Simultaneous-Equation Models

Example: Klien's Model I

System equations model that frequently used is simultaneous equations model. A well know macroeconometrics model is Klien's (1950) Model I, which has the following equations form:

$$C_t = \alpha_0 + \alpha_1 P_t + \alpha_2 P_{t-1} + \alpha_3 \left(W_t^P + W_t^G\right) + \varepsilon_{1t} \qquad \text{(Consumption)},$$

$$I_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + \beta_3 K_{t-1} + \varepsilon_{2t} \qquad \text{(Investment)},$$

$$W_t^P = \gamma_0 + \gamma_1 X_t + \gamma_2 X_{t-1} + \gamma_3 A_t + \varepsilon_{3t} \qquad \text{(Private wages)},$$

$$X_t = C_t + I_t + G_t \qquad \text{(equilibrium demand)},$$

$$P_t = X_t - T_t - W_t^P \qquad \text{(private profits)},$$

$$K_t = K_{t-1} + I_t \qquad \text{(capital stock)},$$

where C_t = Consumption at time *t*
I_t = Investment at time *t*
W_t^P = Private wages at time *t*
X_t = Total demand at time *t*
P_t = Private profits at time *t*
K_t = Capital stock at time *t*
G_t = Government nonwage spending at time *t*
T_t = Indirect business taxes plus net exports at time *t*
W_t^G = Government wage bill at time *t*
A_t = Time trend measured as years from 1931

Endogenous variables are all variables on the left hand side including consumption (C_t), investment (I_t), and private wages (W_t^P). Exogenous variables include government nonwage spending (G_t), indirect business taxes plus net exports (T_t), government wage will (W_t^G), and time trend (A_t). However, there are also predetermined variables including total demand (X_t), private profit (P_t), and capital stock from last year (K_{t-1}). This system model consists of 3 behavior equation and 1 equilibrium condition, and 2 identities. This simultaneous model is a dynamic model for a small economy. Klein (1950) estimated this model using data from 1921 to 1941.

The model is estimated using the data from 1921 to 1941. Methods of estimation include Ordinary Least Squares (OLS), Two Stages Least Squares: 2SLS, (Three stage Least squares: 3SLS, and Iterative Three Stages Least Squares: I3SLS.

```
g w = wg+wp
g k = k1+i
g yr=year-1931
g p1 = p[_n-1]
g x1 = x[_n-1]

reg c p p1 w

      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  3,    17) =  292.71
       Model |  923.549937     3  307.849979           Prob > F      =  0.0000
    Residual |  17.8794524    17  1.05173249           R-squared     =  0.9810
-------------+------------------------------           Adj R-squared =  0.9777
       Total |  941.429389    20  47.0714695           Root MSE      =  1.0255

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |   .1929343   .0912102     2.12   0.049     .0004977     .385371
          p1 |   .0898847   .0906479     0.99   0.335    -.1013658    .2811351
           w |   .7962188   .0399439    19.93   0.000     .7119444    .8804931
       _cons |    16.2366   1.302698    12.46   0.000     13.48815    18.98506
------------------------------------------------------------------------------

reg3 (c p p1 w), 2sls nodfk inst(t wg g yr p1 x1 k1)

Two-stage least-squares regression
----------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"     F-Stat        P
----------------------------------------------------------------------
c                  21      3    1.135659    0.9767   279.0941   0.0000
----------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
c            |
           p |   .0173022   .1180494     0.15   0.885    -.2317603    .2663647
          p1 |   .2162338    .107268     2.02   0.060    -.0100818    .4425495
           w |   .8101827   .0402497    20.13   0.000     .7252632    .8951022
       _cons |   16.55476   1.320793    12.53   0.000     13.76813    19.34139
------------------------------------------------------------------------------
Endogenous variables:  c p w
Exogenous variables:   t wg g yr p1 x1 k1
------------------------------------------------------------------------------
```

