Valuing Stock Options: The Black-Scholes Model

Outline

- Properties of the lognormal distribution
- Assumptions in the Black-Scholes pricing formula
- The Black-Scholes
- Examples
- Implied volatility
- Options of dividend-paying stocks and American options

The Black-Scholes Random Walk Assumption

- Consider a stock whose price is S
- In a short period of time of length Δt the change in the stock price is assumed to follow a distribution with mean $\mu S \Delta t$ and standard deviation

$$\sigma S \sqrt{\Delta t}$$

- μ is expected return and σ is volatility

The Lognormal Property

- From continuous time stochastic mathematics these assumptions imply ln S_T is normally distributed with mean:

$$\ln S_0 + (\mu - \sigma^2 / 2)T$$

and standard deviation: $\sigma\sqrt{T}$

- Because the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Property

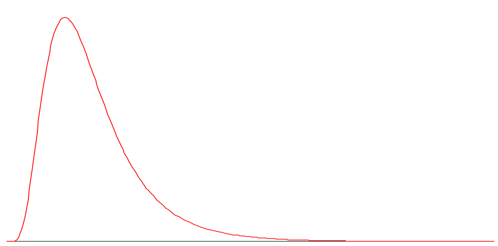
$$\ln S_T \approx \phi\left[\ln S_0 + (\mu - \sigma^2/2)T, \sigma\sqrt{T}\right]$$

or

$$\ln \frac{S_T}{S_0} = \phi\left[(\mu - \sigma^2/2)T, \sigma\sqrt{T}\right]$$

where $\phi\ [m,s]$ is a normal distribution with mean m and standard deviation s

The Lognormal Distribution



$$E(S_T) = S_0\, e^{\mu T}$$

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

The Expected Return

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock with continuous compounding is $\mu - \sigma^2/2$
- The arithmetic mean of the returns over short periods of length Δt is μ
- The geometric mean of these returns is $\mu - \sigma^2/2$

The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in time Δt is

$$\sigma\sqrt{\Delta t}$$

Estimating Volatility from Historical Data

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years
2. Define the continuously compounded return as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s , of the u_i ´s
4. The historical volatility estimate per annum is:

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

Example

1. Take monthly observations of $S_0, S_1, \ldots, S_n$ (end of the month stock price)
2. Define the continuously compounded return as:

$$u_{Jan\ 2006} = \ln\left(\frac{S_{Jan\ 2006}}{S_{Dec\ 2005}}\right)$$

3. Calculate the standard deviation, s , of the u_i ´s. Use the function stdev(u) in excel
4. The historical volatility estimate per annum is:

$$\hat{\sigma} = \frac{s}{\sqrt{1/12}}$$

Nature of Volatility

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued

Example

- Consider a stock with an initial price of $40, an expected return (arithmetic) of 16% per annum, and a volatility of 20% per annum. The probability distribution of the stock price in 6 months is given by

$$\ln S_T \sim \phi\left[\ln 40 + (0.16 - 0.2^2/2)0.5, 0.2\sqrt{0.5}\right]$$

$$\ln S_T \sim \phi\left[3.759, 0.14\right]$$

Concept check

- There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of this mean. 95% of the time the stock price in 6 months will be between S_h and S_l. What are S_h and S_l from the previous example?

G) 56.56 and 32.55

Y) 51.2 and 30.14

R) 50 and 30

The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate

B-S assumptions

- Major assumptions in most versions of B-S
  - Stock price follows a lognormal distribution
  - No restrictions on short sales
  - No transaction costs or taxes
  - No riskless arbitrage opportunities (or they are explored immediately as they occur)
  - Security trading is continuous
  - Interest rate is constant and the same for all maturities
- Assumptions that are later relaxed
  - No dividend during the life of the option

The Black-Scholes Formulas

$$c = S_0 \, N(d_1) - K \, e^{-rT} N(d_2)$$

$$p = K \, e^{-rT} \, N(-d_2) - S_0 \, N(-d_1)$$

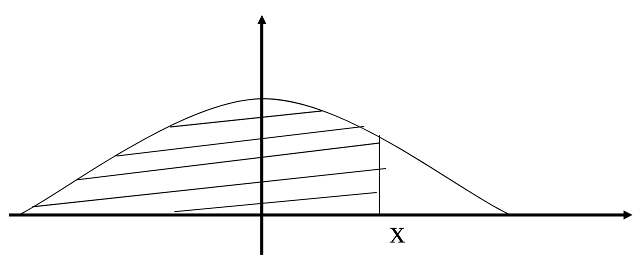
where $$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The *N*(*x*) Function

- *N*(*x*) is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than *x*
- See tables at the end of the book
- You can use the function ‘Normdist’ in excel to calculate N(x)

