which depends only on *t* and the other only on *x*. Then the differential equation takes the special form

$$\dot{x} = g(t)h(x). \qquad \text{e.g.} \qquad \dot{x} = xt$$

For this special case, we say that the differential equation is **separable**. It is important to learn to distinguish between separable and non-separable equations. The reason is that separable equations are among those that can be solved in terms of integrals of known functions.

Ex. 1 Decide which of the following differential equations are separable:

(a) $\dot{x} = xt$ (b) $\dot{x} = t^2 - 1$ (c) $\dot{x} = xt + t$

(d) $\dot{x} = xt + t^2$ (e) $\dot{x} = e^{x+t}\sqrt{1+t^2}$ (f) $\dot{x} = \sqrt[4]{t^2 + x}$

(g) $\dot{x} = F(t) + G(t)$ (h) $\dot{x} = \dfrac{t^3}{x^6 + 1}$

Method for Solving Separable Differential Equations

1. Written a separable differential equation as

$$\frac{dx}{dt} = g(t)h(x) \qquad (*)$$

2. Separate the variables:

$$\frac{dx}{h(x)} = g(t)dt$$

3. Integrate:

$$\int \frac{dx}{h(x)} = \int g(t)dt + C$$

4. If possible, rearrange to obtain a solution $x(t)$.
5. If an initial condition is available, determine the constant C.

Ex. 1.1(a) Solve the differential equation $\dot{x} = \frac{dx}{dt} = xt \quad \& \quad x > 0$ and **prove your answer**.

Separate and integrate $\int \frac{dx}{x} = \int t\, dt$

$$\ln|x| = \frac{t^2}{2} + C_1$$

$$x = Ce^{\frac{t^2}{2}}$$

Note that it is called a *general solution* of the differential equation because x is not unique but depending on C which can be determined using an initial condition.

If $x(0) = 12$, then $C = 12$. Hence,

$$x = 12e^{\frac{t^2}{2}}$$

Prove: $\dot{x} = \left(12e^{\frac{t^2}{2}}\right)\frac{d}{dt}\left(\frac{t^2}{2}\right) = \left(12e^{\frac{t^2}{2}}\right)(t) = xt \qquad \surd \surd \surd$

Ex. 1.1(h) Solve the differential equation $\dot{x} = \frac{4t^3}{7x^6 + 1}$

$$\int (7x^6 + 1)dx = \int 4t^3 dt$$

$$x^7 + x = t^4 + C$$

Ex. 1.2 (Compound Interest) Suppose that $w = w(t)$ is the wealth in an account at time t, and that r(t) is the interest rate, with interest compounded continuously. Then

$\dot{w} = r(t)w$ (Separable equation)

$$\int \frac{dw}{w} = \int r(t)\, dt + C_1$$

$\ln(w) = R(t) + C_1$ where $R(t) = \int r(t)\, dt$ (1)

$w = e^{R(t)+C_1} = e^{C_1}e^{R(t)} = Ce^{R(t)}$ (2) where $C = e^{C_1}$

The initial value of the account is $w(0)$.

$w(0) = Ce^{R(0)} \quad \Rightarrow \quad C = w(0)e^{-R(0)}$

From (2), $w = w(0)e^{R(t)-R(0)}$ (3)

From (1), $R(t) - R(0) = \int_0^t r(t)\, dt$ (4)

Substitute (4) ➔ (3); therefore $w = w(0)e^{\int_0^t r(t)\, dt}$

Ex. 1.3 Solve the differential equation $\frac{dx}{dt} = -2x^2 t$ and the initial value $x(0)$=-1/2.