<u>Alternative Command: `ivreg`</u>

However, this command `ivreg` produces the correct coefficients but the standard errors are wrong.

```
ivreg c p1 (p w = t wg g yr p1 x1 k1)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  3,    17) =  225.93
       Model |  919.504138     3  306.501379           Prob > F      =  0.0000
    Residual |  21.9252518    17  1.28972069           R-squared     =  0.9767
-------------+------------------------------           Adj R-squared =  0.9726
       Total |  941.429389    20  47.0714695           Root MSE      =  1.1357

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |   .0173022   .1312046     0.13   0.897    -.2595153    .2941197
           w |   .8101827   .0447351    18.11   0.000        .7158    .9045654
          p1 |   .2162338   .1192217     1.81   0.087    -.0353019    .4677696
       _cons |   16.55476   1.467979    11.28   0.000     13.45759    19.65192
------------------------------------------------------------------------------
Instrumented:  p w
Instruments:   p1 t wg g yr x1 k1
------------------------------------------------------------------------------

/* additional code to get correct standard errors, suggested by Kit Baum */
mat vpr=e(V)*e(df_r)/e(N)
```

```
mat se=e(b)
local nc=colsof(se)
forv i=1/`nc' { mat se[1,`i']=sqrt(vpr[`i',`i']) }
mat list se

se[1,4]
             p           w          p1       _cons
y1  .11804942  .04024972  .10726797  1.3207925
```

