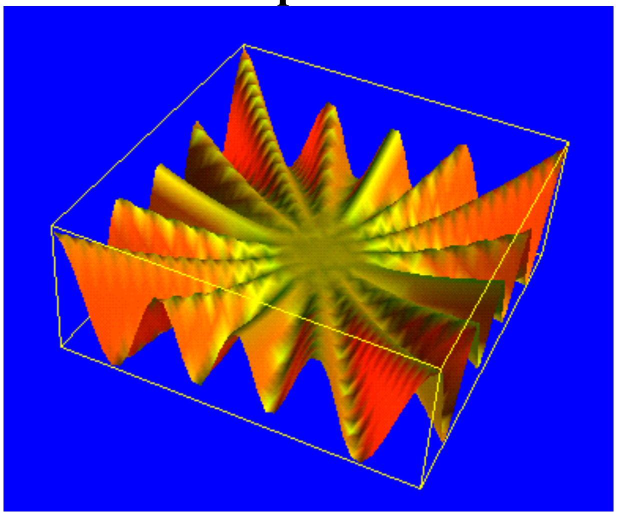
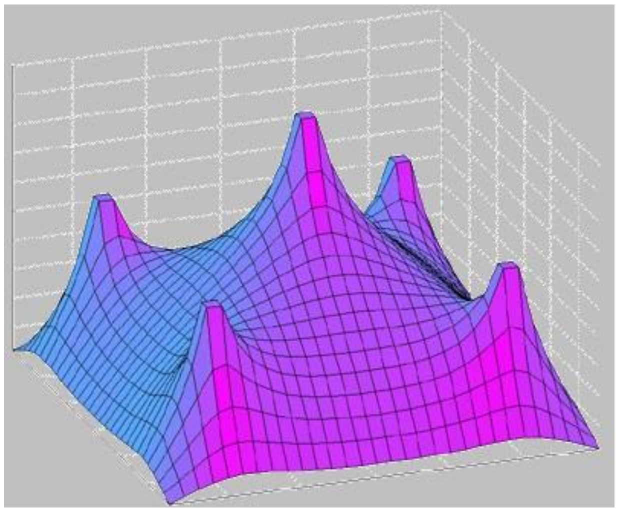
MA216 $y = f(x)$

Optimisation of Multivariable Functions

1. Function of Two or More Variables

Typical relationships in economics are relationships of more than one variable, for example,

$$z = f(x, y)$$

z is a function of two variables x and y. z is dependent variable (dependent on x and y) and x and y are independent variables. It is not stated then the domain is a set of all (x, y) for which the expression $f(x, y)$ is defined (real number).

$f(1,0) = 2(1) + (1)^2(0) = 2$

$f(0,1) = 2(0) + (0)^2(1) = 0$

Ex.1:

(a) $f(x,y) = 2x + x^2y^3$ Find the domain of f, $f(1,0), f(0,1)$ and $f(a+1,b)$.

[Ans: $x = \Re, y = \Re$, 2, 0, $2(a+1)+(a+1)^2b^3$]

$f(a+1,b) = 2(a+1) + (a+1)^2 b^3$

(b) $f(x,y) = \dfrac{3x^2+5y}{x-y}$ Find the domain of f and $f(1,-2)$. → f is not defined when $x - y = 0 \Rightarrow x = y$

$f(1,-2) = \dfrac{3(1)^2 + 5(-2)}{1-(-2)} = \dfrac{3-10}{3} = \dfrac{-7}{3}$

[Ans: $x \neq y$, -7/3]

(c) $f(x,y) = xe^y + \ln x$ Find the domain of f and $f(e^2, \ln 2)$.

[Ans: $x > 0$, $2(e^2+1) \approx 16.78$

(d) $f(x,y,z) = xy + xz + yz$ Find $f(-1,2,5)$.

[Ans: 3]

Ex.2: Production function $F(x,y) = Ax^a y^b$ (Function in this form is called ***'Cobb-Douglas'*** function) where A, a and b are constants. F is number of units produced. x and y are input factors.

Show that

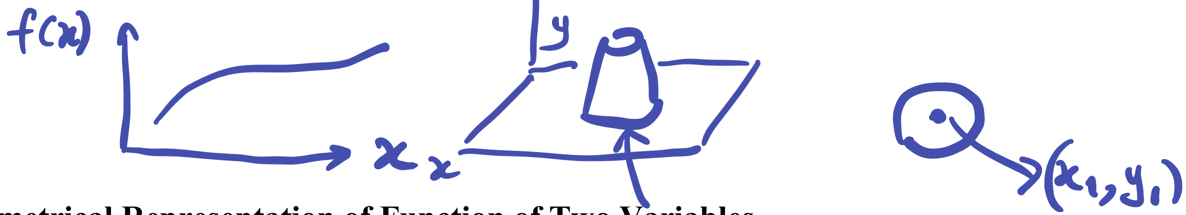
$$F(2x,2y) = 2^{a+b}F(x,y)$$

And

$$F(x+h,y) - F(x,y) = Ay^b\{(x+h)^a - x^a\}.$$

$F(2x,2y) = A(2x)^a(2y)^b$
$= A\,2^a x^a\, 2^b y^b$
$= 2^a \cdot 2^b (Ax^a y^b)$
$= 2^{a+b}(Ax^a y^b)$ #

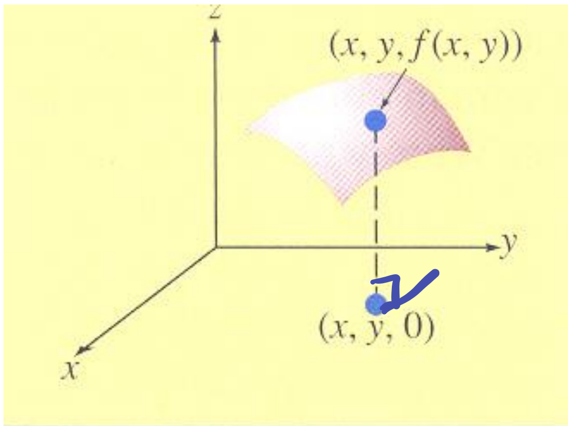
z



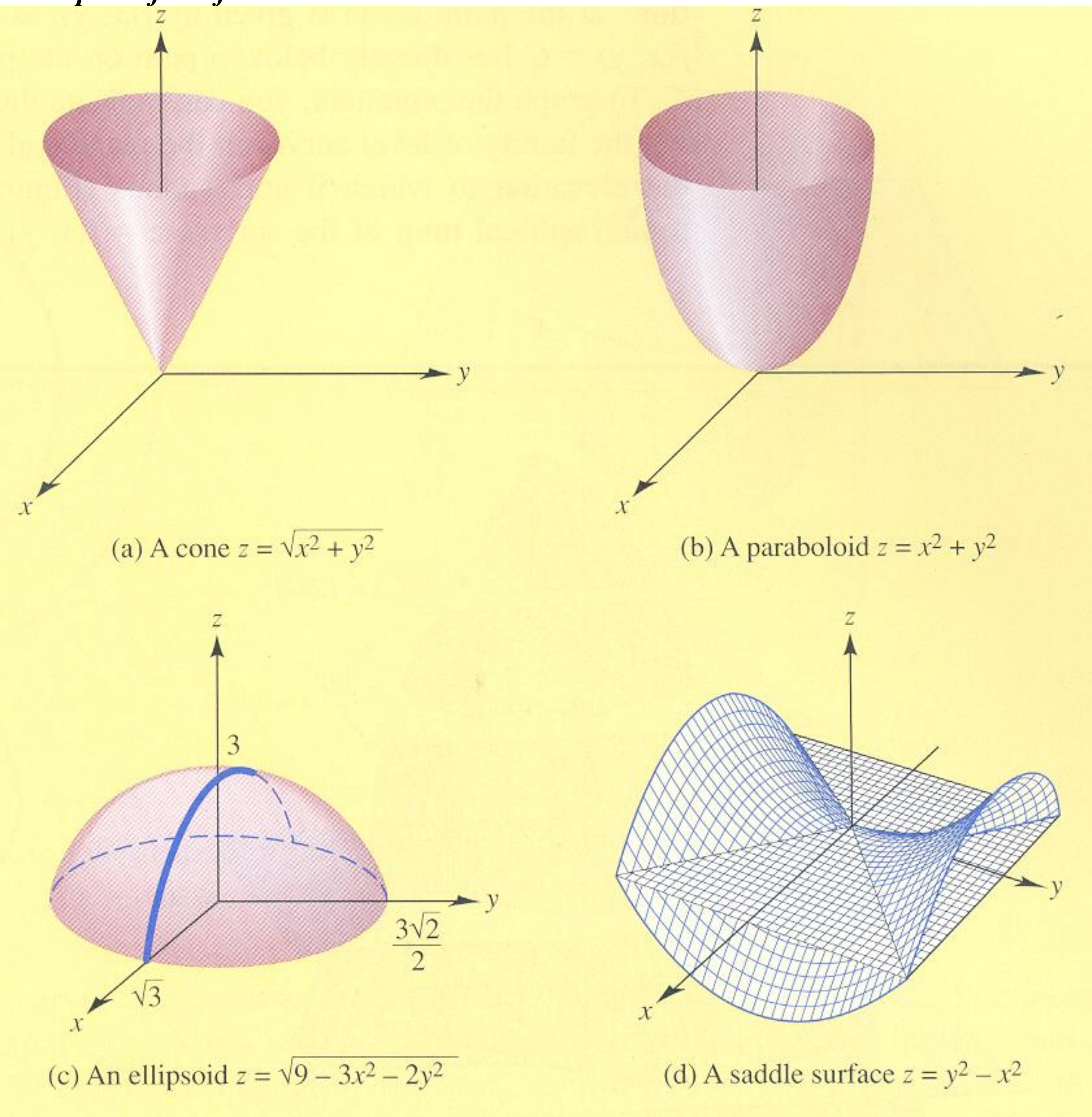
2. Geometrical Representation of Function of Two Variables

The graph of a two-variable function is the set of triple (x, y, z) where (x, y) is in the domain of f and $z = f(x, y)$. $z = f(x, y)$ can be plotted in the three-dimensional rectangular coordinate system.