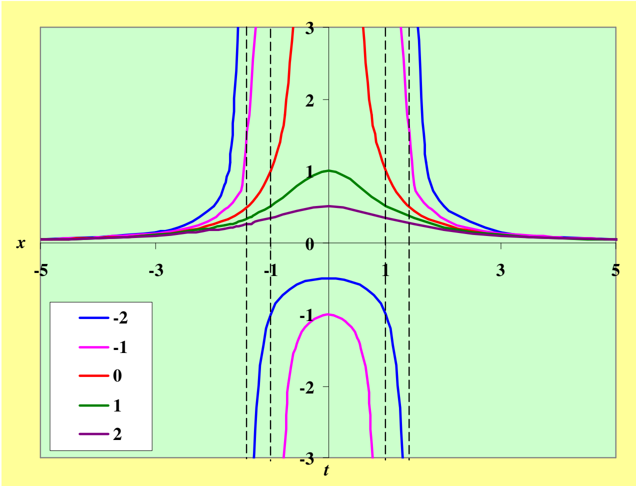
$$-\int \frac{dx}{x^2} = \int 2t\, dt$$

$$\frac{1}{x} = t^2 + C$$

$$x = \frac{1}{t^2 + C} \quad (**)$$

$(t, x) = (0, -1/2)$ ➔ $C = -2$

Thus, the integral curve passing though $(t, x) = (0, -1/2)$ is $x = \frac{1}{t^2 - 2}$.



The constant of integration C is critically affects the shape of the curve as well as its position. In the graph above, C = -2, -1, 0, 1 and 2.

Alternatively, we can solve Ex. 3 using definite integration and the initial condition.

$$-\int_{x_0}^{x} \frac{dx}{x^2} = \int_0^t 2t\, dt$$

$$-\int_{-\frac{1}{2}}^{x} \frac{dx}{x^2} = \int_0^t 2t\, dt \quad ➔ \quad \left[\frac{1}{x}\right]_{-\frac{1}{2}}^{x} = \left[t^2\right]_0^t$$

$$\frac{1}{x} - \frac{1}{\left(-\frac{1}{2}\right)} = t^2 \quad ➔ \quad \frac{1}{x} = t^2 - 2$$

$$x = \frac{1}{t^2 - 2} \quad \text{(same as above).}$$

2. First-Order Linear Differential Equations

The general form of the **first-order linear differential equation** is

$$\dot{x} = a(t)x + b(t)$$

where a and b are continuous functions of t and $x = x(t)$ is the unknown function. This is called ***"linear"*** because $\dot{x}$ is a linear function of x. For example,

$$\dot{x} = x \qquad 5\dot{x} = t - x \qquad \dot{x} = 2tx - 4t$$

Ex. 2.1 $\dot{x} = ax \quad where \quad a(t) = a$ and $b(t) = 0$

Show that the solution is $x = ke^{at}$. If $x(0) = x_0$, then $x = x_0 e^{at}$.

x is converged (stable) to 0 if $a < 0$ or $x_0 = 0$.

Ex. 2.2 $\dot{x} = ax + b \qquad where \quad a(t) = a$ and $b(t) = b$

Show that the solution is $x = -\frac{b}{a} + ke^{at}$. If $x(0) = x_0$, then $x = -\frac{b}{a} + \left(x_0 + \frac{b}{a} \right) e^{at}$.

x is converged (stable) to $-\frac{b}{a}$ if $a < 0$ or $x_0 = -\frac{b}{a}$.

Ex. 2.3 $\dot{x} = a(t)x \qquad where \quad b(t) = 0$

Show that the solution is $x = ke^{\int_0^t a(t)dt}$.

$$\frac{dx}{dt} = a(t)x$$

$$\int \frac{dx}{x} = \int_0^t a(t)dt \qquad \Rightarrow \qquad \ln x = \int_0^t a(t)dt + C$$

$$x = e^{\int_0^t a(t)dt + C} = e^C e^{\int_0^t a(t)dt} = ke^{\int_0^t a(t)dt}$$

Ex. 2.4 $\dot{x} = ax + b(t) \quad where \quad a(t) = a$

Show that the solution is $x = \left[k + \int_0^t b(t)e^{-at}dt \right] e^{at}$. x is converged (stable) to 0 if $a < 0$.

$$\frac{dx}{dt} - ax = b(t)$$

Times both sides of the equation by e^{-at} (integrating factor)

$$\frac{dx}{dt}e^{-at} - axe^{-at} = b(t)e^{-at}$$

From $d(uv) = vdu + udv$ where $u = x$ and $v = e^{-at}$,

$$\frac{d}{dt}\left(xe^{-at} \right) = b(t)e^{-at}$$

Integrate, $\qquad xe^{-at} = \int_0^t b(t)e^{-at}dt + k$

Hence, $\qquad x = \left[k + \int_0^t b(t)e^{-at}dt \right] e^{at}$

Ex. 2.5 $\dot{x} = a(t)x + b(t)$

Show that the solution is $x = \left[k + \int_0^t b(t)e^{-\int_0^t a(t)dt} dt \right] e^{\int_0^t a(t)dt}$. x is converged (stable) to 0 if $a < 0$.

