# 5 Exponential and Logarithmic Functions

## *5.1 Exponential Functions (Revision)*

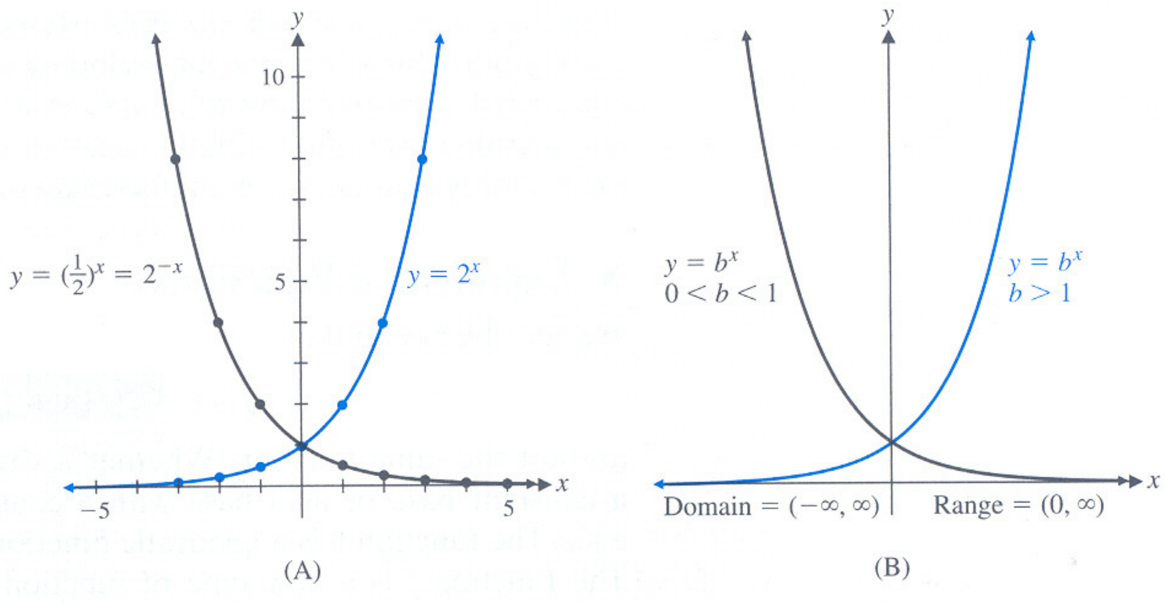
Exponential functions are used extensively in modelling and solving a while variety of real-world problems, including growth of money in compound interest; growth of populations of people, animals, and bacteria; radioactive decay; and learning associated with new technology e.g. computing and manufacturing technology.

### 5.1.1 General form:

$$y = f(x) = b^x \qquad b > 0, b \neq 1$$

Domain is ________________________ and range is ______________________.

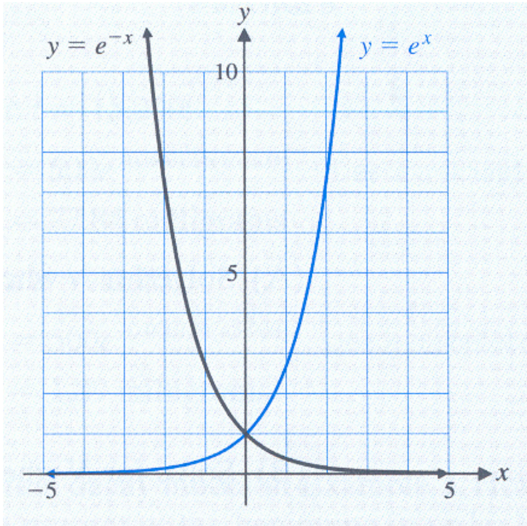
*b* is positive to avoid complex (imaginary) number.




- All graphs will pass through the point (0, 1).
- All graphs are continuous.
- The *x* axis is a horizontal asymptote.
- If $b > 1$, then $b^x$ increases as *x* increases.
- If $0 < b < 1$, then $b^x$ decreases as *x* decreases.

### 5.1.2 Base e Exponential Function

$e = 2.718\ 281\ 828\ 459\ \ldots$




### 5.1.3 Exponential Properties

If $n$ and $m$ are positive integers,

$$b^{n/m} = \left(\sqrt[m]{b}\right)^n = \sqrt[m]{b^n}$$

where $\sqrt[m]{b}$ denotes the positive $m^{\text{th}}$ root

$$b^{-n} = \frac{1}{b^n}$$

$$b^0 = 1$$

$$b^x = b^y \quad \text{if and only if } x = y$$

$$b^m b^n = b^{m+n}$$

$$\left(b^n\right)^m = b^{mn}$$

$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$$

## *5.2 Logarithmic Functions (Revision)*

Logarithmic function is inversed of exponential function.