Alternative Command: `ivgmm0`

However, this command `ivgmm0` produces the same standard errors as in `reg3` but slightly different in the estimated coefficients.

```
ivgmm0 c p1 (p w = t wg g yr p1 x1 k1)

Instrumental Variables Estimation via GMM          Number of obs  =        21
                                                   Root MSE       =    1.0255
                                                   Hansen J       =    4.1098
                                                   Chi-sq( 4) P-val = 0.39135
------------------------------------------------------------------------------
             |               GMM
           c |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |   .0757916   .0615982     1.23   0.219    -.0449386    .1965218
           w |   .8493653   .0292499    29.04   0.000     .7920365    .9066941
          p1 |   .1662683   .0654933     2.54   0.011     .0379039    .2946327
       _cons |   14.74433   .8966058    16.44   0.000     12.98702    16.50165
------------------------------------------------------------------------------
Instrumented:  p w
Instruments:   p1 t wg g yr x1 k1
------------------------------------------------------------------------------

. reg3 (c p p1 w), 2sls nodfk inst(t wg g yr p1 x1 k1)

Two-stage least-squares regression
----------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"     F-Stat        P
----------------------------------------------------------------------
c                  21      3    1.135659    0.9767     279.09   0.0000
----------------------------------------------------------------------
------------------------------------------------------------------------------
             |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
c            |
           p |   .0173022   .1180494     0.15   0.885    -.2317603    .2663647
          p1 |   .2162338    .107268     2.02   0.060    -.0100818    .4425495
           w |   .8101827   .0402497    20.13   0.000     .7252632    .8951022
       _cons |   16.55476   1.320793    12.53   0.000     13.76813    19.34139
------------------------------------------------------------------------------
Endogenous variables:  c p w
Exogenous variables:   t wg g yr p1 x1 k1
------------------------------------------------------------------------------

. reg p t wg g yr p1 x1 k1

      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  7,    13) =    8.82
       Model |  294.248018     7  42.0354311           Prob > F      =  0.0004
    Residual |  61.9500944    13  4.76539188           R-squared     =  0.8261
-------------+------------------------------           Adj R-squared =  0.7324
       Total |  356.198112    20  17.8099056           Root MSE      =   2.183

------------------------------------------------------------------------------
           p |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |  -.9230977   .4337595    -2.13   0.053    -1.860178    .0139827
          wg |  -.0796076   2.533823    -0.03   0.975      -5.5536    5.394385
           g |   .4390162   .3911427     1.12   0.282    -.4059962    1.284029
          yr |   .3194049   .7781286     0.41   0.688     -1.36164    2.000449
          p1 |   .8025008   .5188558     1.55   0.146     -.318419    1.923421
          x1 |   .0220002   .2821641     0.08   0.939    -.5875783    .6315787
          k1 |  -.2161035   .1191134    -1.81   0.093    -.4734323    .0412253
       _cons |   50.38438   31.63026     1.59   0.135    -17.94863    118.7174
------------------------------------------------------------------------------
```

```
. predict phat
(option xb assumed; fitted values)
(1 missing value generated)

. reg p1 t wg g yr p1 x1 k1

      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  7,    13) =       .
       Model |  324.518114     7  46.3597305           Prob > F      =       .
    Residual |           0    13           0           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  324.518114    20  16.2259057           Root MSE      =       0

------------------------------------------------------------------------------
          p1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |   1.52e-16          .        .       .            .           .
          wg |  -4.78e-15          .        .       .            .           .
           g |  -1.05e-15          .        .       .            .           .
          yr |   7.17e-16          .        .       .            .           .
          p1 |          1          .        .       .            .           .
          x1 |   1.94e-15          .        .       .            .           .
          k1 |  -4.21e-16          .        .       .            .           .
       _cons |   5.68e-14          .        .       .            .           .
------------------------------------------------------------------------------

. predict p1hat
(option xb assumed; fitted values)
(1 missing value generated)

. reg w t wg g yr p1 x1 k1

      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  7,    13) =   51.15
       Model |  1101.80519     7  157.400741           Prob > F      =  0.0000
    Residual |  40.0071584    13  3.07747372           R-squared     =  0.9650
-------------+------------------------------           Adj R-squared =  0.9461
       Total |  1141.81235    20  57.0906174           Root MSE      =  1.7543

------------------------------------------------------------------------------
           w |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |  -.6041532   .3485755    -1.73   0.107    -1.357205    .1488984
          wg |   .5562761   2.036218     0.27   0.789    -3.842705    4.955257
           g |   .8662196    .314328     2.76   0.016     .1871552    1.545284
          yr |   .7135829   .6253155     1.14   0.274    -.6373292    2.064495
          p1 |   .8719211   .4169601     2.09   0.057    -.0288665    1.772709
          x1 |   .0953288   .2267512     0.42   0.681    -.3945374    .5851951
          k1 |  -.1229518   .0957213    -1.28   0.221     -.329745    .0838415
       _cons |   43.43552   25.41854     1.71   0.111    -11.47789    98.34894
------------------------------------------------------------------------------

. predict what
(option xb assumed; fitted values)
(1 missing value generated)

. reg c phat p1hat what

      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  3,    17) =   73.65
       Model |  874.172577     3  291.390859           Prob > F      =  0.0000
    Residual |  67.2568126    17  3.95628309           R-squared     =  0.9286
-------------+------------------------------           Adj R-squared =  0.9160
       Total |  941.429389    20  47.0714695           Root MSE      =   1.989

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        phat |   .0173023   .2297973     0.08   0.941    -.4675275    .5021322
       p1hat |   .2162337   .2088099     1.04   0.315    -.2243166    .6567841
        what |   .8101827   .0783509    10.34   0.000     .6448768    .9754886
       _cons |   16.55476    2.57108     6.44   0.000     11.13025    21.97926
------------------------------------------------------------------------------
```

All equations can be estimated in one single command in `reg3` as follows:

OLS:

```
reg3 (c p p1 w) (i p p1 k1) (wp x x1 yr), ols

Multivariate regression
----------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"     F-Stat        P
----------------------------------------------------------------------
c                  21      3     1.02554    0.9810   292.7075   0.0000
i                  21      3    1.009447    0.9313   76.87538   0.0000
wp                 21      3    .7671466    0.9874   444.5687   0.0000
----------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
c            |
           p |   .1929343   .0912102     2.12   0.039     .0098223    .3760464
          p1 |   .0898847   .0906479     0.99   0.326    -.0920987     .271868
           w |   .7962188   .0399439    19.93   0.000      .716028    .8764095
       _cons |    16.2366   1.302698    12.46   0.000     13.62133    18.85188
-------------+----------------------------------------------------------------
i            |
           p |   .4796356   .0971146     4.94   0.000       .28467    .6746012
          p1 |   .3330387   .1008592     3.30   0.002     .1305554     .535522
          k1 |  -.1117947   .0267276    -4.18   0.000    -.1654525   -.0581369
       _cons |   10.12579   5.465546     1.85   0.070    -.8467492    21.09833
-------------+----------------------------------------------------------------
wp           |
           x |   .4394769   .0324076    13.56   0.000      .374416    .5045378
          x1 |     .14609   .0374231     3.90   0.000       .07096      .22122
          yr |   .1302452   .0319103     4.08   0.000     .0661826    .1943077
       _cons |   1.497043   1.270031     1.18   0.244    -1.052651    4.046737
------------------------------------------------------------------------------
```

2SLS:

```
reg3 (c p p1 w) (i p p1 k1) (wp x x1 yr), 2sls nodfk inst(t wg g yr p1 x1 k1)
```

```
Two-stage least-squares regression
----------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"     F-Stat        P
----------------------------------------------------------------------
c                  21      3    1.135659    0.9767   279.0941   0.0000
i                  21      3    1.307149    0.8849   50.89437   0.0000
wp                 21      3    .7671548    0.9874    524.005   0.0000
----------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
c            |
           p |   .0173022   .1180494     0.15   0.884    -.2196919    .2542963
          p1 |   .2162338    .107268     2.02   0.049     .0008844    .4315833
           w |   .8101827   .0402497    20.13   0.000      .729378    .8909874
       _cons |   16.55476   1.320793    12.53   0.000     13.90316    19.20636
-------------+----------------------------------------------------------------
i            |
           p |   .1502219   .1732292     0.87   0.390    -.1975503    .4979941
          p1 |   .6159434   .1627853     3.78   0.000     .2891382    .9427486
          k1 |  -.1577876   .0361262    -4.37   0.000    -.2303141   -.0852612
       _cons |   20.27821   7.542704     2.69   0.010     5.135599    35.42082
-------------+----------------------------------------------------------------
wp           |
           x |   .4388591   .0356319    12.32   0.000     .3673251    .5103931
          x1 |   .1466739   .0388361     3.78   0.000     .0687071    .2246406
          yr |   .1303956    .029141     4.47   0.000     .0718927    .1888985
       _cons |   1.500296   1.147779     1.31   0.197    -.8039674    3.804559
------------------------------------------------------------------------------
Endogenous variables:  c p w i wp x
Exogenous variables:   t wg g yr p1 x1 k1
------------------------------------------------------------------------------
```

3SLS:

```
reg3 (c p p1 w) (i p p1 k1) (wp x x1 yr), 3sls inst(t wg g yr p1 x1 k1)
```

```
Three-stage least squares regression
----------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"       chi2        P
----------------------------------------------------------------------
c                  21      3    .9443305    0.9801   864.5909   0.0000
i                  21      3    1.446736    0.8258   162.9808   0.0000
wp                 21      3    .7211282    0.9863   1594.751   0.0000
----------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
c            |
           p |   .1248904   .1081291     1.16   0.248    -.0870387    .3368194
          p1 |   .1631439   .1004382     1.62   0.104    -.0337113    .3599992
           w |    .790081   .0379379    20.83   0.000      .715724    .8644379
       _cons |   16.44079   1.304549    12.60   0.000     13.88392    18.99766
-------------+----------------------------------------------------------------
i            |
           p |  -.0130791   .1618962    -0.08   0.936    -.3303898    .3042316
          p1 |   .7557238   .1529331     4.94   0.000     .4559805    1.055467
          k1 |  -.1948482   .0325307    -5.99   0.000    -.2586072   -.1310893
       _cons |   28.17785   6.793768     4.15   0.000     14.86231    41.49339
-------------+----------------------------------------------------------------
wp           |
           x |   .4004919   .0318134    12.59   0.000     .3381388     .462845
          x1 |    .181291   .0341588     5.31   0.000     .1143411    .2482409
          yr |    .149674   .0279352     5.36   0.000      .094922    .2044261
       _cons |   1.797216   1.115854     1.61   0.107    -.3898181    3.984251
------------------------------------------------------------------------------
Endogenous variables:  c p w i wp x
Exogenous variables:   t wg g yr p1 x1 k1
------------------------------------------------------------------------------
```

