Exercise 3 (Part 2)

1. (a) Prove the statement:
   "There are distinct integers m and n such that $\frac{1}{m} + \frac{3}{n}$ is an integer. "

   (b) Disprove the statement: " For all integers x and y, if $3x+y$ is odd then x and y are both odd. "

2. Show that "for all integers m and n, if mn is even, then m is even or n is even," by using
   a) a proof by contraposition,
   b) a proof by contradiction.

3. Prove by the **method of exhaustion** that " $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$."

4. Use the **proof by cases** to show that " Prove that for all integers m and n, $m + n$ and $m - n$ are either both odd or both even."

   [Hint: Consider 4 cases of even and odd for m and n]

5. Consider the statement: for any integer $n \geq 0$,

   $$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^n = \frac{1}{2}(5^{n+1} - 1).$$

   Suppose we want to prove the above statement by **mathematical induction**.

   (a) What is $P(n)$?
   (b) Write $P(0)$: Is $P(0)$ true?
   (c) Write $P(k)$:
   (d) Write $P(k+1)$:
   (e) Prove the above statement: $\sum_{j=0}^{n} 2 \cdot 5^j = \frac{1}{2}(5^{n+1} - 1)$ by using **mathematical induction**.

6. Use mathematical induction proof to show that

   $$2^n > n^2$$

   for an integer greater than 4.

7. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

8. (Optional) Use the method of constructive proof to show that:
   if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

9. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

10. (Optional) A sequence $a_1, a_2, \ldots$ is defined recursively by

    $$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$

    Show that

    $$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$