- The origin of the system is where the x-, y- and z-axes intercepts and all three axes are perpendicular.
- The arrows indicate the positive direction.
- Function of two variables can be represented by a 3-D plot.
- Function of more than two variables cannot be visualised.
- The plot of a two-variable function become a surface in three-dimensional space.



Examples of Surfaces



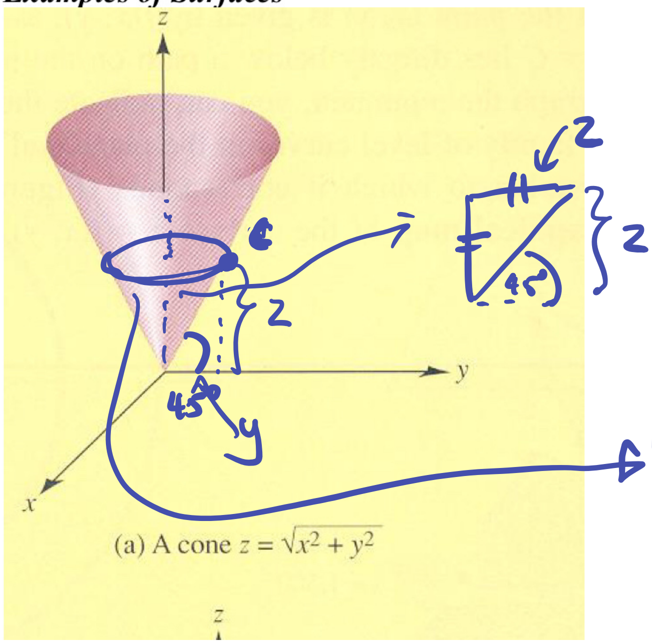
(a) A cone $z = \sqrt{x^2 + y^2}$

(b) A paraboloid $z = x^2 + y^2$

(c) An ellipsoid $z = \sqrt{9 - 3x^2 - 2y^2}$

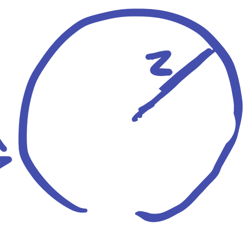
(d) A saddle surface $z = y^2 - x^2$

Examples of Surfaces



(a) A cone $z = \sqrt{x^2 + y^2}$

Then the circle



$\therefore$ radius of this circle $= z$

@ point c coordinate (x, y) height $= z$

point c is a point on a circle

$$\therefore \quad x^2 + y^2 = r^2$$

$$x^2 + y^2 = z^2$$

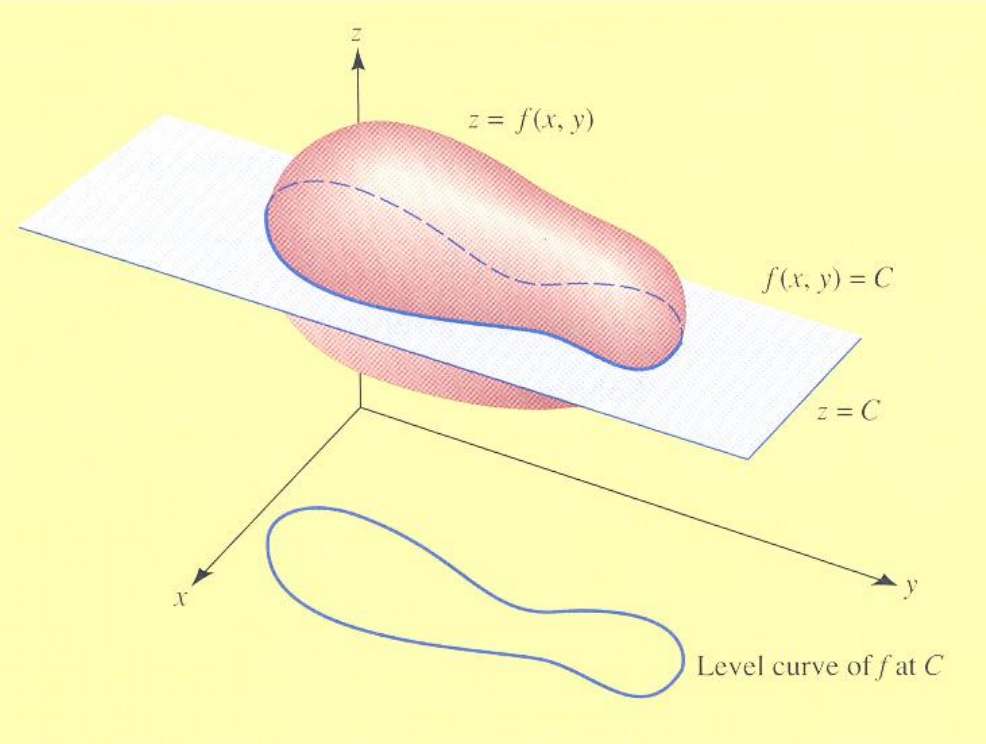
$$z = \sqrt{x^2 + y^2}$$

$\because$ z is (+) in this case

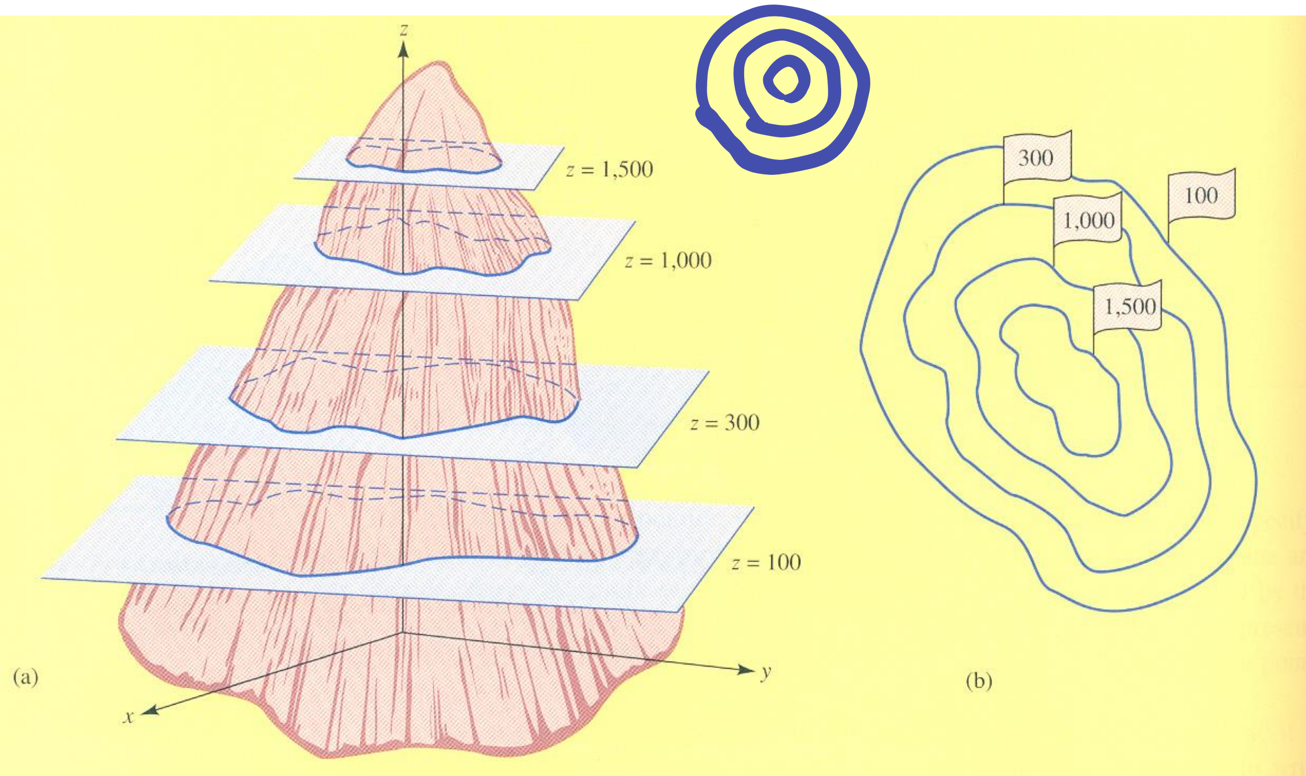
Level Curves: maps

The set of points (x, y) in the plane *xy* plane that satisfy $f(x, y) = C$ is called the **level curve** of *f* at *C*. An entire family of level curves is generated as *C* varies over a set of numbers. These level curves can help to visualise the surface plot.

