

Matrices, Algebra of Matrices & Elementary Operations



What is a Matrix ? (Not a movie trilogy starring Keanu Reeves)

A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called the entries (or elements) of the matrix.

For example,

$$\mathbf{A} = [a_{ij}] = \begin{array}{cccc} \text{Column1} & \text{Column2} & \cdots & \text{Column } n \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & . & \cdots & a_{mn} \end{bmatrix} & \begin{matrix} \text{Row1} \\ \text{Row 2} \\ \\ \text{Row } m \end{matrix} \end{array}$$

$\mathbf{A}$ is a matrix of dimension (size) $m \times n$ (m rows and n columns)

a_{ij} is an entry or an element of the matrix

a_{ii} is a main diagonal entry



$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix} \text{ is a } 2 \times 3 \text{ matrix (Rectangular Matrix)}$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix} \text{ is a } 2 \times 2 \text{ matrix (Square Matrix)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is a } 3 \times 3 \text{ matrix (Square Matrix)}$$

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{ is a diagonal matrix}$$



$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \text{ is an identity matrix or a unit matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix} \text{ is an upper triangular matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 5 \end{bmatrix} \text{ is a lower triangular matrix}$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \text{ is a zero matrix}$$

$[a_1 \ a_2 \ a_3]$ is a 1×3 matrix (we usually call it a row vector)

$$\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \text{ is a } 2 \times 1 \text{ matrix (we usually call it a } \underline{\text{column vector}})$$



Vector

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector. We shall denote vectors by *lowercase* boldface letters $\mathbf{a}$, $\mathbf{b}$, $\dots$ or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \ a_2 \ \dots \ a_n]. \quad \text{For instance,} \quad \mathbf{a} = [-2 \ 5 \ 0.8 \ 0 \ 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$



Basic operations of matrices

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{aligned} a_{11} &= 4, & a_{12} &= 0, \\ a_{21} &= 3, & a_{22} &= -1. \end{aligned}$$



Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ *of the same size* is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

$$\mathbf{A} + \mathbf{B} = [a_{jk} + b_{jk}]_{m \times n}$$

For example;

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{bmatrix}$$



Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c .

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{bmatrix}$$

$$\text{If } \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \text{ then } -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}, \quad \frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$



$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} =$$

$$\mathbf{A} + \mathbf{C} =$$

$$2\mathbf{B} =$$

$$\mathbf{A} - 2\mathbf{B}$$



Properties of Matrix Addition

- Cummulative law** (a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- Associative law of addition** (b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)
- (c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$
- (d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}.$

Properties of Scalar Multiplication

- Distributive laws** (a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$
- (b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$
- (c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)
- (d) $1\mathbf{A} = \mathbf{A}.$



Matrix multiplication

Three different ways with the same answer:

Method 1: Each entry of $\mathbf{AB}$ is the product of a row and a column.

$$(\mathbf{AB})_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} (\mathbf{AB})_{ij} \end{bmatrix}$$

$(\mathbf{AB})_{ij}$ = row i of $\mathbf{A}$ times column j of $\mathbf{B}$

This single entry is the inner product of the two vectors.



Example

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$$



EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute

AB , if it is defined.

Solution: Since A is 2×3 and B is 3×2 , then AB is defined and AB is $\underline{\hspace{1cm}} \times \underline{\hspace{1cm}}$.

$$AB = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 28 & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

EXAMPLE: If A is 4×3 and B is 3×2 , then what are the sizes of AB and BA ?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}$$

$$BA \text{ would be } \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \text{ Which is } \underline{\hspace{2cm}}$$

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.



Method 2: Each column of AB is the product of a matrix and a column

Suppose A is $m \times n$ and B is $n \times p$ where $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_p]$;

$$AB = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_p]$$

$$A_{m \times n} B_{n \times p} = \begin{bmatrix} A_{m \times n} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} & A_{m \times n} \begin{bmatrix} b_{12} \\ b_{21} \\ \vdots \\ b_{n2} \end{bmatrix} & \cdots & A_{m \times n} \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{np} \end{bmatrix} \end{bmatrix}$$

Column j of $AB = A$ times column j of B

The number of columns in A has to equal the number of rows in B .

