Problem 1.In this exercise, we consider a standard utility maximization problem with an unusual (for us) income. The utility function is

$$u(x,y) = \alpha \log(x) + (1-\alpha)\log(y)$$

This function is well-defined for $x > 0$ and for $y > 0$. From now on, assume $x > 0$ and $y > 0$ unless otherwise stated. The price of good x is p_x and the price of good y is p_y.

The unusual part is that consumers' income is given not by a monetary budget M, but by endowments of the goods. To be more precise, the consumer starts off owning $\omega_x > 0$ units of good x and $\omega_y > 0$ units of good y. (These numbers are given, treat them as parameters) The consumer can sell the goods so the initial allocation works as a source of wealth. Given this, the income of the consumer is $p_x\omega_x + p_y\omega_y$.

1. Write down the budget constraint. Notice that the budget constraint is standard other than for the income expression, which is given above.

2. Just for this point, we pick some specific parameter values. Plot the budget lines assuming $\omega_x = 1$, $\omega_y = 1$. Plot first for the case $p_x = 1$, $p_y = 1$, and then plot the budget line for $p_x = 2$,

$p_y = 1$. How does the budget line shift? In a new graph, plot the budget constraint for the same two combinations of prices, but assuming $M = 2$. How does the shift in budget line in the case with endowments ω_x and ω_y compare to the standard case with income M? Provide intuition.

3. Let's go back to the general case. The consumer maximizes utility subject to the budget constraint with endowments as in point (1). Write down the maximization problem of the consumer with respect to x and y. Explain briefly why the budget constraint is satisfied with equality.

4. Write down the Lagrangean function.

5. Write down the first order conditions for this problem with respect to x, y, and λ.

6. Solve explicitly for x^* and y^* as a function of p_x, p_y, ω_x and ω_y.

7. Use the expression for x^* that you obtained in point (6) Differentiate it with respect to

p_x, that is, compute $\partial_{x^*} / \partial p_x$. Is this good a Giffen good?

8. We are now interested in the sign of $\partial_{x^*} / \partial p_y$. That is, we would like to know if how the demand for one good depends on the price of the other good. Based on this, are these goods (gross) substitutes or complements?

9. Consider now how the demand for x^* varies as the endowments ω_x and ω_y vary. That is, compute $\partial_{x^*} / \partial \omega_x$ and $\partial_{x^*} / \partial \omega_y$. Provide an interpretation of these results.

10. So far you derived the optimal demands x^* and y^*. Now, consider the demands net of the endowments, that is $z^*_x = x^* - \omega_x$ and $z^*_y = y^* - \omega_y$ .These net demands represent the net demand for a good. If the net demand is positive, it means that the consumer buys additional quantities relative to the endowment. If the net demand is negative, it indicates that the consumer sells (at least partially) the endowment. Derive expressions for z^*_x and z^*_y

11. Compute how the net demand for good x, that is, z^*_x varies as the endowment for x varies, that is, compute $\partial z^*_x / \partial \omega_x$. Compare to your answer in point (9) regarding

$\partial x^* / \partial \omega_x$ and interpret the difference.

12. Suppose now that you have to solve the same problem as above, that is, same budget constraint as in point (1), but the utility function is $\upsilon(x,y) = x^a y^{1-\alpha}$. Can you derive the solution for this problem without redoing all the algebra? Explain.

Problem 2.

1. Consider an individual with budget constraint $2x + y = 10$= That is, price of x is $p_x = 2$, price of y is $p_y = 1$, and income M equals 10. Plot the budget constraint.

2. Plot indifferent curves for the case $u(x,y) = x^2 + y^2$. (Remember: an indifference curve is defined by $u(u,y) - \bar{u} = 0$)

3. Using the plots of the indifference curves and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y . (Remember that you are maximizing utility subject to the budget constraint and subject to $x \geq 0$ and $y \geq 0$)

4. Why in the above case it would not be a good idea to solve the problem with a Lagrangean?

5. Now, using the same budget constraint $2x + y = 10$, plot indifference curves for the case

$u(x,y) = 4x + y$.

6. Using the plots of the indifference curves and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y^*.

7. What preferences are the ones in points (2) and (5)? What cases do they indicate?

Problem 3. Alice likes three goods: consumption goods x_1 , x_2 and leisure l. He maximizes the utility function

$$u\,(x_1 , x_2 ,x_3) = x_1^{\alpha 1}\, x_2^{\alpha 2}\, l^{\Upsilon},$$

with $0 < \alpha_i < 1$ for $i = 1, 2$ and $0 < \gamma < 1$. The consumption good x_i has price p_i (for $i = 1, 2$), the hourly wage is w.