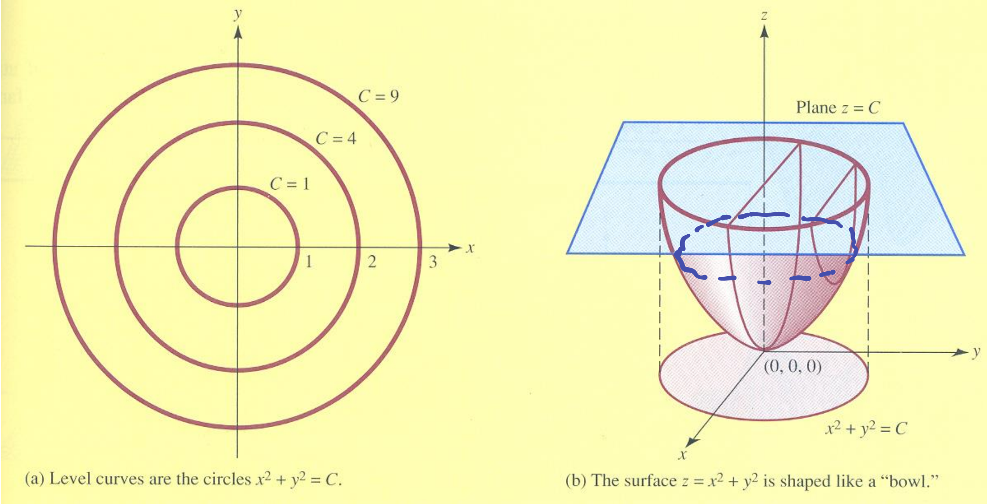
Example of a level curve



Example of the application of level curves in a topographical map for mountain

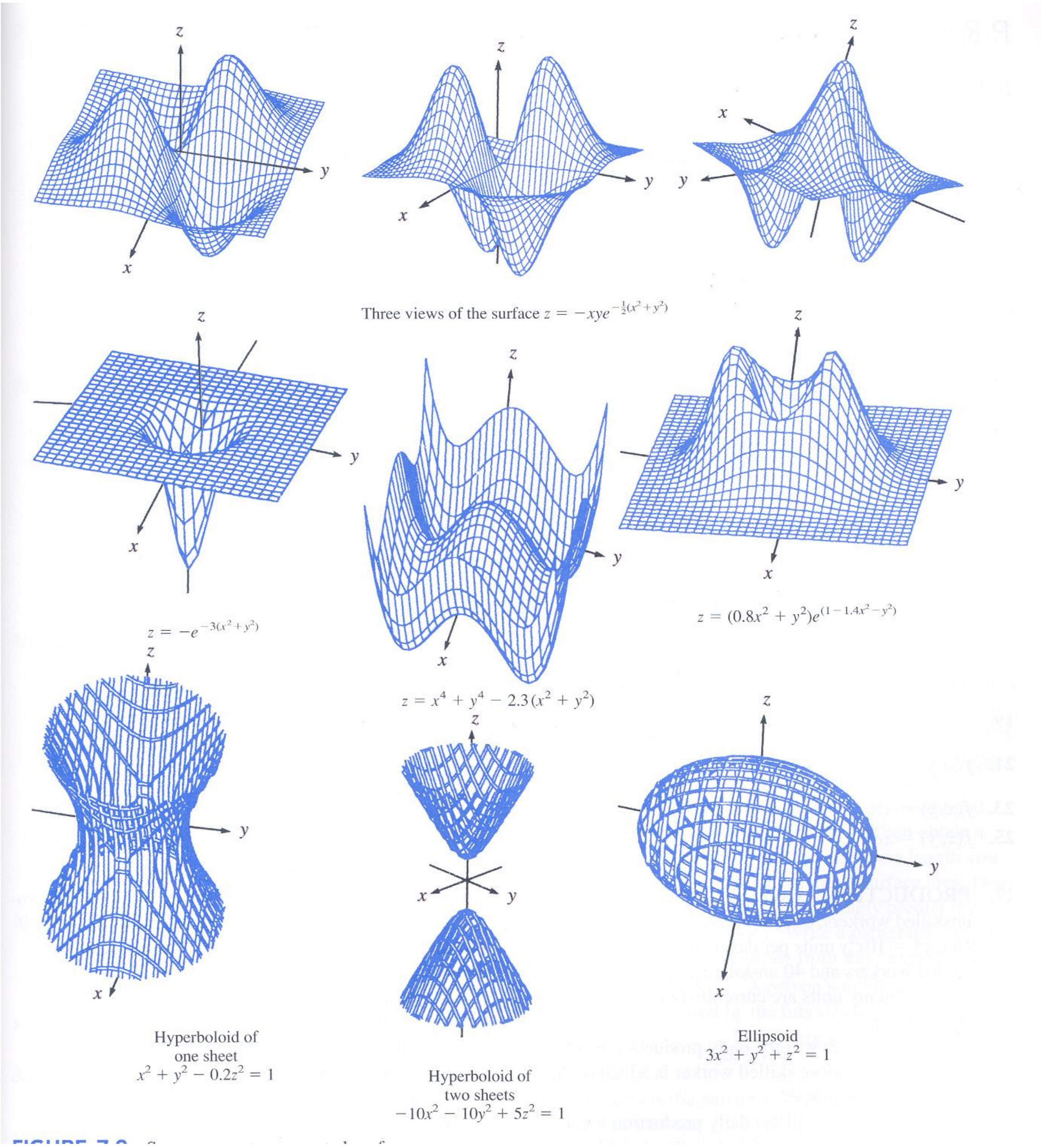


Example of level curves help to visualise the shape of a surface



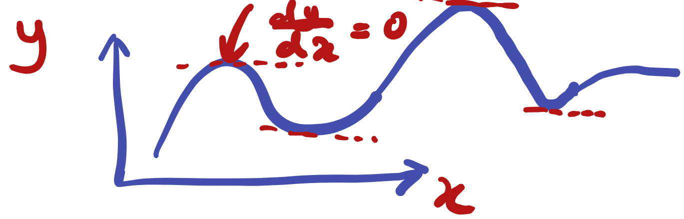
(a) Level curves are the circles $x^2 + y^2 = C$.

(b) The surface $z = x^2 + y^2$ is shaped like a "bowl."



slope = 0

Optimisation → max = ?
→ min = ?



3. Partial Derivatives

Recall a single variable function $y = f(x)$, $\frac{dy}{dx}$ is rate of change of y when x changes.

Now, $z = f(x,y)$, if we want to know how z changes when x **or** y changes, we use partial derivatives.

$$z = x^3 + 2y^2$$

$\rightarrow$ $z = x^3 + 2c^2$
$= 3x^2$

When y is constant, **the rate of change of** z **with respect to** x.

$$\frac{\partial z}{\partial x} = 3x^2 \quad \neq \frac{dz}{dx}$$

Note: We use the notation "∂" not "d" to indicate partial derivative.
When x is constant, **the rate of change of** z **with respect to** y.

$z = c^3 + 2y^2$

$$\frac{\partial z}{\partial y} = 4y$$

Partial Derivative Notations

- Partial derivative of f or z with respect to x $\quad f_x'(x,y), f_x(x,y), \frac{\partial}{\partial x}[f(x,y)]$ and $\frac{\partial z}{\partial x}$
- Partial derivative of f or z with respect to x evaluated at (x_0, y_0)

$$f_x'(x_0,y_0), f_x(x_0,y_0), \left[\frac{\partial z}{\partial x}\right]_{(x_0,y_0)}, \left[\frac{\partial z}{\partial x}\right]_{\substack{x=x_0\\y=y_0}}, \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} \text{ and } \left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}$$

Ex.3: $f(x,y) = x^3y + x^2y^2 + x + y^2$ Find all partial derivatives of $f(x,y)$.

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy^2 + 1 + 0$$
$$\frac{\partial f}{\partial y} = x^3(1) + x^2(2y) + 0 + 2y$$

Ex.4:

Group of 3

(a) $f(x,y) = \frac{xy}{x^2+y^2}$ Find all partial derivatives of $f(x,y)$.

[Ans: $\frac{\partial f}{\partial x} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$ and $\frac{\partial f}{\partial y} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$]

(b) $z = (x^2 + xy + y)^5$ Find all partial derivatives.

[Ans: $\frac{\partial z}{\partial x} = 5(x^2+xy+y)^4(2x+y)$ and $\frac{\partial z}{\partial y} = 5(x^2+xy+y)^4(x+1)$]

↑ Settagorn ↑ Day 1

$= A m^{2.08} p^{-1.5}$

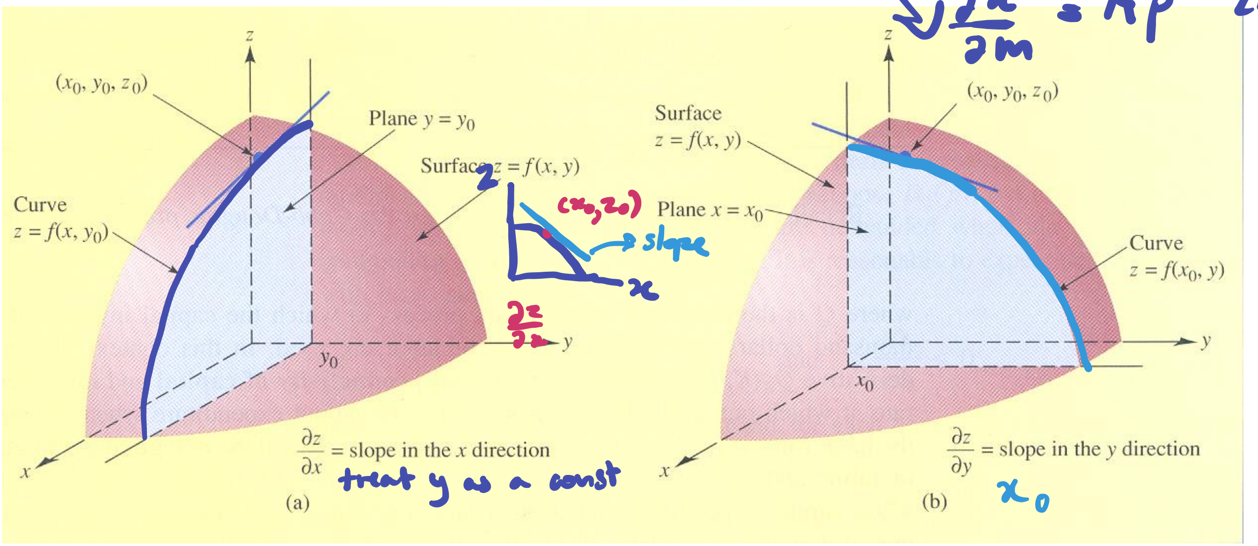
Ex.5: A demand for rice is $x = \dfrac{Am^{2.08}}{p^{1.5}}$ where x is the rice consumption, m is the income per family, p is the price, and A is a constant. Calculate $\dfrac{\partial x}{\partial p}$ and $\dfrac{\partial x}{\partial m}$.

$\dfrac{\partial x}{\partial p} = -1.5\, p^{-2.5} A m^{2.08}$

8 am

$\dfrac{\partial x}{\partial m} = A p^{-1.5}\, 2.08 m^{1.08}$

4. Formal Definition of Partial Derivatives



Slope of tangent line at $(x_0, y_0, f(x_0, y_0))$ = partial derivative of z (or f) with respect to x at (x_0, y_0)

$\dfrac{\Delta z}{\Delta x}$

By definition,
$$\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} = f_x'(x_0,y_0) = \lim_{h\to 0}\left(\frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}\right)$$

Similarly,
$$\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} = f_y'(x_0,y_0) = \lim_{h\to 0}\left(\frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}\right)$$

$\dfrac{\Delta z}{\Delta y}$

Ex.6: A company produces 2 products, the joint cost function per week is given by $C = f(x, y) = 0.07x^2 + 75x + 85y + 6000$. Determine marginal cost $\dfrac{\partial C}{\partial x}$ and $\dfrac{\partial C}{\partial y}$ when $x = 100$ and $y = 50$. [Ans: 89,85]

rate of change of .... w.r.t. ...