$$\frac{dx}{dt} - a(t)x = b(t)$$

Times both sides of the equation by $e^{-\int_0^t a(t)dt}$

$$\frac{dx}{dt}e^{-\int_0^t a(t)dt} - a(t)xe^{-\int_0^t a(t)dt} = b(t)e^{-\int_0^t a(t)dt}$$

From $d(uv) = vdu + udv$ where $u = x$ and $v = e^{-\int_0^t a(t)dt}$,

$$\frac{d}{dt}\left(xe^{-\int_0^t a(t)dt} \right) = b(t)e^{-\int_0^t a(t)dt}$$

Integrate, $$xe^{-\int_0^t a(t)dt} = \int_0^t b(t)e^{-\int_0^t a(t)dt} dt + k$$

Hence, $$x = \left[k + \int_0^t b(t)e^{-\int_0^t a(t)dt} dt \right] e^{\int_0^t a(t)dt}$$

Solve Ex 2.6 – 2.17, determine the stability and verify that your answer is correct.

Ex 2.6 For $\dot{x} = x$, show that $x = ke^t$.

Ex 2.7 For $\dot{x} = -2x$, show that $x = 0$ if $x(0) = 0$, and show that $x = 3e^{-2t}$ if $x(0) = 3$.

Ex 2.8 For $\dot{x} = x + 4$, show that $x = -4 + ke^t$.

Ex 2.9 For $\dot{x} = 5 - x$ and $x(0) = 1$, show that $x = 5 - 4e^{-t}$.

Ex 2.10 For $\dot{x} = 10x + 5$ and $x(0) = 5$, show that $x = -\frac{1}{2} + \frac{11}{2}e^{10t}$.

Ex 2.11 For $\dot{x} = (2t+1)x$, show that $x = ke^{t^2+t}$.

Ex 2.12 For $\dot{x} = t - x$, show that $x = t - 1 + ke^{-t}$.

Ex 2.13 For $\dot{x} = x - t^2$, show that $x = t^2 + 2t + 2 - e^t$ if $x(0) = 1$, and show that $x = t^2 + 2t + 2 - 4e^{t-1}$ if $x(1) = 1$.

Ex 2.14 For $\dot{x} = (2t+1)x - 2t - 1$, show that $x = 1 + ke^{t^2+t}$ using integration by (a) product rule and (b) separated variable.

Ex 2.15 For $\dot{x} + 2tx = 4t$ and $x(0) = -2$, show that $x = 2 - 4e^{-t^2}$.

Ex 2.16 For $\dot{x} = x^2$, show that $x = \dfrac{1}{k-t}$.

Ex 2.17 Solve $\dot{x} = \dfrac{x^3}{t^3}$ and $x(1) = 1$.

3. Applications

Ex. 3.1 Let $X = X(t)$ denote the national product, $K = K(t)$ the capital stock, and $L = L(t)$ the number of workers in a country at time t. Suppose that, for all $t \geq 0$,

Cobb-Douglas production function	$X = AK^{1-\alpha}L^{\alpha}$	(1)
Aggregate investment is proportional to output	$\dot{K} = sX$	(2)
Number of workers grows exponentially	$L = L_0 e^{\lambda t}$	(3)

Where A, α, s, L_0 and λ are all positive constants, with $0 < \alpha < 1$. Derive from these equations a single differential equation to determine $K = K(t)$, and find the solution of the equation when $K(0) = K_0 > 0$.

From (1) – (3), $\dot{K} = sX = sAK^{1-\alpha}L^{\alpha} = sAL_0^{\alpha}e^{\alpha\lambda t}K^{1-\alpha}$ (Separable equation)

$$\int_{K_0}^{K} K^{\alpha-1}dk = \int_0^t sAL_0^{\alpha}e^{\alpha\lambda t}dt$$

$$\frac{1}{\alpha}\left[K^{\alpha}\right]_{K_0}^{K} = \frac{1}{\alpha\lambda}sAL_0^{\alpha}\left[e^{\alpha\lambda t}\right]_0^t$$

$$\frac{1}{\alpha}\left(K^{\alpha} - K_0^{\alpha}\right) = \frac{1}{\alpha\lambda}sAL_0^{\alpha}\left(e^{\alpha\lambda t} - 1\right)$$