Example $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$

Example $\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} =$



EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$\begin{aligned} Ab_1 &= \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}, & Ab_2 &= \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -7 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}, & &= \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix} \\ \Rightarrow AB &= \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix} \end{aligned}$$

Note that Ab_1 is a linear combination of the columns of A and Ab_2 is a linear combination of the columns of A .



Method 3: Each row of AB is the product of a row and a matrix

row i of AB = row i of A times B

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} =$$



Properties of Matrix Multiplication

Cautions !

AB is not always equal to BA

Try

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

If $AB = AC$, B is not necessary equal to C

eg.

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$B \neq C$ but $AB = AC$

If $AB = 0$, A or B is not necessary equal to 0

eg.

$$A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad AB = 0$$



Properties of matrix multiplication

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$ (associative law of multiplication)
- $A(B + C) = AB + AC$ (left - distributive law)
- $(B + C)A = BA + CA$ (right-distributive law)
- $r(AB) = (rA)B = A(rB)$
for any scalar r
- $I_m A = A = A I_n$ (identity for matrix multiplication)



Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $A = [a_{jk}]$ is the $n \times m$ matrix A^T (read A transpose) that has the first row of A as its first column, the second row of A as its second column, and so on.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & . & \cdots & a_{mn} \end{bmatrix} \quad A^T = [a_{ji}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Transposition of Matrices and Vectors

If $A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$.



EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute

AB , $(AB)^T$, $A^T B^T$ and $B^T A^T$.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \quad (AB)^T = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$(AB)^T =$$



Properties of Matrix Transposition

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- $(A^T)^T = A$ (i.e., the transpose of A^T is A)
- $(A + B)^T = A^T + B^T$
- For any scalar r , $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$ (i.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ _____.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T (\quad)^T \\ = C^T (\quad) = \text{_____}.$$



Solution of System of Linear Equations



System of Linear Equations

A linear equation

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

Diagram illustrating the components of a linear equation:

- Unknown Variables:** $x_1, x_2, \dots, x_n$
- Given Coefficients:** $a_1, a_2, \dots, a_n$
- Given Constant:** b

A linear equation

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b \quad (*)$$

eg.

$$2x_1 - x_2 + 5x_3 = 2\sqrt{5}$$

Non-linear equation

Anything that is not in the form of a linear equation (*)

eg.

$$2x_1 - x_2^2 + 5x_3 \sin(x_1) = 25$$

System of Linear Equations

A collection of one or more linear equations involving the same set of variables.

A system of linear equations with 2 variables:

$$ax + by = h \quad \text{E.g.} \quad 2x + y = 8$$

$$cx + dy = k \quad x + 3y = 9$$

A system of linear equations with 3 variables:

$$6x_1 + x_2 + x_3 = 6$$

$$5x_1 + x_2 + 2x_3 = 4$$

$$4x_1 + x_2 - x_3 = -2$$

The whole idea of linear algebra is to solve

$$\underline{\mathbf{A}}\mathbf{x} = \underline{\mathbf{b}}$$

A system of linear equations can be written in matrix form

$$6x_1 + x_2 + x_3 = 6$$

$$5x_1 + x_2 + 2x_3 = 4$$

$$4x_1 + x_2 - x_3 = -2$$

$$\begin{bmatrix} 6 & 1 & 1 \\ 5 & 1 & 2 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

or

$$\left[\begin{array}{ccc|c} 6 & 1 & 1 & 6 \\ 5 & 1 & 2 & 4 \\ 4 & 1 & -1 & -2 \end{array} \right]$$

An augmented matrix form

$$[\underline{\mathbf{A}}|\underline{\mathbf{b}}]$$

A matrix equation:

$$\underline{\mathbf{A}}\mathbf{x} = \underline{\mathbf{b}}$$

A system of linear equations with n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad (1)$$

$x_1, x_2, x_3, \dots, x_n$ Is a set of unknown variables

If all b_i are zero then the system is called "Homogeneous system"

If b_i are not all zero then the system is called "Non homogeneous system"

If the system (1) is homogeneous, it has at least the trivial solution $x_1=0, \dots, x_n=0$



Solutions of System of Linear Equations

A linear system of m equations in n unknowns $x_1, \dots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Can be written in a form of matrix equation as

$$\mathbf{Ax} = \mathbf{b} \longrightarrow [\mathbf{A}|\mathbf{b}]$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times n \qquad n \times 1 \qquad m \times 1$