↳ derivative

Solution: $\dfrac{\partial C}{\partial x} = 0.14x + 75$ $\qquad \dfrac{\partial C}{\partial y} = 85$

$\left[\dfrac{\partial C}{\partial x}\right]_{(100,50)} = 0.14(100) + 75$ $\qquad \left[\dfrac{\partial C}{\partial y}\right]_{(100,50)} = 85$

$= 89$

$z^2 = x^2 + y^2$

$z = (x^2+y^2)^{\frac{1}{2}}$

5. Implicit Partial Differentiation

z is not given in term of x and y, for example,

x cons

$$z^2 - x^2 - y^2 = 0 \qquad \text{Find } \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

Partial differentiate both sides of the equation with respect to x to find $\frac{\partial z}{\partial x}$. treat y as a const

$$2z\frac{\partial z}{\partial x} - 2x + 0 = 0$$

$2z \frac{\partial z}{\partial x} = 2x$

Rearranging, $$\frac{\partial z}{\partial x} = \frac{x}{z}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y}{z}$$

$2z \frac{\partial z}{\partial y} + 0 - 2y = 0$

$2z \frac{\partial z}{\partial y} = 2y$

Next class to revise

Ex.7: If $z = f(x,y)$ and $e^{yz} = -xyz$, find $\frac{\partial z}{\partial x}$ where $x = -\frac{e^2}{2}$, $y = 1$ and $z = 2$. [Ans: $-\frac{4}{e^2}$]

6. Higher-Order Partial Derivative

For $z = f(x,y)$,

1st order partial derivatives are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2nd order partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

2nd 1st   1st y 2nd

Note: Beyond the 2nd order derivatives can also be determined e.g.

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right] = \frac{\partial^3 f}{\partial y \partial x^2} = f_{xxy}$$

$f_y = 3$

$f_{yy} = 0$

const

Ex.8: For $f(x,y) = 7x^2 + 3y$, find f_y, f_{yy} and f_{yyx}. [Ans: 3, 0, 0]

$f_{yyx} = 0$

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Ex.9: For $f(x,y) = (x+y)^2(xy)$, find $f_x, f_y, f_{xx}, f_{yy}, f_{yyx}$ and f_{xyy}. Day 1

[Ans: $3x^2y + 4xy^2 + y^3, x^3 + 4x^2y + 3xy^2, 6xy + 4y^2, 4x^2 + 6xy, 8x + 6y, 8x + 6y$]

Ex.10: For $f(x,y) = y^2e^x + \ln(xy)$, find $f_{xyy}(1,1)$. [Ans: $2e$]

Ex7 $\quad z = f(x,y) \qquad e^{yz} = -xyz$

$y = e^{3x} = e^{u}$

$\frac{dy}{dx} = e^{u}$

Find $\frac{\partial z}{\partial x}$ where $x = -\frac{e^2}{2}$, $y=1$, $z=2$

$$e^{yz} = -xyz = -y(xz)$$

Diff w.r.t. x (treat y as a constant) both sides

$$ye^{yz}\frac{\partial z}{\partial x} = -y\frac{\partial(xz)}{\partial x}$$

$$ye^{yz}\frac{\partial z}{\partial x} = -y\left[(1)z + \frac{\partial z}{\partial x}x\right]$$

$$ye^{yz}\frac{\partial z}{\partial x} + yx\frac{\partial z}{\partial x} = -zy$$

$y = e^{3x} = e^{u}$

$\boxed{u = 3x} \rightarrow \frac{du}{dx} = 3$

$\frac{dy}{dx} = e^{u}3$

$y = 3e^{3x}$

$$\left(ye^{yz} + yx\right)\frac{\partial z}{\partial x} = -zy$$

$$y\left(e^{yz} + x\right)\frac{\partial z}{\partial x} = -zy$$

$$\frac{\partial z}{\partial x} = \frac{-z}{e^{yz} + x}$$

$\rightarrow$ when $x = -\frac{e^2}{2}$

$y = 1$

$z = 2$

$$= \frac{-2}{e^2 + -\frac{e^2}{2}} = \frac{-2}{\frac{e^2}{2}} = \frac{-4}{e^2}$$

Ex.9 $f(x,y) = (x+y)^2(xy)$ $\quad f = uv$

$f' = u'v + v'u$

$f_x = 2(x+y)(1)xy + y(x+y)^2$

$= 2x^2y + 2y^2x + y(x^2+2xy+y^2)$

$= 2x^2y + 2y^2x + x^2y + 2xy^2 + y^3$

$\underline{f_x = 3x^2y + 4xy^2 + y^3} \rightarrow f_{xy} = 3x^2 + 8xy + 3y^2$

$\Downarrow$

$f_{xyy} = 0 + 8x + 6y$

$f_{xx} = 3y(2x) + 4y^2(1) + 0$

$= 6xy + 4y^2$

Ex.9 $f(x,y) = (x+y)^2(xy)$

$f_y = 2(x+y)xy + x(x+y)^2$

$= 2x^2y + 2y^2x + x(x^2+2xy+y^2)$

$= 2x^2y + 2y^2x + x^3 + 2x^2y + xy^2$

$= 4x^2y + 3y^2x + x^3 \rightarrow$

$f_{yy} = 4x^2 + 6yx$

$f_{yyx} = 8x + 6y$

$f_{xyy} = ?$ $\qquad f_x = 3x^2y + 4xy^2 + y^3$

$f_{xy} = 3x^2 + 8xy + 3y^2$

$f_{xyy} = 8x + 6y$

$f = (x)$ $\quad f' =$ slope of a tangent $= \frac{dy}{dx} = \frac{\Delta y}{\Delta x}$

$\lim_{\Delta x \to 0}$

7. Chain Rule

$$z = f_1(x, y),\ x = f_2(r,s) \text{ and } y = f_3(r,s)$$

$$\underbrace{\frac{\partial z}{\partial r} = \frac{\partial z}{\cancel{\partial x}} \cdot \frac{\cancel{\partial x}}{\partial r}}_{\text{consider } x} + \frac{\partial z}{\cancel{\partial y}} \cdot \frac{\cancel{\partial y}}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial s} = \underbrace{\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s}}_{x} + \underbrace{\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}}_{y}$$

Note: In the chain rule, the number of intermediate variables of z (in this case is 2) is the same as the number of terms that compose each of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$.

$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$

Ex.11: For $z = 5x + 3y$, $x = 2r + 3s$ and $y = r - 2s$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$. [Ans: 13,9]

$= 5(2) + 3(1) = \underline{\underline{13}}$

Ex.12: C is the total cost of producing q_C units of digital cameras and q_F units of memory sticks is

$$C = 30q_C + 0.015q_C q_F + q_F + 900.$$

The demand function is $q_C = \frac{9000}{p_C\sqrt{p_F}}$ and $q_F = 2000 - p_C - 400p_F$ where p_C is the price per digital camera and p_F is the price per memory sticks. Find the rate of change of the total cost with respect to price of digital camera when the price per digital camera is 50 and the price per memory stick is 2. [Ans: -123.2]

8. Chain Rule (Extension)

$z : (\ldots x, \ldots y)$

Similar to the chain rule for single-variable functions, $\rightarrow z : (r, s, t)$

- If $z = f(x, y)$, $x = g_1(r,s,t)$ and $y = g_2(r,s,t)$ then

  The partial derivative of z with respect to r is $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$

  The partial derivative of z with respect to s is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$

  The partial derivative of z with respect to t is $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

- If $z = f(w, x, y)$, $w = g_1(r,s)$, $x = g_2(r,s)$ and $y = g_3(r,s)$ then $\rightarrow z = f_2(r, s)$

  $w \quad x \quad y$

  The partial derivative of z with respect to r is $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial r} + \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$

  The partial derivative of z with respect to s is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial s} + \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$

- **General case**

n variables m variables

If $z = f(x_1, x_2, \ldots, x_n)$, $x_1 = g_1(y_1, y_2, \ldots, y_m)$, $x_2 = g_2(y_1, y_2, \ldots, y_m)$ and $x_n = g_n(y_1, y_2, \ldots, y_m)$ where m and n are positive integers then there are m possible partial derivatives of z which are

$$\frac{\partial z}{\partial y_1} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_1} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_1} + \ldots + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_1}$$

$$\frac{\partial z}{\partial y_2} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_2} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_2} + \ldots + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_2}$$

$$\vdots$$

$$\frac{\partial z}{\partial y_m} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_m} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_m} + \ldots + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_m}$$

The change in y_i affects x_i and subsequently affects z.

Ex.13: If $z = \dfrac{x + e^y}{y}$, $x = rs + se^{rt}$ and $y = 9 + rt$, find $\dfrac{\partial z}{\partial s}$ when $r = -2$, $s = 5$ and $t = 4$.

[Ans: $-2 + e^{-8}$]

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

9. Young's Theorem

All the m^{th}-order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ are continuous. If any two involve differentiating with respect to each of the same variables the same number of times, then they are necessary equal. For example, if $m = 2$ and $z = f(x, y)$

$$y = x^4 \qquad \frac{dy}{dx} = 4x^3$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

Provided that slop(e)s are continuous.

$$\frac{\partial z}{\partial x} = 8x^3 + 3y^3 3x^2 + y^2(1)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 9x^2 3y^2 + 2y$$

Ex.14: For $z = f(x, y) = 2x^4 + 3x^3y^3 + xy^2 + y$, show that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = 27x^2y^2 + 2y$.

$$\frac{\partial z}{\partial y} = 9x^3y^2 + 2xy + 1 \rightarrow \frac{\partial^2 z}{\partial y \partial x} = 9y^2 3x^2 + 2y$$

10. Competitive /Complementary Products

For related products, a price change of one affects the demand of the others.

- If an increase in demand for one product results in a decrease in demand for another product, the two products are said to be ***COMPETITIVE*** or ***SUBSTITUTE products***. Examples of competitive or substitute products are Coke and Pepsi; eye glasses and contact lens, ______________________________

- If an increase in demand for one product results in an increase in demand for another product, the two products are said to be ***COMPLEMENTARY products***. Examples of complementary products are eye glasses and lens, digital camera and memory sticks, tennis rackets and tennis balls,

____

rt

$\frac{\partial z}{\partial s} = r + e$ → $y = 9 + (-2)(4) = 1$

Ex.13: If $z = \frac{x+e^y}{y}$, $x = rs + se^{rt}$ and $y = 9 + rt$, find $\frac{\partial z}{\partial s}$ when $r = -2$, $s = 5$ and $t = 4$.