Rearrage, $$K = \left\{K_0^{\alpha} + \left(\frac{s}{\lambda}\right)AL_0^{\alpha}\left(e^{\alpha\lambda t} - 1\right)\right\}^{\frac{1}{\alpha}}$$

Ex. 3.2 When the price of a commodity is P, let $D(P) = a - bP$ denote the demand and $S(P) = \alpha + \beta P$ the supply. Here a, b, α and β are positive constants. Assume that the price $P = P(t)$ varies with time, and that $\dot{P}$ is proportional to excess demand $D(P) - S(P)$. Thus,

$$\dot{P} = \lambda\left[D(P) - S(P)\right]$$

where λ is a positive constant. Show that the general function for the price P is

$$P = Ce^{-\lambda(b+\beta)t} + \frac{a-\alpha}{b+\beta}.$$

Determine that the price P is stable and converged to the equilibrium price $P^* = \dfrac{a-\alpha}{b+\beta}$ as $t \to \infty$.

Ex. 3.3 Consider the following model of economic growth in a developing country: if $X(t)$ is the total production per year, $K(t)$ is the capital stock, $H(t)$ is the flow of foreign aid per year, and $N(t)$ is the size of the population, all measured at time t. Then,

$$X(t) = \sigma K(t) \qquad (1)$$
$$\dot{K}(t) = \alpha X(t) + H(t) \qquad (2)$$
$$N(t) = N_0 e^{\rho t} \qquad (3)$$

From (1), the volume of production is simply proportional to the capital stock, with the factor of proportionality σ being called the *average productivity of capital*. From (2), the total growth of capital per year is equal to internal saving plus foreign aid. The saving are proportional to production, with the factor of proportionality α being called the *saving rate*. Finally, (3) suggests that population increases at a constant proportional rate of growth ρ.

Derive from these equations a differential equation for $K(t)$. Assume that $H(t) = H_0 e^{\mu t}$, and find the solution of the differential equation in this case, given that $K(0) = K_0$ and $\alpha\sigma \neq \mu$. Find an expression for the production per head $x(t) = \dfrac{X(t)}{N(t)}$.

ANS: From (1) and (2); $\quad \dot{K}(t) = \alpha\sigma K(t) + H(t) \qquad (4)$

If $H(t) = H_0 e^{\mu t}$, solve (4) obtain $\quad K(t) = Ce^{\alpha\sigma t} + \dfrac{H_0}{\mu - \alpha\sigma} e^{\mu t}$

For $K(0) = K_0$, $C = K_0 - \dfrac{H_0}{\mu - \alpha\sigma}$.

Hence, $$K(t) = \left(K_0 - \frac{H_0}{\mu - \alpha\sigma} \right) e^{\alpha\sigma t} + \frac{H_0}{\mu - \alpha\sigma} e^{\mu t} \qquad (5)$$

Per capita production is equal to $$x(t) = \frac{X(t)}{N(t)} = \frac{\sigma K(t)}{N_0 e^{\rho t}}. \qquad (6)$$

Substitute (5) → (6), $$x(t) = x(0)e^{(\alpha\sigma - \rho)t} + \left(\frac{\sigma}{\alpha\sigma - \mu} \right) \frac{H_0}{N_0} e^{(\alpha\sigma - \rho)t} \left[1 - e^{(\mu - \alpha\sigma)t} \right]$$

Ex. 3.4 In a macroeconomic model, $C(t)$, $I(t)$, and $Y(t)$, denote respectively the consumption, investment, and national income in a country at time t. Assume that, for all t:

[1] $C(t) + I(t) = Y(t)$ [2] $I(t) = k\dot{C}(t)$ [3] $C(t) = aY(t) + b$

where a, b, and k are positive constants, with a < 1.

(a) Derive the following differential equation for $Y(t)$:

$$\dot{Y}(t) = \frac{1-a}{ka} Y(t) - \frac{b}{ka} - \frac{b}{a}$$

(b) Solve this differential equation when $Y(o) = Y_0 > \dfrac{b}{(1-a)}$, and then find the corresponding function $I(t)$.

(c) Compute $\lim_{t \to \infty} \left[\dfrac{Y(t)}{I(t)} \right]$.

