Optimization with Equality Constraints II

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CW Ch.12

Outline

- Finding the Stationary Values
- Second-Order Conditions
- Utility Maximization and Consumer Demand*
- Homogeneous Functions*
- Least-Cost Combination of Inputs*

Finding the Stationary Values: The *n*-Variable Cases

- The generalization of the Lagrange-multiplier method to n variables can be easily done. Consider the problem

$$\max_{x,y}\ z = f(x_1, x_2, \dots, x_n)$$

$$\text{s.t.}\quad g(x_1, x_2, \dots, x_n) = c.$$

- It follows that the Lagrangian equation will be

$$L = f(x_1, x_2, \dots, x_n) + \lambda[c - g(x_1, x_2, \dots, x_n)],$$

for which the first-order condition will consist of $(n+1)$ equations as follows

$$L_1 = f_1 - \lambda g_1 = 0$$

$$L_2 = f_2 - \lambda g_2 = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$L_n = f_n - \lambda g_n = 0$$

$$L_\lambda = c - g(x_1, x_2, \dots, x_n) = 0$$

- We can solve for stationary values of L from these equations.

Finding the Stationary Values: The Multi-constraint Cases

- When there is more than one constraint, the Lagrange-multiplier method is equally applicable. Let an n-variable function be subject simultaneously to the two constraints

$$g(x_1, x_2, \dots, x_n) = c \text{ and } h(x_1, x_2, \dots, x_n) = d$$

- Then, the Lagrangian equation can be constructed as follows

$$L = f(x_1, x_2, \dots, x_n) + \lambda[c - g(x_1, x_2, \dots, x_n)] + \mu[d - h(x_1, x_2, \dots, x_n)],$$

  where λ and μ are the two Lagrange multipliers.

- The first-order condition will consist of $(n + 2)$ equations

$$L_i = f_i - \lambda g_i - \mu h_i = 0, \quad i = 1,2, \dots, n$$

$$L_\lambda = c - g(x_1, x_2, \dots, x_n) = 0$$

$$L_\mu = d - h(x_1, x_2, \dots, x_n) = 0$$

- We can solve for stationary values of L from these equations.

Example

- Write the Lagrangian equation and the first-order condition for

$$\max_{x,y} \; z = x^2 + 2xy + yw^2$$

$$\text{s.t.} \quad 2x + y + w^2 = 24 \;\text{ and }\; x + w = 8.$$

Second-Order Conditions: Second-Order Total Differential

- Let us come back to the problem

$$\max_{x,y}\ z = f(x,y)$$

$$\text{s.t.}\ \ g(x,y) = c.$$

- The constraint $g(x,y) = c$ implies that

$$dg = g_x dx + g_y dy = 0.$$

- After rearranging, we get

$$dy = -\left(\frac{g_x}{g_y}\right) dx.$$

That is once the value of dx is specified, dy will depend on g_x and g_y, but since the latter derivatives depend on the variables x and y, dy will also depend on x and y.

Second-Order Conditions: Second-Order Total Differential

- Obviously, the earlier formula for d^2z, which is based on the arbitrariness of both dx and dy, can no longer apply.
- To find an appropriate expression of d^2z, we must treat dy as a variable dependent on x and y. That is

$$
\begin{aligned}
d^2z = d(dz) &= \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy \\
&= \frac{\partial}{\partial x}\left(f_x dx + f_y dy\right)dx + \frac{\partial}{\partial y}\left(f_x dx + f_y dy\right)dy \\
&= \left[f_{xx}dx + (f_{xy}dy + f_y\frac{\partial(dy)}{\partial x})\right]dx + \left[f_{yx}dx + (f_{yy}dy + f_y\frac{\partial(dy)}{\partial y})\right]dy \\
&= f_{xx}dx^2 + f_{xy}dydx + f_y\frac{\partial(dy)}{\partial x}dx + f_{yx}dxdy + f_{yy}dy^2 + f_y\frac{\partial(dy)}{\partial y}dy
\end{aligned}
$$

Second-Order Conditions: Second-Order Total Differential

- Since the third and the sixth terms can be reduced to

$$f_y\left[\frac{\partial(dy)}{\partial x}dx+\frac{\partial(dy)}{\partial y}dy\right]=f_y d(dy)=f_y d^2y,$$

  The desired expression for d^2z is

$$d^2z=f_{xx}dx^2+2f_{xy}dxdy+f_{yy}dy^2+f_y d^2y.$$

- The only difference from the previous expression of d^2z (in chapter 11) is the last term, $f_y d^2y$.
- Since the constraint implies that $d^2g=0$, then we can get

$$(d^2g=)g_{xx}dx^2+2g_{xy}dxdy+g_{yy}dy^2+g_y d^2y=0.$$

Second-Order Conditions: Second-Order Total Differential

- Solving this last equation for d^2y and substitute into the above equation of d^2z, we get

$$d^2z = \left(f_{xx} - \frac{f_y}{g_y}g_{xx}\right)dx^2 + 2\left(f_{xy} - \frac{f_y}{g_y}g_{xy}\right)dxdy + \left(f_{yy} - \frac{f_y}{g_y}g_{yy}\right)dy^2$$

$$= (f_{xx} - \lambda g_{xx})dx^2 + 2(f_{xy} - \lambda g_{xy})dxdy + (f_{yy} - \lambda g_{yy})dy^2$$

$$= L_{xx}dx^2 + 2L_{xy}dxdy + L_{yy}dy^2$$

Note that the second expression comes from FOC and the third expression comes from taking the second partial derivatives on L.