[Ans: $-2+e^{-8}$]

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Settagom 8am Day1

$$z = \frac{1}{y}(x + e^y) \qquad z = \frac{x+e^y}{y}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y} \qquad \frac{\partial z}{\partial y} = \frac{-x}{y^2} + \frac{e^y}{y} - \frac{e^y}{y^2}$$

$$= \frac{y e^y - (x+e^y)(1)}{y^2}$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Integ

$$\therefore \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} = \frac{1}{y}(r + e^{rt})$$

$y \rightarrow 1$

$r = -2$
$s = 5$
$t = 4$

$$= \frac{1}{1}(-2 + e^{-2(4)}) = -2 + e^{-8}$$

Partial derivatives can be used to determine whether two products are competitive, complementary, or neither. The demand functions for two products are dependent on the prices for both products p_A and p_B :-

$$q_A = f(p_A, p_B) \qquad \text{Demand function for product } A$$
$$q_B = g(p_A, p_B) \qquad \text{Demand function for product } B$$

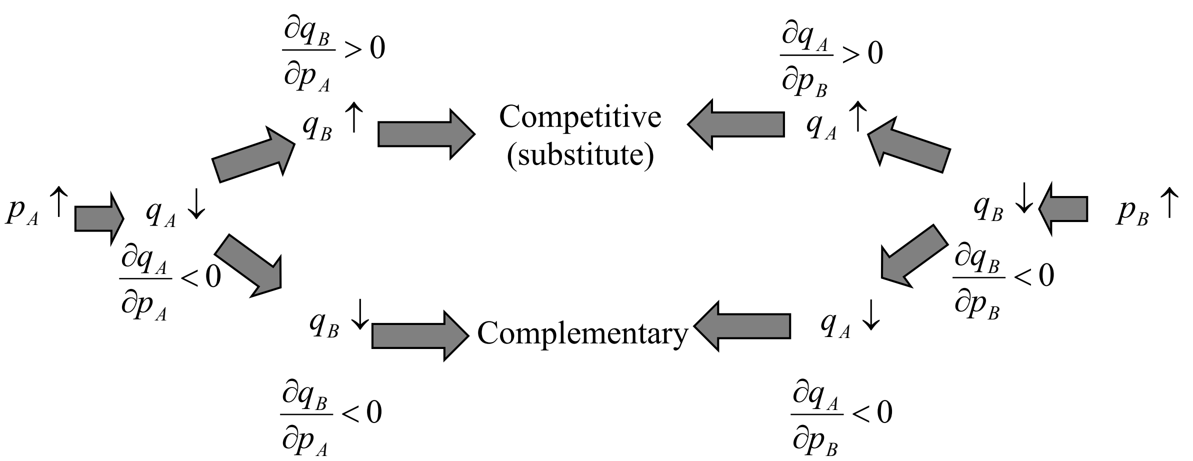
Normally, if the price of A increases while the price of B is held constant, the demand for A will decrease; that is $\frac{\partial q_A}{\partial p_A} < 0$ (i.e. marginal demand of A with respect to the price of A). In addition, if A and B are competitive products, the demand for B will increase; that is, $\frac{\partial q_B}{\partial p_A} > 0$.

Similarly, if the price of B increases while the price of A is held constant, the demand for B will decrease; that is $\frac{\partial q_B}{\partial p_B} < 0$ (i.e. marginal demand of B with respect to the price of B). In addition, if A and B are competitive products, the demand for A will increase; that is, $\frac{\partial q_A}{\partial p_B} > 0$.

Complementary products have similar reasoning, hence, $\frac{\partial q_B}{\partial p_A} < 0$ and $\frac{\partial q_A}{\partial p_B} < 0$.

$p_A \uparrow \quad q_A \downarrow \quad \frac{\partial q_A}{\partial p_A} < 0 \rightarrow \frac{\partial q_B}{\partial p_A} < 0$

Product A and B	**Partial Derivatives**
Competitive (substitute)	$\frac{\partial q_A}{\partial p_B} > 0$ and $\frac{\partial q_B}{\partial p_A} > 0$
Complementary	$\frac{\partial q_B}{\partial p_A} < 0$ and $\frac{\partial q_A}{\partial p_B} < 0$
Neither	$\frac{\partial q_B}{\partial p_A} \ge 0$ and $\frac{\partial q_A}{\partial p_B} \le 0$
Neither	$\frac{\partial q_B}{\partial p_A} \le 0$ and $\frac{\partial q_A}{\partial p_B} \ge 0$



$\frac{\partial q_A}{\partial p_B} = 0.16 p_B$ $\quad$ $\frac{\partial q_B}{\partial p_A} = 0.08 p_A$

$\because p_B, p_A > 0$

Ex.15: Determine whether the indicated products are competitive, complementary or neither where their demand functions are.

$\frac{\partial q_A}{\partial p_B} > 0$ $\quad$ $\frac{\partial q_B}{\partial p_A} > 0$ $\rightarrow$

Settagorn 8 am Day 1

a) $q_A = 8000 - 0.09 p_A^{\ 2} + 0.08 p_B^{\ 2}$ and $q_B = 15000 + 0.04 p_A^{\ 2} - 0.3 p_B^{\ 2}$
b) $q_A = 200 - 5 p_A + 4 p_B^{\ 2}$ and $q_B = 300 + 2 p_A - 4 p_B$
c) $q_A = 800 - 0.004 p_A^{\ 2} - 0.003 p_B^{\ 2}$ and $q_B = 600 - 0.003 p_A^{\ 2} - 0.003 p_B^{\ 2}$
d) $q_A = 500 - 0.5 p_A - p_B^{\ 2}$ and $q_B = 10000 - 0.5 p_A - 100 p_B^{\ 2}$

[Ans: (a) Competitive, (b) competitive, (c) complementary, and (d) complementary]

Ex.16: The weekly demand of A and B are $q_A = e^{-\left(\frac{p_A}{p_B}\right)}$ and $q_B = \dfrac{16}{p_A p_B^{\ 2}}$ where q_A and q_B are the demand of A and B respectively, p_A and p_B are price per unit in thousand of baht of A and B respectively.

a) A and B are competitive, complementary or neither.
b) If unit prices of A and B are 1,000 baht and 2,000 baht respectively, estimate demand change in A when the price of A is held constant and the price of B decreases by 40 baht. [Ans: $-0.01/\sqrt{e}$]

'' '' Δq_A $\quad$ Δp_B

$p_A = 1, \ p_B = 2$

$\Delta y \approx \frac{dy}{dx} \Delta x$

$\Delta q_A = \frac{\partial q_A}{\partial p_B} \Delta p_B$

$= \frac{-40}{1000}$

11. Elasticity (Extension)

Recall, the single-variable demand function,

$$E = \frac{\%\text{change in demand}}{\%\text{change in price}} = \frac{p}{q} \cdot \frac{dq}{dp}$$

$E(q)$	**Demand**	**Interpretation**
$0 < \lvert E(q) \rvert < 1$	Inelastic	Demand is not sensitive to changes in price. A given change in price produces a smaller change in demand.
$\lvert E(q) \rvert > 1$	Elastic	Demand is sensitive to changes in price. A change given in price produces a larger change in demand.
$\lvert E(q) \rvert = 1$	Unit	A given change in price produces the same change in demand.

inelastic

Ex.17: In a supermarket, rice has the elasticity of -1.1 and potato has the elasticity of -0.2. Which product is more essential to customers? elastic

Ex.18: A demand function for a product is $q = 8000 p^{-1.5}$. Find the elasticity and percentage change in demand when the price of the product is increased by 1% from the price of 4.
[Ans: $E = -1.5$, 1.5% decreases in demand]

15 b) $\frac{\partial q_A}{\partial p_B} =$ $\frac{\partial q_B}{\partial p_A} =$

Partial Elasticity

If the demand of a product is a function of two variables (price x and price y) i.e. $q = f(x, y)$, the elasticity of this product can be defined using partial differentiation with respect to each variable.

The percentage change in demand with respect to the percentage change in variable x:

$$E_{qx} = \frac{x}{q}\left(\frac{\partial q}{\partial x}\right)$$

The percentage change in demand with respect to the percentage change in variable y:

$$E_{qy} = \frac{y}{q}\left(\frac{\partial q}{\partial y}\right)$$

Ex.19: A product has the demand function $z = Ax^a y^b$. Show that the partial elasticity of z with respect to x is $E_{zx} = a$.

$$E_{zx} = \frac{x}{z}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{z}\left(Ay^b \cdot ax^{a-1}\right) = \frac{aAx^a y^b}{z}$$

$$E_{zx} = \frac{a}{z}\left(Ax^a y^b\right) = \frac{a}{z}(z) = a$$

Similarly, the partial elasticity with respect to y is $E_{zy} = b$.