Geometric Interpretation. Existence and Uniqueness of Solutions

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

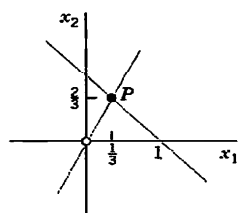
$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

There are 3 possible cases

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - x_2 &= 0 \end{aligned}$$

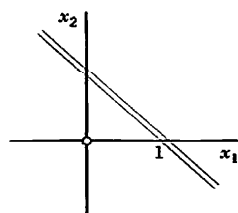
Case (a)



(a) Precisely one solution if the lines intersect.

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned}$$

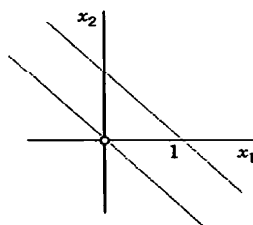
Case (b)



(b) Infinitely many solutions if the lines coincide.

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 0 \end{aligned}$$

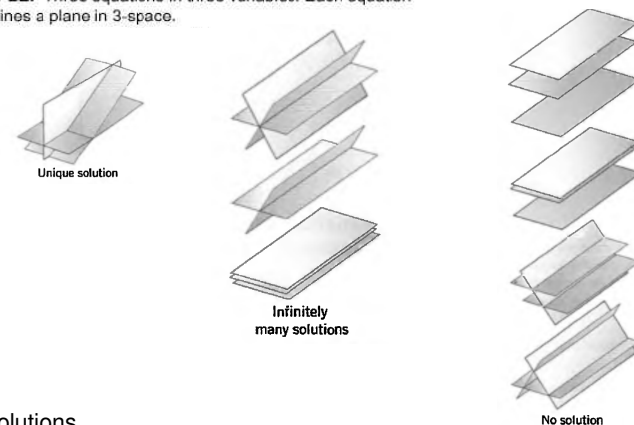
Case (c)



(c) No solution if the lines are parallel



EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.



Solutions

Unique or Infinitely many solutions $\longrightarrow$ Consistent system

No solution $\longrightarrow$ Inconsistent system

How can we know ?



Strategy for solving a linear system

Replace one system with **an equivalent system** (one with the same solution set) that is easier to solve.

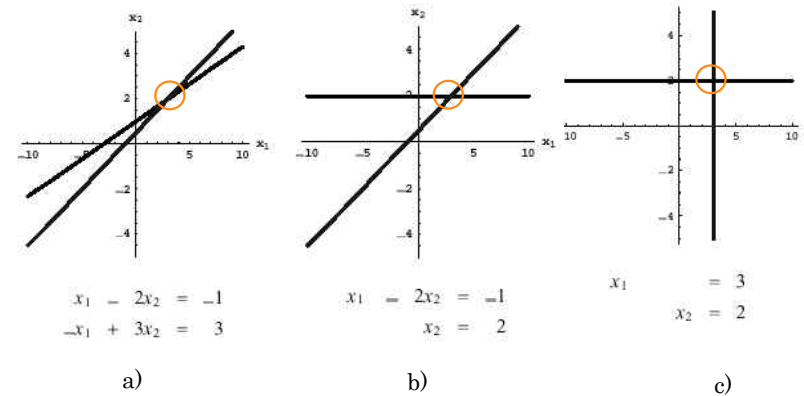
example

$$\begin{array}{l} \text{a)} \quad x_1 - 2x_2 = -1 \\ \quad -x_1 + 3x_2 = 3 \end{array}$$

$$\begin{array}{l} \text{b)} \quad x_1 - 2x_2 = -1 \\ \quad \quad \quad x_2 = 2 \end{array}$$

$$\begin{array}{l} \text{c)} \quad x_1 = 3 \\ \quad \quad \quad x_2 = 2 \end{array}$$

Equivalent systems



Solving a linear system

Elementary Row Operations:

1. (*Replacement*) Add one row to a multiple of another row.
2. (*Interchange*) Interchange two rows.
3. (*Scaling*) Multiply all entries in a row by a nonzero constant.