### 5.2.1 General form:

$y = \log_b x \qquad where \qquad x = b^y, b > 0, b \neq 1$

Domain is ________________________ and range is ______________________.

Note that $\log_b 1 = 0$. If the base $b$ is 10, it is normally omitted. The natural logarithmic of $x$, $\ln(x)$ is $\log_e(x)$.

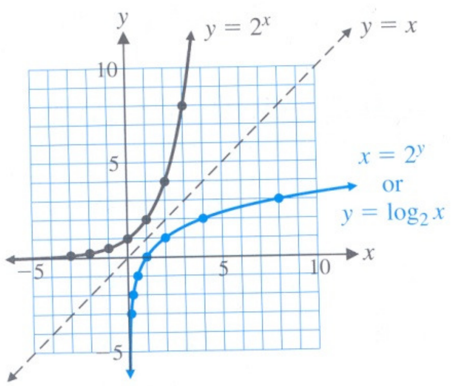


FIGURE 2

| Exponential Function | | Logarithmic Function | |
|---|---|---|---|
| $x$ | $y = 2^x$ | $x = 2^y$ | $y$ |
| $-3$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $-3$ |
| $-2$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $-2$ |
| $-1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ |
| 0 | 1 | 1 | 0 |
| 1 | 2 | 2 | 1 |
| 2 | 4 | 4 | 2 |
| 3 | 8 | 8 | 3 |

Ordered pairs reversed

### 5.2.2 Logarithmic Properties (b > 0 and $b \neq 1$, u and v are also positive numbers)

$\log_b 1 = 0 \qquad$ and $\qquad \log_b b = 1$

$\log_b u = \log_b v$ if and only if $u = v$

$\log_b(uv) = \log_b u + \log_b v$

$\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$

$\log_b u^r = r\log_b u$

$\log_b b^u = u$

$\log_b x = \frac{\ln x}{\ln b}$

## *5.3 Differentiation of Logarithmic and Exponential Functions*

| | |
|---|---|
| $\frac{d}{dx}\left(e^x\right)=e^x$ | $\frac{d}{dx}\left(e^u\right)=e^u\frac{du}{dx}$ |
| $\frac{d}{dx}(\ln x)=\frac{1}{x}$ | $\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{du}{dx}$ |

**Ex. 1:** Find the derivative of the following functions.

(a) $y=2e^x$ Ans: $y'=2e^x$

(b) $y=2e^{3x}$ Ans: $y'=6e^{3x}$

(c) $y=ae^{-bt}$ Ans: $y'=-abe^{-bt}$

(d) $y=5e^{x^4+2x}$ Ans: $y'=5(4x^3+2)e^{x^4+2x}$

(e) $y=\frac{e^{-3x}}{x^2+1}$ Ans: $y'=e^{-3x}\left[\frac{-3x^2-2x-3}{\left(x^2+1\right)^2}\right]$

**Ex.2:** Show that the derivative of $f(x)=xe^{2x}$ is $f'(x)=(2x+1)e^{2x}$. In addition, find the largest and the smallest values of the function $f(x)=xe^{2x}$ on the interval $-1\le x\le 1$.

Ans: Minimum $f\left(-\frac{1}{2}\right)\approx -0.184$ and maximum $f(1)\approx 7.389$

**Ex. 3:** Find the derivative of the following functions.

(a) $y=5\ln x$ Ans: $y'=\frac{5}{x}$

(b) $y=\ln(5x)$ Ans: $y'=\frac{1}{x}$

(c) $y=x\ln x$ Ans: $y'=1+\ln x$

(d) $y=\frac{\ln\sqrt[3]{x^2}}{x^4}$ Ans: $y'=\frac{2}{3}\left[\frac{1-4\ln x}{x^5}\right]$

(e) $y = 3\ln(x^2 - 5x)$ Ans: $y' = \dfrac{3(2x-5)}{x^2 - 5x}$

(f) $y = (x + \ln x)^{\frac{3}{2}}$ Ans: $y' = \dfrac{3}{2}(x + \ln x)^{\frac{1}{2}}\left(1 + \dfrac{1}{x}\right)$

(g) $y = \log_b x$ Ans: $y' = \dfrac{1}{(\ln b)x}$

(h) $y = \log_b u$ ($u$ is a function of $x$) Ans: $y' = \dfrac{1}{(\ln b)u} \cdot \dfrac{du}{dx}$

***Generalised Derivative of Logarithmic Functions***

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u} \cdot \frac{du}{dx}$$

**Ex. 4:** Show that the derivative of $f(x) = x - \ln\sqrt{x}$ is $f'(x) = 1 - \dfrac{1}{2x}$. In addition, show that an equation for the tangent line to the graph of $f(x) = x - \ln\sqrt{x}$ at the point where $x = 1$ is $y = 0.5x + 0.5$.