HW

Ex.20: The annual demand function of rice in Thailand is estimated to be $q_r = Ap_r^{-0.28} m^{0.34}$ where p_r is the price of rice per kilogram and m is the average income of the people per year. The annual demand for orange is $q_o = Bp_o^{-1.27} m^{1.32}$ where p_o is the price of orange per kg. Show that the elasticity of demand with respect to the price and income are

$$E_{q_r p_r} = -0.28, E_{q_o p_o} = -1.27, E_{q_r m} = 0.34, E_{q_o m} = 1.32.$$

Explain the significance of each elasticity.

$$E_{q_r p_r} = \frac{p_r}{q_r}\left(\frac{\partial q_r}{\partial p_r}\right) = \frac{p_r}{\cancel{A}p_r^{-0.28}\cancel{m^{0.34}}}\cancel{A}\cancel{m^{0.34}}(-0.28)p_r^{-1.28}$$

$$= -0.28 \rightarrow \text{“inelastic”}$$

$$E_{q_o, p_o} = \frac{p_o}{q_o}\left(\frac{\partial q_o}{\partial p_o}\right) = \frac{\cancel{p_o}}{\cancel{Bp_o^{-1.27}}\cancel{m^{1.32}}}\cancel{B}\cancel{m^{1.32}}(-1.27)\cancel{p_o^{-2.27}}$$

$$= -1.27 \qquad |E_{q_o, p_o}| = 1.27 > 1 \rightarrow \text{elastic}$$

$$E_{q_r, m} = \frac{m}{q_r}\left(\frac{\partial q_r}{\partial m}\right) = \frac{m}{\cancel{A\rho_r^{-0.28}}\, m^{0.34}}\ \cancel{A\rho_r^{-0.28}}\ 0.34\, m^{-0.66}$$

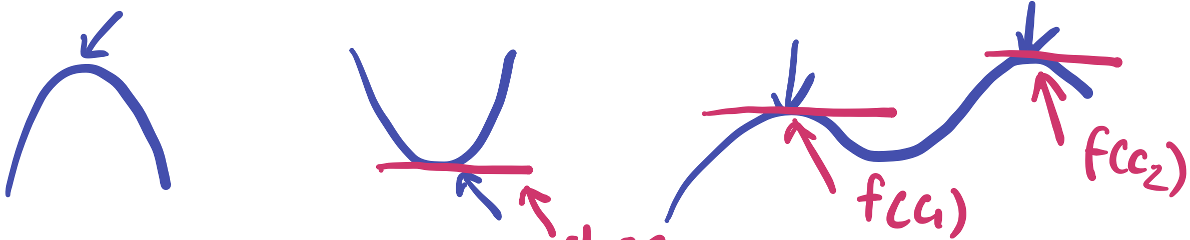
$$= 0.34 \longrightarrow \underline{\text{inelastic}}$$

$$E_{q_o, m} = \frac{m}{q_o}\left(\frac{\partial q_o}{\partial m}\right) = \frac{m}{B\rho_o^{-1.27} m^{1.32}} \cdot B\rho_o^{-1.27}\, 1.32\, m^{0.32}$$

$$= 1.32 \longrightarrow \text{elastic} \longrightarrow m\uparrow\ q_o\uparrow$$

$$E_{q_o, \rho_o} -$$

$$= -1.27$$



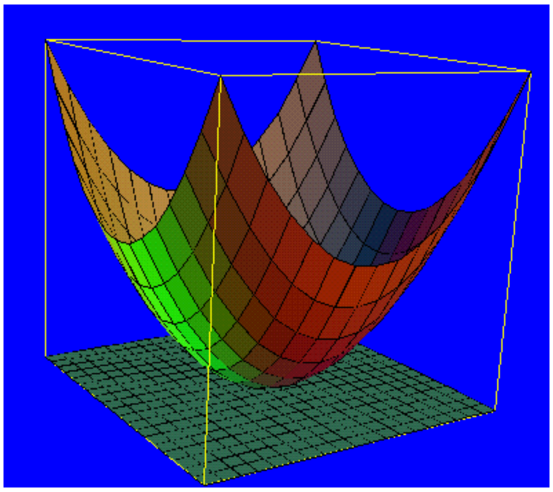
12. Multivariable Optimisation

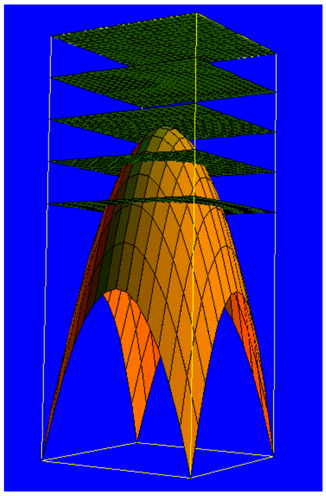
1st find critical point

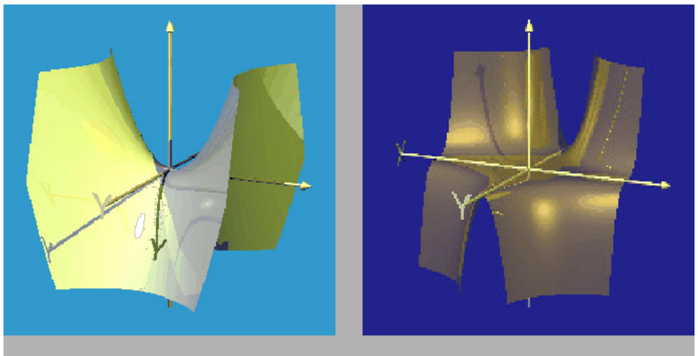
2nd check if that is max or min

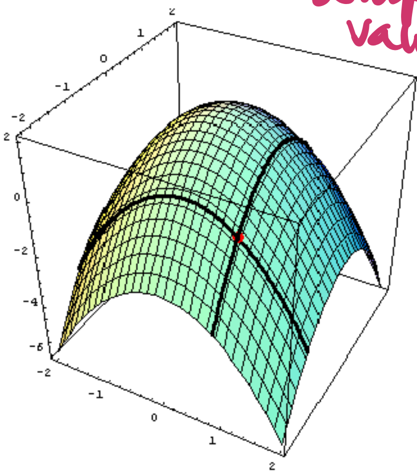
2nd derivative

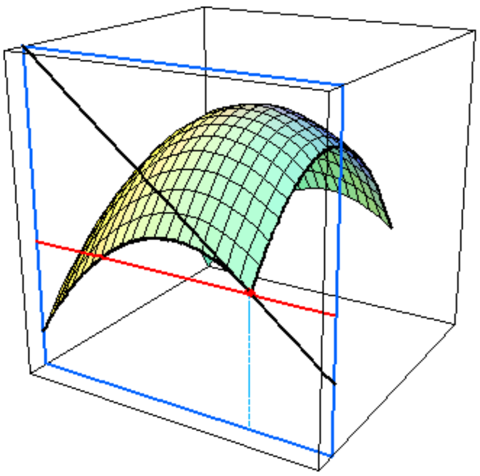
compare values of f at all critical points

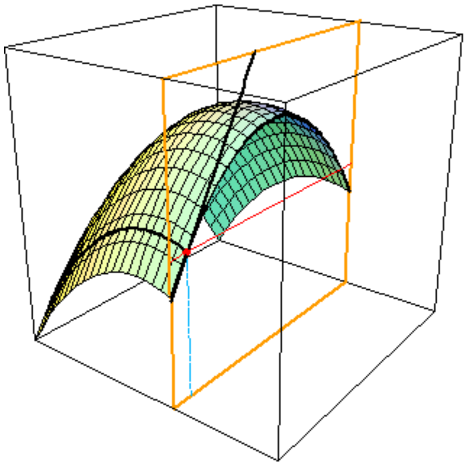





(q) $z = x^2 - y^2$ (r) $z = x^2 y^3$







- $f(x_0, y_0)$ is a relative maximum if $f(x_0, y_0) \geq f(x, y)$ for all x and y closed to (x_0, y_0).
- $f(x_0, y_0)$ is a relative minimum if $f(x_0, y_0) \leq f(x, y)$ for all x and y closed to (x_0, y_0).

proof is not in exam → Next Friday, 27^{th} → 8 – 9

→ post a video

Necessary Conditions for Relative Extrema

$y = x^n$

$\frac{dy}{dx} = nx^{n-1}$

A two-variable function, $z = f(x, y)$, has a relative maximum or minimum at (x_0, y_0) if the point (x_0, y_0) satisfies

$$f_x'(x,y) = 0 \quad \left(\frac{\partial z}{\partial x} = 0\right) \qquad \text{and} \qquad f_y'(x,y) = 0 \quad \left(\frac{\partial z}{\partial y} = 0\right)$$

Hence, (x_0, y_0) is a critical point if $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$.

$\frac{\partial f}{\partial x} = -2(2x) - 2y + 0 + 36 + 0 + 0 = 0 \Rightarrow -4x - 2y + 36 = 0$ — (1)

Ex.21: $z = f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$ Assume that z has a relative maximum point, find this point. [Ans:(5,8)]

$\frac{\partial f}{\partial y} = -2x - 4y + 0 + 42 - 0 = 0 \Rightarrow -2x - 4y + 42 = 0$ — (2)

Second-Derivative Test for Relative Extrema

Let $z = f(x, y)$ has continuous $f_{xx}''(x,y), f_{yy}''(x,y)$ and $f_{xy}''(x,y)$ at all points (x, y) near the critical point (x_0, y_0). Let D be the function defined by

$$\boxed{D(x,y) = f_{xx}''(x,y) f_{yy}''(x,y) - \left[f_{xy}''(x,y)\right]^2}$$

- If $f_{xx}''(x,y) < 0$ and $D(x_0, y_0) > 0$, f has a relative maximum at (x_0, y_0).
- If $f_{xx}''(x,y) > 0$ and $D(x_0, y_0) > 0$, f has a relative minimum at (x_0, y_0).
- If $D(x_0, y_0) < 0$, f has neither a relative maximum nor minimum at (x_0, y_0). This point (x_0, y_0) is known as a **saddle point**.
- If $D(x_0, y_0) = 0$, no conclusion about an extremum at (x_0, y_0) can be drawn, and further analysis is required..

$3x^2 - 2x = 0$

Ex.22: $f(x, y) = x^3 - x^2 - y^2 + 8$ Find all critical points and classify them. $x(3x - 2) = 0$

1st find critical point

$$f_x'(x,y) = 3x^2 - 2x = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = \frac{2}{3}$$

$$f_y'(x,y) = -2y = 0 \quad \Rightarrow \quad y = 0$$

Hence, critical points are (0, 0) and $\left(\frac{2}{3}, 0\right)$.

2nd derivative test

$$f_{xx}''(x,y) = 6x - 2$$
$$f_{yy}''(x,y) = -2$$
$$f_{xy}''(x,y) = 0$$

$$D(x,y) = f_{xx}''(x,y) f_{yy}''(x,y) - \left[f_{xy}''(x,y)\right]^2$$

Consider the critical point (0, 0),

$$f_{xx}''(0,0) = -2 < 0, \quad f_{yy}''(0,0) = -2, \quad f_{xy}''(0,0) = 0, \quad D(0,0) = (-2)(-2) - 0^2 = 4 > 0$$

$\therefore$ (0,0) is a relative maximum.