Note: **Row equivalent matrices:** Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Replacement: $k \times \text{Row } i$ adds to Row j and **replace Row j (the one that is not multiplied.)**



Example Solving a system of linear equations using augmented matrix methods.

$$3x_1 + 4x_2 = 1$$

$$x_1 - 2x_2 = 7$$

1. Augmented matrix corresponding to the system of linear equations.

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array} \right]$$

2. $R_1 \leftrightarrow R_2$ (To get a 1 in the upper left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array} \right]$$



3. $(-3)R_1 + R_2 \rightarrow R_2$ (To get a 0 in the lower left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array} \right]$$

4. $\left(\frac{1}{10}\right)R_2 \rightarrow R_2$ (To get a 1 in the lower right corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array} \right]$$

5. $(2)R_2 + R_1 \rightarrow R_1$ (To get a 0 in the upper right corner.)

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array} \right]$$

Hence, $x_1 = 3$ and $x_2 = -2$.



We can stop the row operation process at step 4 and perform back substitution to obtain the solution set. This method is known as "**Gauss Elimination**" method.

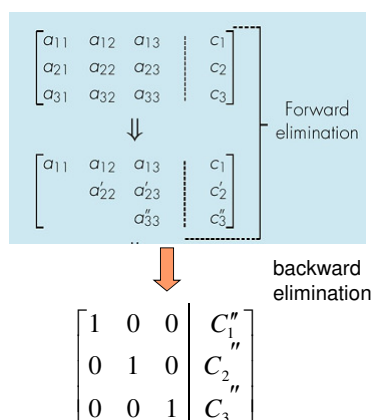
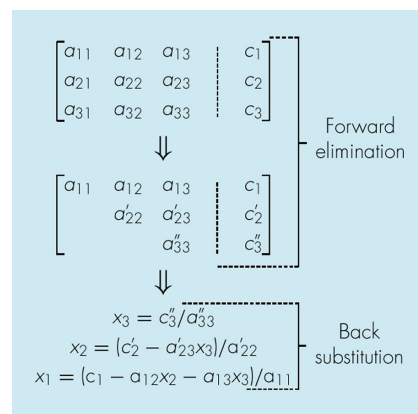
4. $\left(\frac{1}{10}\right)R_2 \rightarrow R_2$ (To get a 1 in the lower right corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array} \right]$$

$$x - 2y = 7 \quad (1)$$

$$y = -2 \quad (2)$$

Solve for y first in eq. (2)
and then substitute y into eq.(1)
to solve for x



Example of system of linear equations with 3 variables

An augmented matrix

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ -4x_1 + 5x_2 + 9x_3 & = & -9 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \quad \begin{array}{l} R_2 + 4R_1 \rightarrow R_2 \\ \\ R_3 + 4R_1 \rightarrow R_3 \end{array}$$

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ -3x_2 + 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \quad \begin{array}{l} R_2/2 \\ \\ R_3 + 3R_2 \rightarrow R_3 \end{array}$$

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ x_2 - 4x_3 & = & 4 \\ -3x_2 + 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \quad \begin{array}{l} \\ \\ R_3 + 3R_2 \rightarrow R_3 \end{array}$$



$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ x_2 - 4x_3 & = & 4 \\ -3x_2 + 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \quad R_2 + 4R_3 \rightarrow R_2$$

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ x_2 - 4x_3 & = & 4 \\ x_3 & = & 3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \quad R_1 + 2R_2 \rightarrow R_1$$

$$\begin{array}{rcl} x_1 - 2x_2 & = & -3 \\ x_2 & = & 16 \\ x_3 & = & 3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$



$$\begin{array}{rcl} x_1 & = & 29 \\ x_2 & = & 16 \\ x_3 & = & 3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Check: Is (29, 16, 3) a solution of the *original* system?

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$(29) - 2(16) + 3 = 29 - 32 + 3 = 0$$

$$2(16) - 8(3) = 32 - 24 = 8$$

$$-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9$$



Row operations are reversible.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution sets.