1. Write down the budget constraint. Consider that Alice has H hours to work and, if he does not work, he takes leisure. For example, H could be 24 hours if the time period is a day. Hence, the hours worked h equal $H - l$. There are no sources of income other than income from hours worked. Write down the budget constraint as a function of x_1 , x_2 and l. [Hint: Money spent on goods has to be smaller than or equal to money earned]

2. Write down the maximization problem of the worker with respect to x_1 , x_2 and l with all the relevant constraints Assume that the budget constraint is satisfied with equality. Why can we assume that the budget constraint is satisfied with equality? Provide as complete an explanation as you can.

3. Write down the Lagrangean and derive the first order conditions with respect x_1 , x_2 and l, and λ.

4. Solve for x_1^*as a function of the prices p_1, p_2, w, the total number of hours H, and the parameters α_1, α_2, and γ. [Hint: combine the first and second first-order condition, then combine the first and third first-order condition, and finally plug in budget constraint] Similarly solve for x_2^* and l^*.

5. Plot the Engel function relating the demand for good 1 x_1^* (H) to the number of hours available H. (Plot x_1 in the x axis and H in the y axis) In what sense H plays the role of income? Explain.

6. Plot the demand function for good 1 x_1^* (p_1) as a function of p_1. (Put x_1 in the x axis and price p_1 in the y axis) Is the demand function downward sloping? Interpret.

7. Are goods $x_1$and x_2 gross complements, gross substitutes, or neither? Define and answer.

8. Plot the demand function for leisure l^* (w) as a function of its (shadow) price w. (Put l in the x axis and price w in the y axis) Is the demand function downward sloping? Interpret.

9. Relate the response to question 8 (leisure l and price w) to substitution and income effects. Be careful here, this is not exactly the case we saw in class.

10. Using the envelope theorem, compute how the indirect utility υ $(p_1, p_2, w, \alpha_1, \alpha_2, \gamma, H)$ changes as H changes: $\partial\upsilon/\partial H$. Remember that the indirect utility is the utility of Yingyi at the optimum level of the parameters: $\upsilon\,(p_1, p_2, w, \alpha_1, \alpha_2, \gamma, H) = u\,(x_1^*, x_2^*, l^*)$. What is the sign of $\partial\upsilon/\partial H$? Interpret

11. Solve for the Lagrangean multiplier λ^* $(p_1, p_2, w, \alpha_1, \alpha_2, \gamma)$ using the first order conditions above. Comment on what this implies for $\partial\upsilon/\partial H$.

Problem 4.

1. Complements and Substitutes.

(a) Define when two goods x_1 and x_2 are gross substitutes/complements and net substitutes/complements.

(b) Is it possible for two goods to be gross substitutes and net complements if both goods are normal goods? And if both goods are inferior goods? Use the general form of the Slutsky equation and provide an explanation for the result.

2. Kim has utility function u $(x_1, x_2) = \exp(x_1 + x_2)$ for $x_1 \geq 0$, $i = 1, 2$.

(a) Plot the indifference curves of Kim. What kind of goods do they represent?

(b) Are the preferences represented by this utility function monotonic? Define.

(c) Are they rational? Define.

Problem 5. Bob likes three goods: x_1, x_2 and x_3. He maximizes the utility function

$u(x_1, x_2, x_3) = x_1^{\alpha 1} x_2^{\alpha 2} x_3^{\alpha 3}$,

with $0 < \alpha_i < 1$ for $i = 1, 2, 3$. The consumption good x_1 has price p_i (for $i = 1, 2, 3$) and the individual has total income M.

1. Compute the marginal utility of consumption with respect to good x_1, $\partial u(x_1, x_2, x_3)/\partial x_i$.

2. What is the limit of the marginal utility for $x_1 \to 0$ and for $x_1 \to \infty$? Interpret the economic intuition behind this feature of this utility function.

3. Write the budget constraint.

4. Write the maximization problem of Bob. Bob wants to achieve the highest utility subject to the budget constraint. Write down the boundary constraints for x_1, x_2, x_3, and neglect them for now.

5. Assuming that the budget constraint holds with equality, write down the Lagrangean and derive the first order conditions with respect to x_1, x_2, x_3, and λ.

6. Solve for x_1^* as a function of the prices p_1, p_2, p_3, the income M, and the parameters α_1, α_2, and α_3. [Hint: combine the first and second first-order condition, then combine the first and third first-order condition, and finally plug in budget constraint] Similarly solve for x_2^* and x_3^*

7. True or false? Show your work: "Cobb-Douglas preferences have the feature that the share of money spent on each good does not depend on the income, or on prices"

8. Are the boundary conditions for x_1, x_2 and x_3 satisfied?

9. Is good x_1 a normal good (for all values of M and prices p_i)? Compute and answer.

10. Plot the implied demand function for x_1,that is plot x_1 as a function of p_1 (Put p_1 on the y axis and x_1 on the x axis)

11. Is good x_1 a Giffen good? Why did you know this already from the answer to question 9?

12. Are goods x_1 and x_2 gross complements, gross substitutes, or neither? Define and answer.

Problem 6.