***Derivative of Exponential Functions to the Base 'a'*** ($a$ is a real number)

We can use the formula for the derivatives of $e^u$ to differentiate a more general exponential function $a^u$. Replacing $a$ with its equivalent from $e^{\ln a}$. $[a = e^{\ln a}]$

$$\begin{aligned} \frac{d(a^u)}{dx} &= \frac{d}{dx}\left[\left(e^{\ln a}\right)^u\right] \\ &= \frac{d}{dx}\left(e^{u\ln a}\right) \\ &= e^{u\ln a}\frac{d}{dx}(u\ln a) \\ &= e^{u\ln a}\left(\frac{du}{dx}\right)(\ln a) \end{aligned}$$

$$\frac{d(a^u)}{dx} = a^u \ln a\left(\frac{du}{dx}\right)$$

**Ex. 4:** Determine the derivative of the following functions.

(a) $y = a^x$.  Ans: $y' = a^x \ln a$

(b) $y = 20^x$  Ans: $y' = 20^x \ln 20$

(c) $y = x2^{3x}$  Ans: $y' = 2^{3x}(1 + 3x \ln 2)$

(d) $y = \alpha 10^{-x} + \beta 10^{-x^2}$  Ans: $y' = -\ln 10\left(\alpha 10^{-x} + 2\beta x 10^{-x^2}\right)$

## *5.4 Logarithmic Differentiation*

This technique is often useful to simplify the differentiation of $y = f(x)$ when $f(x)$ involves products, quotients or powers.

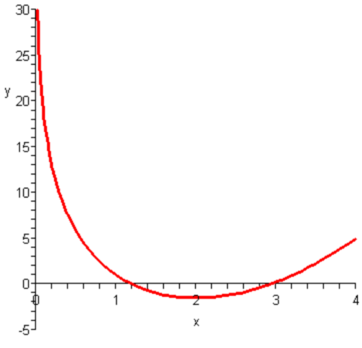
**Ex. 5:** Show that the derivative of $y = 2^x$ is $y' = (\ln 2)2^x$.

**Ex. 6:** Show that the derivative of $y = (x+1)^2(x+2)$ is $y' = (x+1)(3x+5)$.

**Ex. 7:** Show that the derivative of $y = x^{2x+1}$ is $y' = x^{2x+1}\left[2\ln x + \frac{2x+1}{x}\right]$.

## *5.5 Curve Sketching*

**Ex. 8:** $f(x) = x^2 - 8\ln x$ intercepts the $x$ axis at $x \approx 1.2$ and $x \approx 2.9$. Sketch the graph of $f(x) = x^2 - 8\ln x$.




Ans: Domain is positive real number.

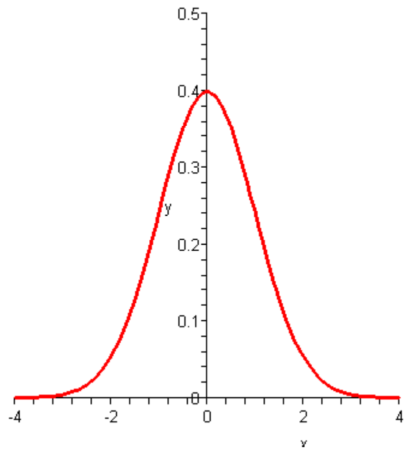
$$f'(x) = \frac{2x^2 - 8}{x}$$

$f(x)$ is decreasing between (0,2) and is increasing between $(2, \infty)$. The minimum point is (2,-1.5). The function is always concave up and there is no inflection point. $x = 0$ is the vertical asymptote and there is no horizontal asymptote.

**Ex. 9:** Determine where the function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is increasing and decreasing, and where its graph is concave up and concave down. Find the relative extrema and inflection points and draw the graph.




Ans: $f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$,

$f''(x) = \frac{-x}{\sqrt{2\pi}} \left(x^2 - 1\right) e^{-\frac{x^2}{2}}$ (0, 0.4) is the relative maximum. (1, 0.24) and (-1, 0.24) are inflection points. The function is increasing for negative $x$ and decreasing for positive $x$. The function is concave up where $x < -1$ and $x > 1$; and concave down where $-1 < x < 1$. There is no $x$ intercept. $y$ = 0 is the horizontal asymptote.