4. Second-Order Linear Differential Equations

Second-order differential equations can usually be written in the form

$$\ddot{x} = f(t, x, \dot{x})$$

The general second-order linear differential equation is

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t) \qquad \text{(I)}$$

where $a(t)$, $b(t)$, and $f(t)$ are all continuous functions of *t*. In this lecture, we will only consider the case with constant coefficients hence (I) becomes

$$\ddot{x} + a\dot{x} + bx = f(t) \qquad \text{(II).}$$

In contrast to first-order linear equations, generally (II) has no explicit solution in terms of functions in this lecture. However, it is still possible to determine the general solution in some cases using the method similar to what we have done for difference equations.

General Solution = Complementary Solution + Particular Solution

4.1 Complementary Solution

The complementary solution is the solution to the homogeneous equation. The homogeneous equation of (II) obtains by replacing $f(t)$ by 0. Hence,

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \text{(III)}$$

If $u_1 = u_1(t)$ and $u_2 = u_2(t)$ both satisfy (III), then $x = Au_1 + Bu_2$ also satisfy (III) for all choices of *A* and *B*. Differentiation gives $\dot{x} = A\dot{u}_1 + B\dot{u}_2$ and $\ddot{x} = A\ddot{u}_1 + B\ddot{u}_2$. Inserting these expressions for x, $\dot{x}$ and $\ddot{x}$ into the left-hand side of (III) yields

$$\ddot{x} + a\dot{x} + bx = (A\ddot{u}_1 + B\ddot{u}_2) + a(A\dot{u}_1 + B\dot{u}_2) + b(Au_1 + Bu_2) = 0$$
$$\ddot{x} + a\dot{x} + bx = A(\ddot{u}_1 + a\dot{u}_1 + bu_1) + B(\ddot{u}_2 + a\dot{u}_2 + bu_2) = 0$$

In order for $x = Au_1 + Bu_2$ to be the solution of (III), u_1 and u_2 must not be constant multiples of each other – that is, they must not be proportional. Now, we need to guess u_1 and u_2. Because the coefficients in (III) are constants, it seems a good idea to try possible solutions x with the property that x, $\dot{x}$ and $\ddot{x}$ are all constant multiples of each other. The exponential function $x = e^{mt}$ has this property, because $\dot{x} = me^{mt} = mx$ and $\ddot{x} = m^2e^{mt} = m^2x$. Inserting these expressions for x, $\dot{x}$ and $\ddot{x}$ into the left-hand side of (III) yields

$$m^2e^{mt} + ame^{mt} + be^{mt} = 0$$

Canceling e^{mt} gives $\qquad m^2 + am + b = 0 \qquad$ (IV)

which is the **characteristic equation** of the homogenous second-order differential equation (III). This is a quadratic equation and the two **characteristic roots** are

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

There are 3 possible solution of the homogenous second-order differential equation $\ddot{x} + a\dot{x} + bx = 0$ (III) **Case 1** $a^2 - 4b > 0$ when the characteristic equation has two distinct real roots. $x = Ae^{m_1 t} + Be^{m_2 t}$ **Case 2** $a^2 - 4b = 0$ when the characteristic equation has one distinct real root $m = -\frac{a}{2}$. $x = (A + Bt)e^{mt}$ **Case 3** $a^2 - 4b < 0$ when the characteristic equation has no real roots (complex roots). $x = Ae^{\alpha t}\cos(\beta t + B)$ where $\alpha = -\frac{a}{2}$ and $\beta = \sqrt{b - \frac{1}{4}a^2}$

For case 1 ($a^2 - 4b > 0$), The functions $e^{m_1 t}$ and $e^{m_2 t}$ satisfy (III). These functions are not proportional when $m_1 \neq m_2$, so the general solution is $x = Ae^{m_1 t} + Be^{m_2 t}$.

For case 2 ($a^2 - 4b = 0$), then $m = -\frac{a}{2}$ is one double root of (IV), and that $u_1 = e^{mt}$ satisfies (III). We cannot use $u_2 = e^{mt}$ because it is proportional to u_1. Then, we try $u_2 = te^{mt}$. Differentiation gives $\dot{u}_2 = e^{mt} + tme^{mt}$ and $\ddot{u}_2 = me^{mt} + me^{mt} + tm^2e^{mt} = 2me^{mt} + tm^2e^{mt}$. Inserting these expressions for x, $\dot{x}$ and $\ddot{x}$ into the left-hand side of (III) yields

$$\ddot{u}_2 + a\dot{u}_2 + bu_2 = \left(2me^{mt} + tm^2e^{mt}\right) + a\left(e^{mt} + tme^{mt}\right) + bte^{mt} = 0$$

$$\ddot{u}_2 + a\dot{u}_2 + bu_2 = (2m + a)e^{mt} + \left(m^2 + m + b\right)te^{mt} = 0. \qquad \text{(V)}$$

(V) is true because $m = -\frac{a}{2}$ and $m^2 + am + b = 0$. Hence $u_2 = te^{mt}$ satisfies (III). These two solution is not proportional so the solution is $x = (A + Bt)e^{mt}$.