Example 1

$$6x_1 + x_2 + x_3 = 6$$

$$5x_1 + x_2 + 2x_3 = 4$$

$$4x_1 + x_2 - x_3 = -2$$

$$x_1 = 3, x_2 = -13, x_3 = 1$$



Example 2

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$x_1 = -1 + \frac{4}{3}x_3$$

$$x_2 = 2$$



Example 3

$$3x_2 - 6x_3 = 8$$

$$x_1 - 2x_2 + 3x_3 = -1$$

$$5x_1 - 7x_2 + 9x_3 = 0$$

Inconsistent



Example 4

$$2x_1 - 4x_2 + x_3 = -4$$

$$4x_1 - 8x_2 + 7x_3 = 2$$

$$-2x_1 - 4x_2 - 3x_3 = 5$$

Inconsistent



Example 5 : For what values of h will the following system be consistent?

$$3x_1 - 9x_2 = 4$$

$$-2x_1 + 6x_2 = h$$

Solution: Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h + \frac{8}{3} \end{array} \right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = -\frac{8}{3}$.



Example 7 Give an example of $\mathbf{b}$ that will make the linear system consistent.

$$\begin{array}{l} x_1 + 2x_2 + 2x_3 = b_1 \\ 2x_1 + 4x_2 + 6x_3 = b_2 \\ 3x_1 + 6x_2 + 8x_3 = b_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 2 & 4 & 6 & b_2 \\ 3 & 6 & 8 & b_3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & 2 & b_3 - 3b_1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & -b_2 + 2b_1 + b_3 - 3b_1 \end{array} \right]$$

For consistent system $0 = b_3 - b_2 - b_1$

Eg. $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$



Echelon form (or row echelon form):

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

EXAMPLE: Echelon forms

$$(a) \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

$$(c) \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The whole purpose of doing the elimination is to get an upper triangular matrix (U) (echelon form)



Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$



Important terms

Pivot position: a position of a leading entry in an echelon form of the matrix.

Pivot: a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's

Pivot column: a column that contains a pivot position.

Pivot row: a row that contains a pivot position.

Number of pivots = **Rank of a matrix**

Example : 3 equations 3 unknowns

$$\begin{matrix} \text{(coefficient matrix)} \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$



$$\begin{bmatrix} \boxed{2} & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

The first pivot
Pivot row
Pivot column

$$\text{Replacement} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 8 & 3 \end{bmatrix} \xrightarrow{\text{Replacement}} \begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

U

By definition, pivots cannot be zero. (need to divide them!)



Example Row reduce to echelon form and locate the pivot columns.

$$\begin{array}{c}
 \begin{matrix} & & \text{pivot} \\ & \nearrow & \\ \left[\begin{array}{cccccc} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array} \right] & \sim & \left[\begin{array}{cccccc} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right] & \begin{matrix} R_2+R_1 \rightarrow R_2 \\ R_3+2R_1 \rightarrow R_3 \\ R_1 \leftrightarrow R_4 \end{matrix} & \left[\begin{array}{cccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right] \\
 & & \text{pivot column} & & \text{Possible Pivots:}
 \end{matrix}
 \end{array}$$

$$\begin{array}{c}
 \begin{matrix} R_2-[5/2]R_2 \rightarrow R_2 \\ R_4+[3/2]R_2 \rightarrow R_4 \end{matrix} \left[\begin{array}{cccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array} \right] \sim \left[\begin{array}{cccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]
 \end{array}$$

Note: There is no more than one pivot in any row.
There is no more than one pivot in any column.

Original Matrix: $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

pivot columns: $\begin{matrix} \uparrow & \uparrow & & \uparrow \\ 1 & 2 & & 4 \end{matrix}$



EXAMPLE: Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ R_2-R_1 \rightarrow R_2 \end{matrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$



Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{R_2/2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} R_1-6R_3 \rightarrow R_1 \\ R_2-R_3 \rightarrow R_2 \end{matrix} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Echelon form

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1+9R_2 \rightarrow R_1} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Reduced echelon form



Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

How many pivot variables?



How could this 3x3 case fail to give answer?

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

-Fail to come up with 3 pivots.

Example

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

If this entry is changed to 0
→ matrix would not have been invertible



The breakdown of elimination

Under what circumstances could the elimination break down?

If the algorithm produces n pivots (a full set of pivots), then there is only one solution to the equations (nonsingular case)

If a **zero** appears in a pivot position, elimination has to stop!

Stop temporarily → there is possibility to exchange with a lower row for a proper pivot.

Stop permanently → there is no exchange of row that can avoid zero.

However, we do not know whether a zero will appear until we try.



