

**EE 325 Section 1 (Aj.Wanwiphang) Homework Assignment 1**

**Due date: 31 January 2020 before 11pm**

**\*\* Please submit this assignment on Moodle. For those who work on paper, please scan or submit the pictures of your work. \*\***

1. Find the answers following questions (please also show your calculation)

$$a. \sum_{i=1}^5 (a + bx_i) = \sum_{i=1}^5 a + \sum_{i=1}^5 bx_i = 5a + b \sum_{i=1}^5 x_i = 5a + b(x_1 + x_2 + x_3 + x_4 + x_5)$$

$$b. \sum_{y=0}^5 f(x+y) = f(x+0) + f(x+1) + f(x+2) + f(x+3) + f(x+4) + f(x+5)$$

$$c. \sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + \dots + 10^2 = 385$$

$$d. \sum_{x=1}^2 \sum_{y=2}^3 (2x+y) = \sum_{x=1}^2 (2x+2 + 2x+3) = \sum_{x=1}^2 (4x+5) = 4(1)+5+4(2)+5 = 22$$

2. Given  $X$  is discrete random variable. The probability distribution function (PDF) of this variable is shown in the table

$X$	-2	-1	0	1	2	3	4
$f(x)$	0.5b	b	2.25b	2b	1.5b	0.5b	0.25b

\*\* when b is constant number

a. Find the value of b

$$0.5b + b + 2.25b + 2b + 1.5b + 0.5b + 0.25b = 1$$

$$8b = 1$$

$$b = 0.125$$

b. Find the answer for  $P(X \leq 2)$

$$P(X \leq 2) = P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$$

$$= 0.5(0.125) + 1(0.125) + 2.25(0.125) + 2(0.125) + 1.5(0.125)$$

$$= 0.90625$$

c. Find the answer for  $P(-2 \leq X \leq 3)$

$$P(-2 \leq X \leq 3) = 1 - P(X = 4)$$

$$= 1 - 0.25(0.125)$$

$$= 0.96875$$

d. Find the answer for  $P(X \geq 1)$

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

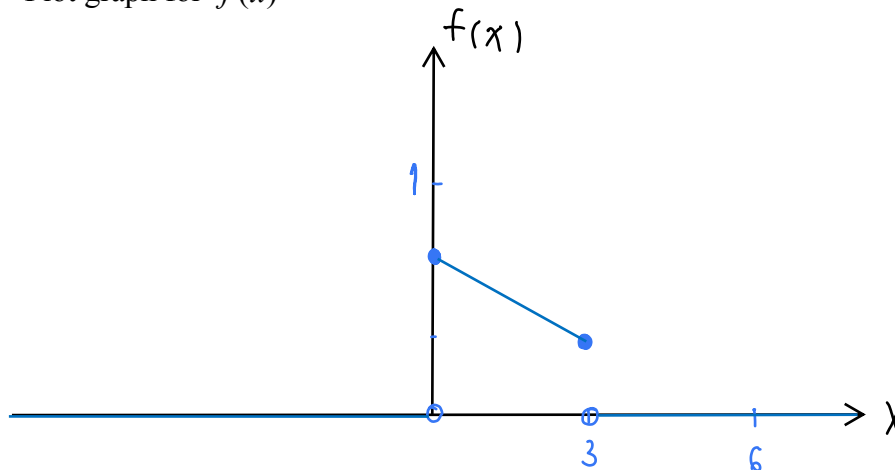
$$= 2(0.125) + 1.5(0.125) + 0.5(0.125) + 0.25(0.125)$$

$$= 0.53125$$

3. Given  $X$  is continuous random variable. The probability distribution function (PDF) of this variable is

$$f(x) = -\frac{1}{9}x + \frac{6}{9}, 0 \leq x \leq 3$$

- a. Plot graph for  $f(x)$



- b. Find the answer for  $P(1 \leq X \leq 3)$

$$\begin{aligned} P(1 \leq X \leq 3) &= \int_1^3 \left(-\frac{1}{9}x + \frac{6}{9}\right) dx \\ &= \left[-\frac{1}{9}\left(\frac{x^2}{2}\right) + \frac{6}{9}x\right]_{x=1}^3 \\ &= -\frac{1}{2} + 2 + \frac{1}{18} - \frac{6}{9} = \frac{8}{9} \end{aligned}$$