1. Angela has utility function u $(x_1, x_2) = 2x_1 + 2x_2$.

   (a) Plot the indifference curves of Angela. What kind of goods do they represent?

   (b) Using the plot you did, find the utility-maximing solution x_1^*, x_2^* for prices $p_1 = 1$, $p_2 = 2$ and income M. Argue the steps you make.

2. Kim has utility function u $(x_1, x_2) = \min(x_1, 2x_2)$

   (a) Plot the indifference curves of Kim. What kind of goods do they represent?

   (b) Are the preferences represented by this utility function monotonic? Define.

   (c) Are they strictly monotonic? Define.

Problem 7. Consider a farmer that produces corn using labor. The labor cost in dollar to produce y bushels of corn is $c(y) = Cy^2$, with $C > 0$. There are 100 identical farms which all behave competitively. Corn sells at a price p per bushel.

1. Derive the marginal cost $c'(y)$ and the average cost $c(y)/y$. Plot them. Derive graphically the supply curve. (Have the quantity y on the horizontal axis).

2. Write the expression for the supply of corn $y(p)$ for each firm, and derive the expression for the aggregate supply function $Y^S(p)$

3. From now on, suppose that the demand curve of corn is $D(p) = 200 - 50p$ and assume

$C = 1$. Derive the equilibrium price p^* and total quantity sold Y^S.

4. Derive the profits of each firm.

5. Now assume that land is not free, and that the farmers are renting the land from a monopolist (that is, there is only one land owner, and it is impossible to rent land somewhere else). How much will the land-owner charge each of the farms? Explain.

6. Taking the rent of the land into account, how much are the profits of each firm?

7. In what sense there is a parallel between the presence of rents in the land and the entry of firms in the long-run equilibrium?

Problem 8. We consider the market for widgets, which is characterized by the aggregate (inverse) demand function $p(X) = a - bX$, where X is the total quantity of widgets demanded in the market. The cost function of each company is $c(y) = cy^{\alpha}$, with $c > 0$.

1. Assume perfect competition (that is, the price p of the widget is given) and set up the profit maximization of each firm.

2. Solve for the profit-maximizing level of production $y^*(p)$ (that is, the supply function) using the first-order condition.

3. Check the second-order conditions. Under what values of the parameters are they satisfied? Interpret the economic significance of this parameter restriction.

4. Now consider the conditions for the market equilibrium. For points 4-7, assume that the parameters are such that the second order conditions are satisfied. Assume that N firms produce and write the equation for the equilibrium price p^* that equates aggregate supply and demand. Do not attempt to solve explicitly for p^*.

5. Now introduce taxation. Denote by p the price inclusive of tax that the consumer pays, and by $p - t$ the price net of tax that accrues to the producer. Rewrite the market equilibrium condition. [Note: If you get stuck here, you can move on to point 8]

6. Use the implicit function theorem to compute $\partial p^*/\partial t$.

7. Show that $0 < \partial p^*/\partial t < 1$. What does it mean economically?

8. From now on, consider the case $\alpha = 1$. Assume for now no taxes ($t = 0$). What is the economic interpretation of this special case?

9. Characterize mathematically the supply function $y^*(p)$ for an individual company, and plot it.

10. Solve for the market equilibrium price p_* and total quantity X^* produced in the market, assuming no taxes. [Note: A figure may help you here]

11. Compute the aggregate consumer surplus and the producer surplus. You can help yourself with a plot of market demand and supply. [Note: Do not worry here about the distinction between compensated and uncompensated demand]

12. Solve now for the market equilibrium price p_* and total quantity X^* produced in the market assuming a tax t. How much of the tax do the companies pass through in prices (that is, what is $\partial p^*/\partial t$)?

13. Compute the consumer surplus and the producer surplus for the case with a tax t.

14. Compute the total surplus adding the consumer surplus, the producer surplus, and the revenue raised with the taxes.

15. Using what you just found, argue that the deadweight loss from taxation is given by $t^2/2b$. Do you agree or disagree with the following statement? Provide a precise argument: 'Small taxes are not very distortionary, but large taxes can induce very large deadweight loss'

16. Consider now the case of monopoly. Keep assuming $\alpha = 1$ and a tax t. [Note: As above, the price p denotes the price that consumers pay inclusive of taxes, and $p - t$ is the price that producers receive] Set-up the maximization problem and solve for the profit-maximizing price p^*_M and quantity X^*_{M}

17. How much of the tax does the firm pass through to the consumer (that is, what is $\partial p^*/\partial t$)? Compare to the case of perfect competition for $\alpha = 1$.

18. Compute the producer surplus and the consumer surplus and compare to the case of perfect competition for $\alpha = 1$.