Two Fundamental Questions (Existence and Uniqueness)

- 1) Is the system consistent; (i.e. does a solution **exist**?)
- 2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

If a solution exists either a unique solution or infinitely many solutions, the system is said to be consistent. Otherwise the system is inconsistent.

How an echelon form (or rref) of a matrix (obtained by performing Row operations on a matrix) tell us about the nature of the solution (e.g. Unique solution, infinitely many solutions or no solution)?



EXAMPLE: Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

In the last example, this system was reduced to the triangular form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

This is sufficient to see that the system is consistent and unique. Why?



EXAMPLE: Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \quad \left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array} \right]$$

Solution: Row operations produce:

$$\left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array} \right]$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \leftarrow \text{Never true} \end{aligned}$$

The original system is inconsistent!



EXAMPLE: For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

Solution: Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h + \frac{8}{3} \end{array} \right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = -\frac{8}{3}$.



example

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 1 & 5 & 6 & 6 \\ 2 & 7 & 8 & 8 \\ 4 & 14 & 16 & 16 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 3 & 4 & 4 \\ 0 & 3 & 4 & 8 \\ 0 & 6 & 8 & 8 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$m-n=2$
These 2 rows are linear combination of other rows

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 1 & 5 & 6 & 6 \\ 2 & 7 & 8 & 8 \\ 3 & 12 & 14 & 14 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ Unique solutions

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 1 & 5 & 6 & 6 \\ 2 & 7 & 8 & 8 \\ 3 & 1 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 3 & 4 & 4 \\ 0 & 3 & 4 & 8 \\ 0 & -5 & -5 & -5 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5/3 & 5/3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ No solutions

(c) $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 \\ 3 & 6 & 6 & 6 \\ 5 & 10 & 10 & 10 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ Many solutions



- Solution to $Ux=0$ is the same as solution to $Ax=0$
- Number of pivot variables (basic variables) = r
- Number of free variables = $n-r$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & x_1 \\ 0 & 0 & 2 & 4 & x_2 \\ 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_4 \end{array} \right] = 0$$

- Assign any number to x_2 and x_4
- Solve for x_1 and x_3



$$\begin{aligned} x_1 &= -2x_2 - 2x_3 - 2x_4 \\ x_3 &= -2x_4 \end{aligned}$$

Parametric description of general solutions



Example

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 2 & 4 & 6 & 8 & 5 \\ 3 & 6 & 8 & 10 & 6 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Ans

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$



Example

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

General solutions

$$x_1 = -1 + \frac{4}{3}x_3$$

$$x_2 = 2$$



Example

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \quad \& \quad b = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}$$

RREF

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 6 & 1 & 7 \\ 5 & 1 & 6 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -5 & -5 \\ 0 & -17 & -17 \\ 0 & -14 & -14 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$r=n < m$
(full column rank)



Solutions of linear systems

Pivot variable (Basic variable) :

→ any variables that corresponds to a pivot column

Free variable: all nonbasic variables

EXAMPLE:

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{array}{rclcl} x_1 + 6x_2 + 3x_4 & = & 0 \\ x_3 - 8x_4 & = & 5 \\ x_5 & = & 7 \end{array}$$

Pivot columns:

Pivot variables:

Free variables



Final step in solving a consistent linear system: After the augmented matrix is in reduced echelon form and the system is written down as a set of equations

Solve each equation for the pivot variables in terms of the free variables (if any) in the equations

EXAMPLE:

$$\begin{array}{rclcl} x_1 + 6x_2 & + 3x_4 & = & 0 \\ x_3 - 8x_4 & = & 5 \\ x_5 & = & 7 \end{array} \quad \left\{ \begin{array}{l} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \\ \text{(general solution)} \end{array} \right.$$

The system is consistent and has infinite many solutions



EXAMPLE:

$$\begin{bmatrix} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{bmatrix}$$

In an earlier example, we obtained the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

Free variables: x_3 and x_4

Consistent system with free variables $\Rightarrow$ infinitely many solutions.



EXAMPLE:

$$\begin{array}{rcl} 3x_1 + 4x_2 & = & -3 \\ 2x_1 + 5x_2 & = & 5 \\ -2x_1 - 3x_2 & = & 1 \end{array} \rightarrow \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

Consistent system, no free variables

Full column rank $\Rightarrow$ unique solution.

$$\begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Inverse of Matrices



Matrix Inverses

The inverse of a real number a is denoted by a^{-1} . For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A .

example

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} ; \quad C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$AC =$

$CA =$

The inverse of A is usually denoted by A^{-1} .

