Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

$\cancel{2}, \cancel{3}, \cancel{4}, \cancel{6}$

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E\left[m X_a\right] \end{aligned}$$

$U(C_0) + \delta E[U(C_1)]$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

From $p_s = \pi_s m_s (1)$, ← pay 1 in state s

$$m_s = \frac{\delta U'(C_s^*)}{U'(C_0^*)} = \frac{\delta\left(\frac{1}{C_s^*}\right)}{\frac{1}{C_0^*}} = \frac{\delta C_0^*}{C_s^*}$$

Hence, $p_s(C_0^*, C_s^*) = \delta \pi_s \left(\frac{C_0^*}{C_s^*}\right)$ #

3. Consider the one-period consumption - portfolio choice problem. The individual's first order conditions leads to the general relationship

$$\text{FOC} \quad : \quad 1 = E\left[m_{01} R_s\right] \iff 1 = \pi_s m_{01} R_s$$

where m_{01} is the stochastic discount factor between dates 0 and 1 and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s, that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\tilde{C}_1\right)\right] \qquad U'(C) = C^{\gamma-1}$$

where utility displays constant relative risk aversion, $\boxed{U(C) = C^\gamma/\gamma}$, solve for the elasticity of intertemporal substitution, ε^I.

From $m_{01} = \dfrac{\delta U'(C_s)}{U'(C_0)} = \dfrac{\delta C_s^{\gamma-1}}{C_0^{\gamma-1}} = \delta\left(\dfrac{C_s}{C_0}\right)^{\gamma-1}$,

therefore, the first order condition can be rewritten as

$$1 = \delta\left(\frac{C_s}{C_0}\right)^{\gamma-1} \pi_s R_s \qquad \text{——} \quad (1)$$

Total differentiate equation (1), we have

$$0 = \delta \pi_s \left[\left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s + R_s(\gamma-1)\left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right) \right]$$

$$0 = \left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s + R_s(\gamma-1)\left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right)$$

$$-\left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s = R_s(\gamma-1)\left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right)$$

$$-\left(\frac{C_s}{C_0}\right) = (\gamma-1) R_s \frac{d\left(\frac{C_s}{C_0}\right)}{d(R_s)}$$

$$-\frac{1}{\gamma-1} = \frac{R_s}{\left(\frac{C_s}{C_0}\right)} \frac{d\left(\frac{C_s}{C_0}\right)}{d(R_s)} = \varepsilon^I \quad \#$$

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d(C_s/C_0)}{dR_s}$$

4. [16]Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state. There are two assets, a risk free asset with $R_f = 1.05$, and a second risky asset that pays cashflows

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

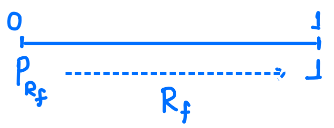
The current price of the risky asset is 6.

4.a Solve for the prices of the elementary securities, p_1 and p_2 and the risk-neutral probabilities of the two states.

$$\mathbb{X} = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix} = [\mathbb{X}_1 \ \ \mathbb{X}_2]$$

$$\mathbb{P} = \begin{bmatrix} 1/R_f \\ 6 \end{bmatrix} = \begin{bmatrix} 1/1.05 \\ 6 \end{bmatrix}$$

Risk-free asset



$$: \ P_{R_f} = \frac{1}{R_f}$$

thus, $p_1 = \mathbb{P}^T \mathbb{X}^{-1} e_1 = [1/1.05 \quad 6] \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.2476$ #

$p_2 = \mathbb{P}^T \mathbb{X}^{-1} e_2 = [1/1.05 \quad 6] \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.7048$ #

Risk-neutral probabilities ($\hat{\pi}_s = p_s R_f$)

$\hat{\pi}_1 = p_1 R_f = 0.2476(1.05) = 0.26$ #

$\hat{\pi}_2 = p_2 R_f = 0.7048(1.05) = 0.74$ #

4.b Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$.

What is the stochastic discount factor for the two states?