$-4x-2y+36=0 \quad (1)$

$-2x-4y+42=0 \quad (2)$

$(1)\times 2 \rightarrow \quad -8x-4y+72=0 \quad (3)$

$(2)-(3) \quad 6x-30=0$

$6x=30$

$x=5$

Sub $x=5$ into (1) $\quad -4(5)-2y+36=0$

$-20+36=2y$

$16=2y$

$y=8$

$\therefore (x,y)=(5,8)$

$$f_{xx}''(x,y)=6x-2$$
$$f_{yy}''(x,y)=-2$$
$$f_{xy}''(x,y)=0$$

$$D(x,y)=f_{xx}''(x,y)f_{yy}''(x,y)-\left[f_{xy}''(x,y)\right]^2$$

Ex.24: A production function of a firm is $p = 1.08l^2 - 0.03l^3 + 1.68k^2 - 0.08k^3$. Find quantities of *l* and *k* which maximise the output *p*.

[Ans: Critical points are (0,0), (0,14), (24,0), (24,14). Maxima $l = 24$ and $k = 14$]

$$\frac{\partial p}{\partial l} = 2.16l - 0.09l^2 = 0$$

$$l(2.16 - 0.09l) = 0$$

$$l = 0, \quad 0.09l = 2.16$$

$$l = 24$$

$$\frac{\partial p}{\partial k} = 3.36k - 0.24k^2 = 0$$

$$k(3.36 - 0.24k) = 0$$

$$k = 0, \quad 3.36 = 0.24k$$

$$k = 14$$

(0, 0) (0, 14) (24, 0) (24, 14)

Day1 8 am Settagom

et $z = f(x, y)$ has continuous $f_{xx}''(x, y), f_{yy}''(x, y)$ and $f_{xy}''(x, y)$ at all points (x, y) near the itical point (x_0, y_0). Let *D* be the function defined by

$l = x$
$k = y$

$$\boxed{D(x, y) = f_{xx}''(x, y) f_{yy}''(x, y) - \left[f_{xy}''(x, y)\right]^2}$$

- If $f_{xx}''(x, y) < 0$ and $D(x_0, y_0) > 0$, f has a relative maximum at (x_0, y_0).
- If $f_{xx}''(x, y) > 0$ and $D(x_0, y_0) > 0$, f has a relative minimum at (x_0, y_0).
- If $D(x_0, y_0) < 0$, f has neither a relative maximum nor minimum at (x_0, y_0). This point (x_0, y_0) is known as a **saddle point**.
- If $D(x_0, y_0) = 0$, no conclusion about an extremum at (x_0, y_0) can be drawn, and further analysis is required..

Ex.24: A production function of a firm is $p = 1.08l^2 - 0.03l^3 + 1.68k^2 - 0.08k^3$. Find quantities of *l* and *k* which maximise the output *p*.

[Ans: Critical points are (0,0), (0,14), (24,0), (24,14). Maxima $l = 24$ and $k = 14$]

$$\frac{\partial p}{\partial l} = 2.16l - 0.09l^2 \qquad \frac{\partial p}{\partial k} = 3.36k - 0.24k^2$$

$$\frac{\partial^2 p}{\partial l^2} = 2.16 - 0.18l \qquad \frac{\partial^2 p}{\partial k^2} = 3.36 - 0.48k$$

$$D(l,k) = \frac{\partial^2 p}{\partial l^2}\frac{\partial^2 p}{\partial k^2} - \left(\frac{\partial^2 p}{\partial l \partial k}\right)^2$$

$$\frac{\partial^2 p}{\partial l \partial k} = 0 \Rightarrow D(l,k) = (2.16 - 0.18l)(3.36 - 0.48k)$$

$$(0,0) \quad \frac{\partial^2 p}{\partial l^2} = 2.16 - 0.18(0) = 2.16$$

$\downarrow$ minimum $D(0,0) = (2.16)(3.36) > 0$

et $z = f(x,y)$ has continuous $f_{xx}''(x,y), f_{yy}''(x,y)$ and $f_{xy}''(x,y)$ at all points (x,y) near the itical point (x_0, y_0). Let D be the function defined by

$$\boxed{D(x,y) = f_{xx}''(x,y) f_{yy}''(x,y) - [f_{xy}''(x,y)]^2}$$

- If $f_{xx}''(x,y) < 0$ and $D(x_0,y_0) > 0$, f has a relative maximum at (x_0, y_0).
- If $f_{xx}''(x,y) > 0$ and $D(x_0,y_0) > 0$, f has a relative minimum at (x_0, y_0).
- If $D(x_0,y_0) < 0$, f has neither a relative maximum nor minimum at (x_0, y_0). This point (x_0, y_0) is known as a **saddle point**.
- If $D(x_0,y_0) = 0$, no conclusion about an extremum at (x_0, y_0) can be drawn, and further analysis is required..

min	saddle	min	no conclusion	
(0, 0)	(0, 14)	(24, 0)	(24, 14)	"max"
	Day1	8am	Setfagom	

Ex.24: A production function of a firm is $p = 1.08l^2 - 0.03l^3 + 1.68k^2 - 0.08k^3$. Find quantities of l and k which maximise the output p.

[Ans: Critical points are (0,0), (0,14), (24,0), (24,14). Maxima $l = 24$ and $k = 14$]

$$
\left.\begin{array}{l}
P(0,0) = 0 \\
P(0,14) = 329.28 - 219.52 = 109.76 \\
P(24,0) = 622.08 - 414.72 = 207.36 \\
P(24,14) = 207.36 + 109.76 = 317.12
\end{array}\right\}
$$

Consider the critical point $\left(\frac{2}{3},0\right)$,

$$f_{xx}''\left(\frac{2}{3},0\right)=6\left(\frac{2}{3}\right)-2=2>0, \quad f_{yy}''\left(\frac{2}{3},0\right)=-2, \quad f_{xy}''\left(\frac{2}{3},0\right)=0,$$

$$D\left(\frac{2}{3},0\right)=(2)(-2)-0^2=-4<0$$

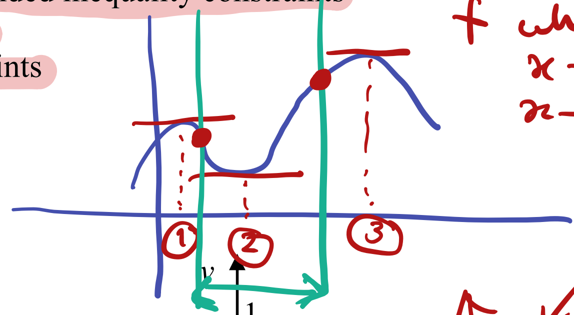
$\therefore \left(\frac{2}{3},0\right)$ is a saddle point.

Ex.23: $f(x,y)=\frac{x^3}{3}+y^2-2x+2y-2xy$ Determine all critical points and classify them.
[Ans: $f_x'(x,y)=x^2-2-2y=0$, $f_y'(x,y)=2y+2-2x=0$, $(0,-1)$ saddle point, $(2,1)$ rel. min.]

Ex.24: A production function of a firm is $p=1.08l^2-0.03l^3+1.68k^2-0.08k^3$. Find quantities of l and k which maximise the output p.
[Ans: Critical points are (0,0), (0,14), (24,0), (24,14). Maxima l = 24 and k = 14]

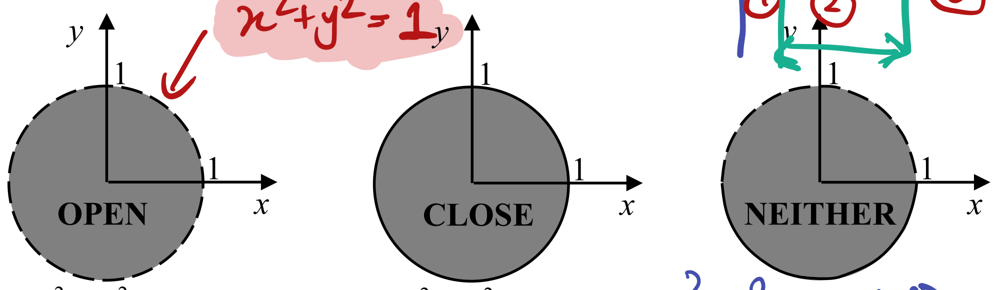
13. Optimisation of Multivariable Function with Constraints
Three methods will be introduced:-
1. Extreme Value Theorem for closed and bounded inequality constraints
2. Substitution Method for equality constraints
3. Lagrange Multiplier Method for any constraints

need to check f when $x \to +\infty$, $x \to -\infty$

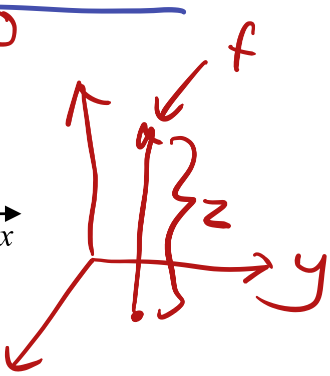


14.1 Domain – Range of consideration
1-variable function ➔ the domain is an interval
2-variable function ➔ the domain is a plane (set)

$x^2+y^2=1$



$x^2+y^2<1$ $x^2+y^2\le 1$ $x^2+y^2\le 1$ AND when $x^2+y^2=1$ $y<0$



Ex.25: Classify if the following sets are open, close or neither.
a) $2x+y\le 2, x\ge 0$ and $y\ge 0$ → close
b) $2x+y<2, x>0$ and $y>0$ → open
c) $\underbrace{2x+y\le 2}_{\text{close}}, \underbrace{x>0}_{\text{open}}$ and $\underbrace{y\ge 0}_{\text{close}}$ → neither

open – domain contains none of its boundary points
close – domain contains ALL of its boundary points

A set (domain) is considered bounded if the whole set is contained within a closed area.

Ex.23: $f(x,y)=\frac{x^3}{3}+y^2-2x+2y-2xy$ Determine all critical points and classify them.