We have

$$AA^{-1} = A^{-1}A = I_n$$

Not all $n \times n$ matrices are invertible. A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.



Fact 1 If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A . Then

$$B = BI = B(\text{---}) = (\text{---})\text{---} = I\text{---} = C.$$

So the inverse is unique since any two inverses coincide. ■

Fact 2 The inverse of A^{-1} is A itself.

Fact 3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.



Assume A is any invertible matrix and we wish to solve $AX = b$. Then

$$\text{---}AX = \text{---}b \quad \text{and so}$$

$$IX = \text{---} \text{ or } X = \text{---}.$$

Suppose w is also a solution to $AX = b$. Then $Aw = b$ and

$$\text{---}Aw = \text{---}b \quad \text{which means} \quad w = A^{-1}b.$$

So, $w = A^{-1}b$, which is in fact the same solution.

We have proved the following result:

Fact 4 If A is an invertible $n \times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $AX = b$ has the unique solution $x = A^{-1}b$.



EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$x = A^{-1}b = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$



Properties of Inverses

Suppose A and B are invertible. Then the following results hold:

- A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\text{_____})A^{-1}$$

$$= A(\text{_____})A^{-1} = \text{_____} = \text{_____}.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Proof part c

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Matrix inversion algorithm

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ Take a column at a time, that equation determines the columns of A^{-1}

A times column j of A^{-1} = column j of I

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$$

Carry out elimination on all systems simultaneously.



$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$$

$$[A \mid I]$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$[I \mid A^{-1}]$$

The Gauss-Jordan method

Matrix inversion algorithm

Place A and I side-by-side to form an augmented matrix $[A \mid I]$. Then perform row operations on this matrix (which will produce identical operations on A and I).

$[A \mid I]$ will row reduce to $[I \mid A^{-1}]$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \mid I] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

$$\text{So } A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$$



Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists



Determinant of Matrices



Determinants

Determinant of a square matrix $\mathbf{A}$ is denoted by $|\mathbf{A}|$ or $\det \mathbf{A}$ is a uniquely defined SCALAR associated with that matrix. Determinants are defined only for square matrices

A second-order determinant

For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad \text{is a scalar}$$

Example : Given

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 5 & 9 \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (6)(9) - (-3)(5) = 69$$



A Third-order determinant

For a 3x3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}) - (a_{13}a_{31}a_{22} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

or

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \\ a_{31} & a_{31} \end{vmatrix} a_{13}$$



OR

Subdeterminant or Minor

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \quad [= \text{a scalar}]$$



A Third-order determinant

Example :

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (2)(5)(9) + (1)(6)(7) + (3)(8)(4) - (2)(8)(6) - (1)(4)(9) - (3)(5)(7) = -9$$

$$\begin{vmatrix} -7 & 0 & 3 \\ 9 & 1 & 4 \\ 0 & 6 & 5 \end{vmatrix} = (-7)(1)(5) + (0)(4)(0) + (3)(6)(9) - (-7)(6)(4) - (0)(9)(5) - (3)(1)(0) = 295$$



An n^{th} -order determinant

For an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{n \times n}$$

By expanding

Any row i $\det A$ or $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in} = \sum_{j=1}^n a_{ij}C_{ij}$

Any column j $\det A$ or $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj} = \sum_{i=1}^n a_{ij}C_{ij}$

C_{ij} is the "cofactor" of the element

M_{ij} is the "minor" of the element a_{ij}

Obtained by deleting the i th row and j th column of a given determinant



The (i,j) -cofactor of A is the number C_{ij} where

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

A cofactor expansion across the first row of A

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$



Example

Evaluate determinant of $\mathbf{A}$, given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Choose row 1 for expansion, since there is 0 in row 1

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$C_{12} = (-1)^{1+2}(M_{12}) \rightarrow M_{12} = \begin{vmatrix} 1 & 4 \\ 5 & 7 \end{vmatrix} = (1)(7) - (4)(5) = -13$$

$$C_{13} = (-1)^{1+3}(M_{13}) \rightarrow M_{13} = \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} = (1)(6) - (3)(5) = -9$$

$$\det \mathbf{A} = (1)(3) + (2)(-9) = -5$$



EXAMPLE: Compute the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\det \mathbf{A} = ?$$

(-2)



EXAMPLE: Compute the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

(14)



Basic properties of determinants

(1) $\det \mathbf{A}^T = \det \mathbf{A}$

Example 1 $\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 3 & 6 \end{vmatrix} = 9$

Example 2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

Hence, column operations = row operations in determinant. (2)-(4)

(2) The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

Example $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc)$

$$\begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{vmatrix} = -26, \quad \begin{vmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 26.$$



- (3) The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k -fold.