Second-Order Conditions

- The second-order conditions still revolve around the algebraic sign of the second-order total differential d^2z, evaluated at a stationary point.
- However, in the present context, we are concerned with the sign definiteness of d^2z, ***not*** for all possible values of dx and dy (not both zero), but ***only*** for those dx and dy values (not both zero) satisfying the linear constraint $dg = 0$.
- Thus, the second-order sufficient conditions are

For maximum of z: d^2z is negative definite, subject to $dg = 0$.

For minimum of z: d^2z is positive definite, subject to $dg = 0$.

Second-Order Conditions: The Bordered Hessian

- Previously, we have

$$d^2z = L_{xx}dx^2 + 2L_{xy}dxdy + L_{yy}dy^2,$$

and

$$(dg =)\ g_x dx + g_y dy = 0.$$

- From the second equation, we get $dy = -\left(g_x/g_y\right)dx$. Substitute dy into the first equation, we have

$$d^2z = L_{xx}dx^2 - 2L_{xy}\left(\frac{g_x}{g_y}\right)dx^2 + L_{yy}\frac{g_x^2}{g_y^2}dx^2$$

$$= \left(L_{xx}g_y^2 - 2L_{xy}g_xg_y + L_{yy}g_x^2\right)\frac{dx^2}{g_y^2}.$$

- It is obvious that d^2z is positive (negative) definite if and only if the expression in parenthesis is positive (negative).

Second-Order Conditions: The Bordered Hessian

- It happens that the determinant

$$\begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = 2L_{xy}g_xg_y - L_{xx}g_y^2 - L_{yy}g_x^2$$

  is exactly the negative of the mentioned expression in parenthesis.

- We, therefore, can conclude that

$$d^2z \text{ is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \text{ subject to } dg = 0 \text{ iff } \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} \begin{Bmatrix} < 0 \\ > 0 \end{Bmatrix}.$$

  The above determinant, often referred to as a ***bordered Hessian***, shall be denoted by $|\overline{H}|$.

Example 1

- Check the SOC for the following problem

$$\max_{x,y}\ z = xy$$

$$\text{s.t.}\ \ x + y = 6.$$

Example 2

- Check the SOC for the following problem

$$\max_{x,y}\ z = x_1^2 + x_2^2$$

$$\text{s.t.}\quad x_1 + 4x_2 = 2.$$

Second-Order Conditions: *n*-Variable Case

- Now, consider the problem $\max_{x,y} \; z = f(x_1, x_2, \dots, x_n) \text{ s.t. } \; g(x_1, x_2, \dots, x_n) = c.$
- We can still apply the bordered Hessian determinant. But this time the conditions must be expressed in terms of the bordered leading principal minors of the Hessian.
- Given a bordered Hessian

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix},$$

Its bordered leading principal minors can be defined as

$$|\overline{H_2}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}, |\overline{H_3}| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}, \dots$$

Second-Order Conditions: *n*-Variable Case

- The conditions for positive and negative definiteness of d^2z are then

$$d^2z \text{ is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \text{ subject to } dg = 0 \text{ iff } \begin{cases} |\overline{H_2}|, |\overline{H_3}|, \ldots, |\overline{H_n}| < 0. \\ |\overline{H_2}| > 0, |\overline{H_3}| < 0, |\overline{H_4}| > 0, \ldots. \end{cases}$$

- Thus, we can conclude the conditions for constrained optimizations as

Condition	Maximum	Minimum
First-order necessary	$L_1 = L_2 = \cdots = L_n = L_\lambda = 0$	$L_1 = L_2 = \cdots = L_n = L_\lambda = 0$
Second-order sufficient	$(-1)^n \lvert\overline{H_n}\rvert > 0, n = 2,3, \ldots$	$\lvert\overline{H_2}\rvert, \lvert\overline{H_3}\rvert, \ldots, \lvert\overline{H_n}\rvert < 0$

Example

- Check for the second-order condition of

$$\max_{x,y}\ z = x^2 + 2xy + yw^2$$

$$\text{s.t.}\quad 2x + y + w^2 = 24\ \text{ and }\ x + w = 8.$$