

Quiz 4

1. Show that $n^2 - 7n + 12 \geq 0$ when $n = 4$.
2. Use **mathematical induction** to prove that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \geq 3$.

Solution:

1. For $n = 4$,

$$n^2 - 7n + 12 = 4^2 - 7(4) + 12 = 16 - 28 + 12 = 0$$

which is nonnegative. I.e. $n^2 - 7n + 12 \geq 0$ for $n = 4$.

2. Proof by mathematical induction:

Let $P(n)$ be the statement “ $n^2 - 7n + 12$ is nonnegative.”

We want to prove that $P(n)$ is true for all integer $n \geq 3$.

(I) **Basis step:** Show that $P(3)$ is true.

$P(3)$: $3^2 - 7(3) + 12$ is nonnegative.

Since $3^2 - 7(3) + 12 = 9 - 21 + 12 = 0$, $3^2 - 7(3) + 12$ is nonnegative and $P(3)$ is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 3$.

Assume that $P(k)$: “ $k^2 - 7k + 12$ is nonnegative.”

—————(★) “inductive hypothesis”

We want to show that $P(k+1)$: “ $(k+1)^2 - 7(k+1) + 12$ is nonnegative.” Consider

$$\begin{aligned}
 (k+1)^2 - 7(k+1) + 12 &= \underbrace{(k^2 + 2k + 1)}_{\geq 0} - 7(k+1) + 12 \\
 &= k^2 - 7k + 12 + (2k + 1 - 7) \\
 &\geq 0 + (2k + 1 - 7) \quad \text{by (★) “inductive hypothesis : } k^2 - 7k + 12 \geq 0 \text{ ”} \\
 &= 2k - 6 \\
 &= 2(k - 3) \\
 &\geq 0 \quad \text{since } k \geq 3, \text{ then } k - 3 \geq 0 \text{ and } 2(k - 3) \geq 0.
 \end{aligned}$$

Therefore $(k+1)^2 - 7(k+1) + 12$ is nonnegative and $P(k+1)$ is true.

Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 3$ by mathematical induction proof. ■

Remark: It is possible to use different approaches in the inductive step (II).