Example

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} 15a & 7b \\ 12c & 2d \end{vmatrix} = 3 \begin{vmatrix} 5a & 7b \\ 4c & 2d \end{vmatrix} = 3(2) \begin{vmatrix} 5a & 7b \\ 2c & d \end{vmatrix} = 6(5ad - 14bc)$$



- (4) The addition (subtraction) of a multiple of any row (or column) to (from) another row will leave the value of the determinant unaltered.

Example

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$



EXAMPLE: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$



EXAMPLE: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.



EXAMPLE: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

(-10)



Compute $\text{Det}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

$\text{Det}(\mathbf{A}) = 0$ when $\mathbf{A}$ is not invertible (singular).

(0)



- (5) If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0 \quad \begin{vmatrix} c & c \\ d & d \end{vmatrix} = cd - cd = 0$$

- (6) A zero row or column renders the value of a determinant zero.

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 = \begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 8 \\ 0 & 3 & 9 \end{vmatrix}$$



Further Properties

(7) $\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$

EXAMPLE: Compute $\det \mathbf{A}^3$ if $\det \mathbf{A} = 5$.

Solution: $\det \mathbf{A}^3 = \det(\mathbf{A}\mathbf{A}\mathbf{A}) = (\det \mathbf{A})(\det \mathbf{A})(\det \mathbf{A})$
 $= \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$



(8) If $\mathbf{A}$ is an $n \times n$ upper or lower triangular matrix

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \quad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

$$\det \mathbf{A} = a_{11}a_{22}a_{33}\dots a_{nn}$$

Example

$$[A] = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 10 & 0 \\ 4 & 8 & 9 \end{bmatrix} \rightarrow |A| = (2)(10)(9) = \mathbf{180}$$



Applications of determinants

Cramer's Rule

Cramer's rule can be used to study how the solution of $\mathbf{Ax}=\mathbf{b}$ affected by the changes in the entries of $\mathbf{b}$.

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in R^n , the unique solution $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, 3, \dots, n$$

where

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \underset{\substack{\uparrow \\ \text{Col } i}}{\mathbf{b}} \quad \cdots \quad \mathbf{a}_n]$$



Example

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$



Find the solution of the equation system

$$\begin{aligned} 7x_1 - x_2 - x_3 &= 0 \\ 10x_1 - 2x_2 + x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$



The computation of A^{-1}

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \quad (4)$$

The matrix of cofactors on the right side of (4) is called the **adjugate** (or classical **adjoint**) of A , denoted by $\text{adj } A$. (The term *adjoint* also has another meaning in advanced texts on linear transformations.) The next theorem simply restates (4).

The adjoint of $A_{n \times n}$ is defined to be the transpose of the matrix of cofactors:

$$\text{adj} A = [C_{ij}(A)]^T$$

$$\text{adj} A \Rightarrow n \times n$$

Example

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 7 & 3 \\ -2 & 8 & 5 \end{bmatrix}$$

$$\text{adj} A = \begin{bmatrix} 11 & -26 & 46 \\ -8 & 13 & -24 \\ 7 & -13 & 21 \end{bmatrix}^T = \begin{bmatrix} 11 & -8 & 7 \\ -26 & 13 & -13 \\ 46 & -24 & 21 \end{bmatrix}$$

Theorem

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$

EXAMPLE 3 Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

Solution The nine cofactors are

$$C_{11} = + \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = + \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = - \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = - \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance, C_{12} is in the (2, 1) position.] Thus

$$\text{adj } A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

We could compute $\det A$ directly, but the following computation provides a check the calculations above and produces $\det A$:

$$(\text{adj } A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since $(\text{adj } A)A = 14I$, Theorem 8 shows that $\det A = 14$ and

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$