- c. Find the answer for  $P(X \geq 2)$

$$\begin{aligned} P(X \geq 2) &= \int_2^3 f(x) dx + \int_3^{\infty} f(x) dx \\ &= \int_2^3 \left(-\frac{1}{9}x + \frac{6}{9}\right) dx + 0 \\ &= \left[-\frac{1}{9}\left(\frac{x^2}{2}\right) + \frac{6}{9}x\right]_{x=2}^3 = -\frac{1}{2} + 2 + \frac{2}{9} - \frac{12}{9} = \frac{7}{18} \end{aligned}$$

- d. Find the expected value of  $X$

$$\begin{aligned} E(X) &= \int_0^3 x f(x) dx = \int_0^3 x \left(-\frac{1}{9}x + \frac{6}{9}\right) dx \\ &= \int_0^3 \left[-\frac{1}{9}x^2 + \frac{6}{9}x\right] dx \\ &= \left[-\frac{x^3}{27} + \frac{6x^2}{18}\right]_{x=0}^3 \\ &= -1 + 3 \\ &= 2 \end{aligned}$$

4. Let random variable  $X$  be the outcome of throwing one dice and random variable  $Y$  be the outcome of tossing one coin. Coin has two sided that has valued 1 and 0.

a. Construct the joint probability distribution function (PDF) table of  $X$  and  $Y$

$Y \backslash X$	1	2	3	4	5	6
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

b. Find the marginal probability distribution function (PDF) of  $X$

$$P(X=1) = \frac{2}{12} = \frac{1}{6} \quad P(X=3) = \frac{2}{12} = \frac{1}{6} \quad P(X=5) = \frac{2}{12} = \frac{1}{6}$$

$$P(X=2) = \frac{2}{12} = \frac{1}{6} \quad P(X=4) = \frac{2}{12} = \frac{1}{6} \quad P(X=6) = \frac{2}{12} = \frac{1}{6}$$

c. Find the marginal probability distribution function (PDF) of  $Y$

$$P(Y=0) = \frac{6}{12} = \frac{1}{2} = 0.5$$

$$P(Y=1) = \frac{6}{12} = \frac{1}{2} = 0.5$$

d. Find the conditional probability distribution function (PDF) of

$X$  given  $Y$  is equal to 1

$$P(X|Y=1) = \frac{P(X, Y=1)}{P(Y=1)} = \frac{P(X, Y=1)}{0.5} = 2 P(X, Y=1)$$

e. Find the expected value of  $X$  given  $Y$  is equal to 1

$$E(X|Y=1) = \sum_i x_i P(X=x_i | Y=1) = \sum_i x_i (2 P(X, Y=1))$$

$$= 2 \left[ 1\left(\frac{1}{12}\right) + 2\left(\frac{1}{12}\right) + 3\left(\frac{1}{12}\right) + 4\left(\frac{1}{12}\right) + 5\left(\frac{1}{12}\right) + 6\left(\frac{1}{12}\right) \right]$$

$$E(X|Y=1) = 2 \left(\frac{21}{12}\right) = \frac{21}{6} = 3.5$$

f. Find the variance of  $X$  given  $Y$  is equal to 1

$$Var(X|Y=1) = E[X^2|Y=1] - [E(X|Y=1)]^2$$

$$= \sum_i x_i^2 P(X_i | Y=1) - (3.5)^2$$

$$= (1^2)(2)\left(\frac{1}{12}\right) + 2^2(2)\left(\frac{1}{12}\right) + \dots + 6^2(2)\left(\frac{1}{12}\right) - (3.5)^2$$

$$= \frac{1}{6} (91) - (3.5)^2 = 2.9667$$

5. If  $X_1, X_2, X_3$  is a random sample from a population with mean  $\mu$  and variance  $\sigma^2$ .  $X_1, X_2, X_3$  are not independent

$$\text{Cov}(X_1, X_2) = \text{Cov}(X_1, X_3) = \text{Cov}(X_2, X_3) = \frac{1}{4}\sigma^2$$

$\bar{X}$  is estimator used to estimate mean value.  $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$