The *N*(*x*) Function



• N(x) equals the area under the normal probability density function up to x

• The area under this probability density function sums to a total of 1

The *N*(*x*) Function

- For the problem sets you can obtain N(x) using excel
- For the final exam, I will provide you with a look-up table for N(x)
- Note that N(-x) = 1-N(x)

Example

The current stock price is $42. Value a call option with maturity of 6 months and strike price of 40. The risk free rate is 10% per annum. The volatility of the continuous time return of this stock is 20% per annum

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2)0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.7693 - 0.2\sqrt{0.5} = 0.6278$$

$$c = 42N(0.7693) - 40\ e^{-0.1*0.5}N(0.6278)$$

$$=4.76$$

Properties of Black-Scholes Formula

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

- As S_0 becomes very large c tends to $S - Ke^{-rT}$ and p tends to zero

Properties of Black-Scholes Formula

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

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- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S$

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- This is consistent with the risk-neutral valuation principle
- The price of risk of the underlying asset is reflected in the current stock price

Applying Risk-Neutral Valuation

1. Assume that the expected return from an asset is the risk-free rate
2. Calculate the expected payoff from the derivative
3. Discount at the risk-free rate

Valuing a Forward Contract with Risk-Neutral Valuation

- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $Se^{rT} - K$
- Present value of expected payoff is

$$e^{-rT}[Se^{rT} - K] = S - Ke^{-rT}$$

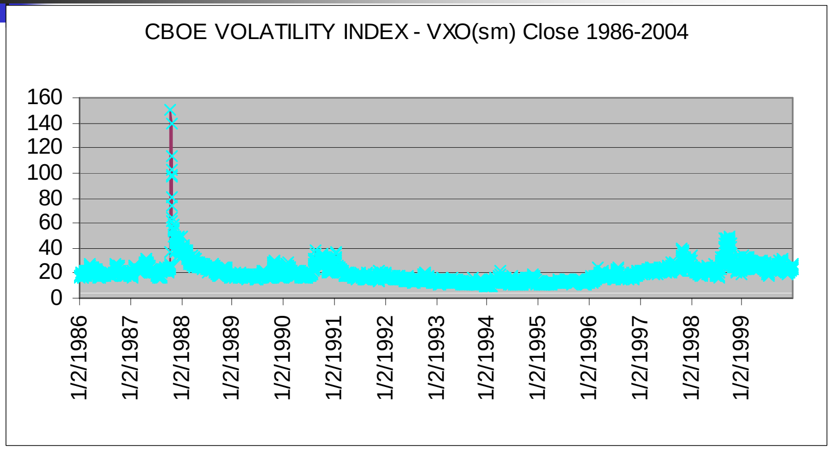
Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- The is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

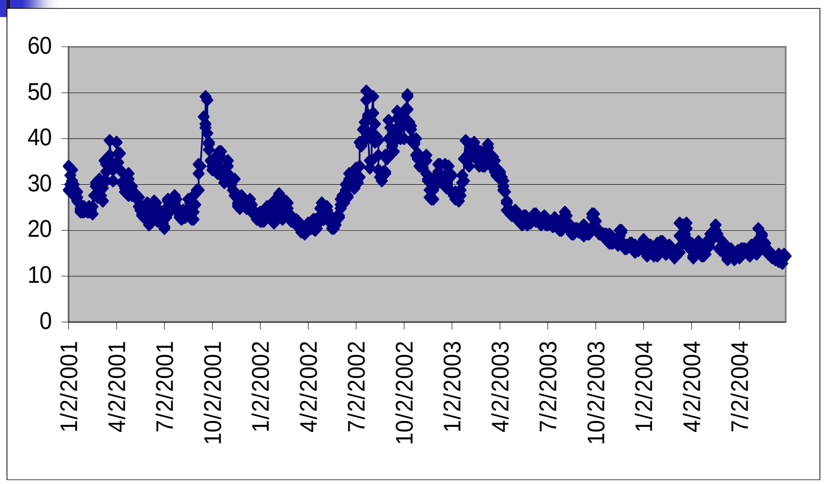
Implied Volatility

- To obtain the implied volatility we solve the non-linear B-S pricing equation for the volatility given an option price
- Implied volatility shows the market consensus of volatility of the underlying asset looking forward

Implied Volatility



Implied Volatility



Concept check

Option of the same stock, same maturity date, same strike price, A with higher implied volatility than B, which one is more richly priced?

G) A

R) B

Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into the Black-Scholes formula
- Only dividends with ex-dividend dates during life of option should be included
- The “dividend” should be the expected reduction in the stock price expected
- If the underlying asset pays dividend yield (q) then the dividend is subtracted from interest rate r

American Calls

- An American call on a non-dividend-paying stock should never be exercised early
- An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date

Black's Approach to Dealing with Dividends in American Call Options

Set the American price equal to the maximum of two European prices:

1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date

Summary

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