From the lecture, stochastic discount factor for state s is

$m_s = \frac{p_s}{\pi_s}$, then

$m_1 = \frac{p_1}{\pi_1} = \frac{0.2476}{0.5} = 0.4952$ #

$m_2 = \frac{p_2}{\pi_2} = \frac{0.7048}{0.5} = 1.4096$ #

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

6.a Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

From $$E[m(R_i - R_f)] = 0$$

$$E[(a + bR_m)(R_i - R_f)] = 0$$

$$E[aR_i - aR_f + bR_iR_m - bR_mR_f] = 0$$

$$aE[R_i] - aR_f + bE[R_iR_m] - bR_fE[R_m] = 0$$

$$a(E[R_i] - R_f) + b(E[R_i]E[R_m] + \mathrm{cov}[R_i, R_m] - R_fE[R_m]) = 0$$

$$a(E[R_i] - R_f) + b(E[R_m](E[R_i] - R_f) + \mathrm{cov}[R_i, R_m]) = 0$$

$$(E[R_i] - R_f)(a + bE[R_m]) + b\,\mathrm{cov}[R_i, R_m] = 0$$

$$E[R_i] - R_f = \frac{-b\,\mathrm{cov}[R_i, R_m]}{a + bE[R_m]}$$

Since $\beta_i = \dfrac{\mathrm{cov}[R_i, R_m]}{\sigma_m^2}$, multiply RHS of the above equation by $\dfrac{\sigma_m^2}{\sigma_m^2}$

we'll have $$E[R_i] - R_f = \frac{-b\sigma_m^2}{a + bE[R_m]} \cdot \frac{\mathrm{cov}[R_i, R_m]}{\sigma_m^2} = \frac{-\beta_i b \sigma_m^2}{a + bE[R_m]}$$

From CAPM, $E[R_i - R_f] = \beta_i \gamma$.

We, then, have $\boxed{\gamma = \dfrac{-b\sigma_m^2}{a + bE[R_m]}}$ #

6.b Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

Risk-free asset

$$1 = E[mR_f]$$

$$\frac{1}{R_f} = E[m] = E[a + bR_m] = a + bE[R_m] \quad \text{——— (1)}$$

Market Portfolio

$$1 = E[mR_m] = E[(a + bR_m)R_m]$$

$$1 = aE[R_m] + bE[R_m^2]$$

Note that $\sigma_m^2 = E[R_m^2] - (E[R_m])^2$, so

$$1 = aE[R_m] + b(\sigma_m^2 + (E[R_m])^2) \quad \text{——— (2)}$$

From (1), it can be rewritten as

$$a = \frac{1}{R_f} - bE[R_m] \quad \text{——— (3)}$$

then substitute in (2).

$$1 = \left(\frac{1}{R_f} - bE[R_m]\right)E[R_m] + b(\sigma_m^2 + (E[R_m])^2)$$

$$1 = \frac{E[R_m]}{R_f} + b\sigma_m^2$$

$$b = \left(1 - \frac{E[R_m]}{R_f}\right)\frac{1}{\sigma_m^2} = \boxed{\frac{-(E[R_m] - R_f)}{R_f\sigma_m^2}}$$ #

Substitute b into (3), we'll have

$$a = \frac{1}{R_f} + \frac{(E[R_m] - R_f)}{R_f \sigma_m^2} E[R_m]$$

$$a = \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f \sigma_m^2} \quad \#$$

6.c Using the formula for a and b in part b, show that $\gamma = E[R_m] - R_f$.

From 6.a), It is shown that $\gamma = \dfrac{-b\sigma_m^2}{a + bE[R_m]}$.

So,
$$\gamma = \frac{\frac{(E[R_m] - R_f)}{R_f \cancel{\sigma_m^2}} \cdot \cancel{\sigma_m^2}}{\frac{\sigma_m^2 + \cancel{E[R_m](E[R_m] - R_f)}}{R_f \sigma_m^2} - \frac{\cancel{E[R_m](E[R_m] - R_f)}}{R_f \sigma_m^2}}$$

$$\gamma = \frac{\frac{(E[R_m] - R_f)}{\cancel{R_f}}}{\frac{\cancel{\sigma_m^2}}{\cancel{R_f}\cancel{\sigma_m^2}}} = E[R_m] - R_f \quad \#$$