Find  $E(\bar{X})$  and  $\text{var}(\bar{X})$

$$\begin{aligned} E(\bar{X}) &= E\left(\frac{1}{3}(X_1 + X_2 + X_3)\right) \\ &= \frac{1}{3} [E(X_1) + E(X_2) + E(X_3)] \\ &= \frac{1}{3} (3\mu) \end{aligned}$$

$$E(\bar{X}) = \mu$$

$$\begin{aligned} \text{Var}(\bar{X}) &= \text{Var}\left[\frac{1}{3}(X_1 + X_2 + X_3)\right] \\ &= \left(\frac{1}{3}\right)^2 \text{Var}(X_1 + X_2 + X_3) \\ &= \left(\frac{1}{3}\right)^2 \left[ \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + 2\text{Cov}(X_1, X_2) + 2\text{Cov}(X_2, X_3) + 2\text{Cov}(X_1, X_3) \right] \\ &= \left(\frac{1}{3}\right)^2 \left[ 3\sigma^2 + 2\left(\frac{3\sigma^2}{4}\right) \right] \\ &= \frac{1}{9} \left(\frac{9}{2}\sigma^2\right) \end{aligned}$$

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{2}$$

6. Given  $X_1, X_2, X_3, X_4$  are independent identically distributed random variables from population with mean  $\mu$  and variance  $\sigma^2$ .  $\bar{X}$  is estimator used to estimate mean value.  $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$

- a. Find  $E(\bar{X})$  and  $\text{var}(\bar{X})$  in term of  $\mu$  and  $\sigma$

$$\begin{aligned} E(\bar{X}) &= E\left[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right] \\ &= \frac{1}{4} [E(X_1) + E(X_2) + E(X_3) + E(X_4)] \\ &= \frac{1}{4} (4\mu) \\ &= \mu \end{aligned}$$

$$\begin{aligned}
 \text{Var}(\bar{X}) &= \text{Var}\left[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right] \\
 &= \left(\frac{1}{4}\right)^2 \text{Var}(X_1 + X_2 + X_3 + X_4) \\
 &= \left(\frac{1}{4}\right)^2 [\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(X_4)] \\
 &= \left(\frac{1}{16}\right) (4\sigma^2) \\
 \text{Var}(\bar{X}) &= \frac{\sigma^2}{4}
 \end{aligned}$$

- b. Given  $\tilde{X} = \frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4$  is another estimator of  $\mu$ . Show that  $\tilde{X}$  is an unbiased estimator of  $\mu$

$$\begin{aligned}
 E(\tilde{X}) &= E\left[\frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4\right] \\
 &= E\left(\frac{1}{8}X_1\right) + E\left(\frac{1}{4}X_2\right) + E\left(\frac{1}{8}X_3\right) + E\left(\frac{1}{2}X_4\right) \\
 &= \frac{1}{8}E(X_1) + \frac{1}{4}E(X_2) + \frac{1}{8}E(X_3) + \frac{1}{2}E(X_4) \\
 &= \frac{1}{8}\mu + \frac{1}{4}\mu + \frac{1}{8}\mu + \frac{1}{2}\mu
 \end{aligned}$$

$$E(\tilde{X}) = \mu$$

Therefore,  $\tilde{X}$  is an unbiased estimator of  $\mu$

- c. Between  $\bar{X}$  and  $\tilde{X}$ , which one is the better estimator for  $\mu$ ? Why?

$$\begin{aligned}
 \text{Var}(\tilde{X}) &= \text{Var}\left(\frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4\right) \\
 &= \left(\frac{1}{8}\right)^2 \text{Var}(X_1) + \left(\frac{1}{4}\right)^2 \text{Var}(X_2) + \left(\frac{1}{8}\right)^2 \text{Var}(X_3) + \left(\frac{1}{2}\right)^2 \text{Var}(X_4) \\
 &= \frac{1}{64}\sigma^2 + \frac{1}{16}\sigma^2 + \frac{1}{64}\sigma^2 + \frac{1}{4}\sigma^2 \\
 \text{Var}(\tilde{X}) &= \frac{11}{32}\sigma^2
 \end{aligned}$$

Since  $\text{Var}(\bar{X}) < \text{Var}(\tilde{X})$ ,  $\bar{X}$  is the better estimator for  $\mu$ .