4.2 Particular Solution (x_{ps})

The general second-order linear differential equation is

$$\ddot{x} + a\dot{x} + bx = f(t) \qquad \text{(II).}$$

We need to guess a particular solution that satisfies (II) and use the *method of undetermined coefficients* is a simple method to determine unknowns of the guessed particular solution.

$f(t)$	Guessed x_{ps}
A	k
pa^{qt}	ka^{qt}
pt^n (Polynomial)	$k_0 + k_1t + k_2t^2 + \ldots + +k_nt^n$
$p\sin rt + q\cos rt$	$k\sin rt + j\cos rt$

If $f(t) = A$ (a constant), the guess $x = k$. Then $\dot{x} = \ddot{x} = 0$, so the equation (II) reduces to $bk = A$. Hence, $k = A/b$.

$\ddot{x} + a\dot{x} + bx = A$ **has a particular solution $x_{ps} = A/b$.**

4.3 General Solution

General Solution = Complementary Solution + Particular Solution

4.4 Stability

The stability concepts for second-order linear differential equation are closely related to those for difference equations. If small changes in the initial conditions has no effect on the long-run behaviour of the solution, the system is called **stable**. On the other hand, if small changes in the initial conditions can lead to significant differences in the long-run behaviour of the solution, then the system is **unstable**.

> The second-order linear differential equation
> $$\ddot{x} + a\dot{x} + bx = f(t)$$
> has a general solution
> $$x = Au_1 + Bu_2 + x_{ps}.$$
> The equation is **stable** if both roots of the characteristic equation $m^2 + am + b = 0$ has negative real parts.
>
> $\ddot{x} + a\dot{x} + bx = f(t)$ is stable $\Leftrightarrow$ $a > 0$ and $b > 0$.

Examples: Solve Ex. 4.1 – 4.7, determine the stability.

Ex. 4.1 For $\ddot{x} - 3x = 0$, show that $x = Ae^{t\sqrt{3}} + Be^{-t\sqrt{3}}$.

Ex. 4.2 For $\ddot{x} - 4\dot{x} + 4x = 0$, show that $x = (A + Bt)e^{2t}$.

Ex. 4.3 For $\ddot{x} - 4\dot{x} + 4x = 44$, show that $x = (A + Bt)e^{2t} + 11$.

Ex. 4.4 Find the general solutions and determine the stability

(a) $\ddot{x} - 3x = 0$ (b) $\ddot{x} + 4\dot{x} + 8x = 0$ (c) $3\ddot{x} + 8\dot{x} = 0$

(d) $4\ddot{x} + 4\dot{x} + x = 0$ (e) $\ddot{x} + \dot{x} - 6x = 8$

Ex. 4.5 Solve the following differential equation $\ddot{x} + 2\dot{x} + x = t^2$ for the specific initial conditions $x(0) = 0$, $\dot{x}(0) = 1$.

Example of past exam questions:

9. For each of the following differential equations, determine the solution and **<u>prove</u>** that your solution is correct.

(a) $x.\dot{x} = 2xt + 2t$ and $x(1) = 0$ (8 marks)

(You can leave your answer with all the x(s) on one side and all the t(s) on the other side)

(b) $\dot{x} = \dfrac{1 - xt}{t^2}$ and $x(1) = 5$ (8 marks)

10. A second order linear differential equation is

$$a\ddot{x} - 2\dot{x} + \frac{x}{2} = 3 \text{ and } x(0) = 2, \dot{x}(0) = -5$$

(a) Find the value of a that will give the characteristic equation **<u>one distinct real root</u>**.

(b) Use the value of a from (a), determine the general solution

(c) **<u>Discuss with reasons</u>** the stability of the solution.

(11 marks)