[Ans: $f_x'(x,y)=x^2-2-2y=0$, $f_y'(x,y)=2y+2-2x=0$, $(0,-1)$ saddle point, $(2,1)$ rel. min.]

$$\frac{\partial f}{\partial x} = x^2-2-2y = 0 \Rightarrow x^2-2y-2=0 \quad \text{— (1)}$$

$$\frac{\partial f}{\partial y} = 2y+2-2x = 0 \Rightarrow y-x = -1$$

$$y = x-1 \quad \text{— (2)}$$

$$x^2-2(x-1)-2 = 0$$

$$x^2-2x+\cancel{2}-\cancel{2} = 0$$

$$x^2-2x = 0$$

$$x(x-2) = 0$$

$$x = 0, 2$$

$$y = -1, 1$$

critical points are $(0,-1)$ and $(2,1)$

$$D(x,y)=f_{xx}''(x,y)f_{yy}''(x,y)-\left[f_{xy}''(x,y)\right]^2$$

$f''_{xx} = 2x$

$f''_{yy} = 2$

$f''_{xy} = -2$

at $(0,-1)$ $f''_{xx}(0,-1) = 0$

$D(0,-1) = 0 - (-2)^2 < 0$

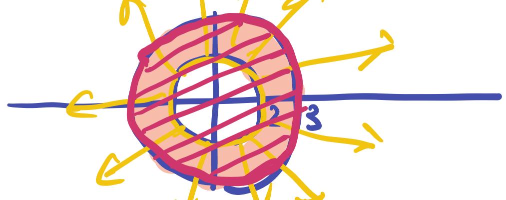
" saddle point "

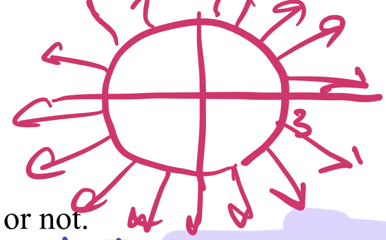
at $(2,1)$ $\quad f''_{xx}(2,1) = 4 > 0 \quad f''_{yy} = 2$

$f''_{xy} = -2$

$D(2,1) = 4(2) - 4 > 0$

$(2,1)$ is a relative minimum





Ex.26: Determine if the following sets is bounded or not.

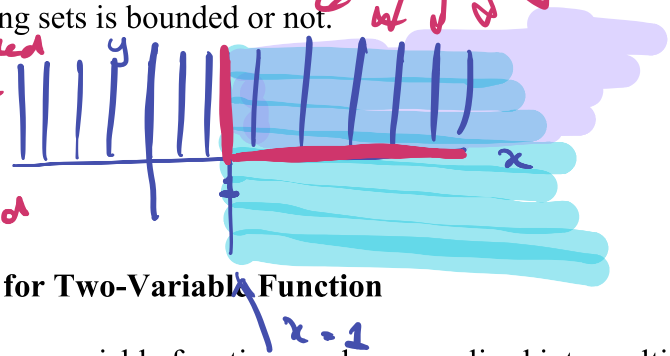
a) $x \geq 1$ and $y \geq 0$ not bounded

bounded

b) $4 \leq x^2 + y^2 \leq 9$ $x^2+y^2 \geq 4$

c) $4 < x^2 + y^2 \leq 9$ $x^2+y^2 \leq 9$

d) $x^2 + y^2 \geq 9$ not bounded



14.2 Extreme-Value Theorem for Two-Variable Function

The extrema-value theorem for one-variable function can be generalised into multi-variable function. If a function $f(x,y)$ is continuous throughout **a closed bounded set *S*** in a plane, then there exists both a point (a,b) in *S* where $f(a,b)$ has an absolute minimum and a point (c,d) is *S* where $f(c,d)$ has an absolute maximum, i.e.

$$f(a,b) \leq f(x,y) \leq f(c,d) \quad \text{for all } (x,y) \text{ in } S.$$

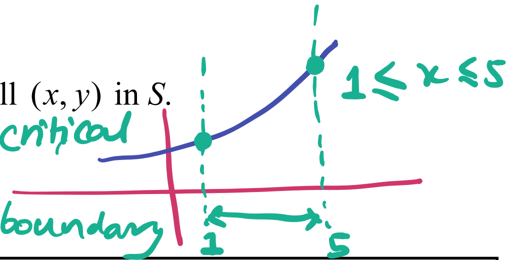
Candidates for absolute max

This is sufficient but not necessary.

absolute min

critical

boundary

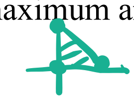


To find maxima and minima of a differentiable function $f(x,y)$ defined on a closed bounded set *S* in a plane:

(1) Find all critical points of $(x,y)_{c,i}$ in *S* by differentiating with respect to each variable and equating to zero. $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

(2) If asked, classify relative maximum and minimum points. **(ONLY if ASKED)**

(3) Compare relative maximum and minimum points to find absolute maximum and minimum points by comparing



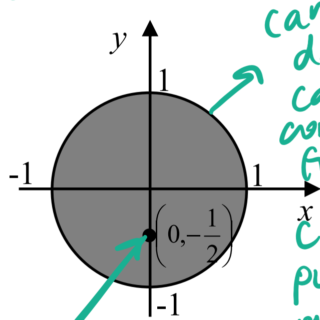
a. the **critical points** of $f(x,y)$ on **the boundary of *S*** and the **corner points** (same as in one-variable case).

b. the values of $f(x,y)_{c,i}$ in (1)

Ex.27: Find absolute maxima and minima of $f(x,y)$ defined over *S* when $z = f(x,y) = x^2 + y^2 + y - 1$ subject to $S = \{(x,y) : x^2 + y^2 \leq 1\}$

closed and bounded

Procedures:



candidate can come from critical points on S

(1) Find critical points

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x = 0 \quad \Rightarrow \quad x = 0 \\ \frac{\partial f}{\partial y} &= 2y + 1 = 0 \quad \Rightarrow \quad y = -\frac{1}{2} \end{aligned}\right\}$$

1st order condition gives **ONLY** one critical point $\left(0,-\frac{1}{2}\right)$.

(2) Don't need to classify the nature of the critical point.

(3) (a) Find the absolute maximum or minimum points on the boundary $S: x^2 + y^2 = 1$

The domain of x is [-1, 1] and the domain of y is also [-1, 1] and $x^2 + y^2 = 1$.

Hence, on the boundary of S, $f(x, y)$ becomes

$$z = f(x,y) = x^2 + y^2 + y - 1 = 1 + y - 1 \quad \Rightarrow f(x,y) = y \quad \text{and} \quad y \in [-1, 1]$$

$\frac{df}{dy} = 1 \neq 0$

Hence, $f(x, y)$ is **maximised** when $y = 1$ and $x = 0$, $f(0, 1) = 1$

$f(x, y)$ is **minimised** when $y = -1$ and $x = 0$, $f(0, -1) = -1$

(b) Calculate the value of the critical point $\left(0, -\frac{1}{2}\right)$.

$$f\left(0, -\frac{1}{2}\right) = 0^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 1 = -\frac{5}{4}$$ *

Compare (a) and (b): $f(0, 1) = 1$, $f(0, -1) = -1$ and $f\left(0, -\frac{1}{2}\right) = -\frac{5}{4}$.

Hence, maximum value of $f(x, y)$ in S is 1 at (0, 1).

Minimum value of $f(x, y)$ in S is $-\frac{5}{4}$ at $\left(0, -\frac{1}{2}\right)$.

Ex.28.a: For a two-variable function,

$$f(x, y) = x^2 - xy + y^2 + 4$$

(a) Find critical point(s) using the first derivative. [Ans: (0,0)]

(b) Find the maximum and minimum value of the above function $f(x, y)$ on the closed area in the first quadrant bounded by the triangle formed by the line $x = 0$, $y = 4$ and $y = x$.

[Ans: Maximum value $f(0,4) = f(4,4) = 20$ and minimum value $f(0,0) = 4$]

Ex.28.b: Thai government is promoting transportation vehicles e.g. taxi, van and trucks to use natural gas (NG). Currently the government is importing NG from Burma (x) and Malaysia (y). The benefit function of import NG is given by $f(x, y) = 9x + 8y - 6(x + y)^2$. Due to the limitation of capacity $0 \le x \le 5$ and $0 \le y \le 3$. For potential reason, importing from Burma should not be too small, so that $x \ge 2(y - 1) \quad \Rightarrow \quad y \le \frac{x}{2} + 1$. Find the import value from Burma and Malaysia that will maximise the benefit.

$$\boxed{\text{Max. } f(x, y) = 9x + 8y - 6(x + y)^2 \text{ subject to } 0 \le x \le 5;\ 0 \le y \le 3 \text{ and } y \le \frac{x}{2} + 1}$$

Ex.28.a: For a two-variable function,

$$f(x,y) = x^2 - xy + y^2 + 4$$

(a) Find critical point(s) using the first derivative. [Ans: (0,0)]

$\frac{\partial f}{\partial x} = 2x - y = 0$ — (1)

$\frac{\partial f}{\partial y} = -x + 2y = 0$ — (2)

(b) Find the maximum and minimum value of the above function $f(x,y)$ on the closed area in the first quadrant bounded by the triangle formed by the line $x=0$, $y=4$ and $y=x$.

[Ans: Maximum value $f(0,4)$ = $f(4,4)$ = 20 and minimum value $f(0,0) = 4$]

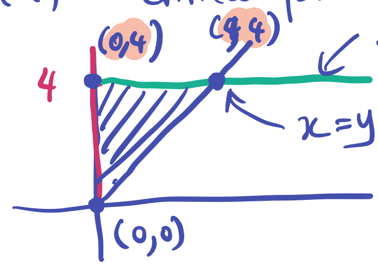
$2 \times (2) \Rightarrow -2x + 4y = 0$ — (3)

$(1) + (3) \Rightarrow 0 + 3y = 0 \Rightarrow y = 0 \rightarrow (1)$

$2x = 0$

$x = 0$

(a) critical point (0,0)



$f(x,y) = x^2 - xy + y^2 + 4$

$x = 0$ $\quad f(0,y) = y^2 + 4$

$\frac{df}{dy} = 2y = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

$y = 4$ $\quad f(x,4) = x^2 - 4x + 20$

$\frac{df}{dx} = 2x - 4 = 0$

$x = 2 \Rightarrow (2,4)$

$y = x$ $\quad f(x,y) = x^2 - xy + y^2 + 4$

$f(x,x) = \cancel{x^2} - \cancel{x^2} + x^2 + 4$

$\frac{df}{dx} = 2x = 0 \Rightarrow x = 0$

$\because y = x \Rightarrow y = 0$

(0,0)

Candidates for ab max ab min are
$(0,0)$, $(0,4)$, $(2,4)$ and $(4,4)$

$f(0,0) = 4$

$f(0,4) = 20$ max

$f(2,4) = 16$

$f(4,4) = 20$ max