<u>I3SLS</u>:

```
reg3 (c p p1 w) (i p p1 k1) (wp x x1 yr), 3sls ireg3 inst(t wg g yr p1 x1 k1)

Iteration 1:   tolerance =  .37125491
..
Iteration 24:   tolerance =  7.049e-07

Three-stage least squares regression, iterated
----------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"       chi2        P
----------------------------------------------------------------------
c                  21      3    .9565088    0.9796   970.3072   0.0000
i                  21      3    2.134327    0.6209   56.77951   0.0000
wp                 21      3    .7782334    0.9840   1312.188   0.0000
----------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
c            |
           p |   .1645096   .0961979     1.71   0.087    -.0240348    .3530539
          p1 |   .1765639   .0901001     1.96   0.050    -.0000291    .3531569
           w |   .7658011   .0347599    22.03   0.000     .6976729    .8339294
       _cons |   16.55899   1.224401    13.52   0.000     14.15921    18.95877
-------------+----------------------------------------------------------------
i            |
           p |  -.3565316   .2601568    -1.37   0.171    -.8664296    .1533664
          p1 |   1.011299   .2487745     4.07   0.000     .5237098    1.498888
          k1 |    -.2602    .0508694    -5.12   0.000    -.3599022   -.1604978
       _cons |   42.89629   10.59386     4.05   0.000     22.13271    63.65987
-------------+----------------------------------------------------------------
wp           |
           x |   .3747792   .0311027    12.05   0.000     .3138191    .4357394
          x1 |   .1936506   .0324018     5.98   0.000     .1301443     .257157
          yr |   .1679262   .0289291     5.80   0.000     .1112263    .2246261
       _cons |   2.624766   1.195559     2.20   0.028     .2815124    4.968019
------------------------------------------------------------------------------
Endogenous variables:  c p w i wp x
Exogenous variables:   t wg g yr p1 x1 k1
------------------------------------------------------------------------------
```

<u>LIML</u>:

```
. ivregress liml c p1 (p w=t wg g yr p1 x1 k1)

Instrumental variables (LIML) regression               Number of obs =      21
                                                       Wald chi2(3)  =  438.84
                                                       Prob > chi2   =  0.0000
                                                       R-squared     =  0.9566
                                                       Root MSE      =  1.3953

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |  -.2225133   .2017479    -1.10   0.270    -.6179318    .1729053
           w |   .8225587   .0553782    14.85   0.000     .7140194     .931098
          p1 |   .3960272   .1735978     2.28   0.023     .0557818    .7362727
       _cons |   17.14766   1.840296     9.32   0.000     13.54074    20.75457
------------------------------------------------------------------------------
Instrumented:  p w
Instruments:   p1 t wg g yr x1 k1
```

Hausman Test:

```
. reg c p w p1

      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  3,    17) =  292.71
       Model |  923.549937     3  307.849979           Prob > F      =  0.0000
    Residual |  17.8794524    17  1.05173249           R-squared     =  0.9810
-------------+------------------------------           Adj R-squared =  0.9777
       Total |  941.429389    20  47.0714695           Root MSE      =  1.0255

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |   .1929343   .0912102     2.12   0.049     .0004977     .385371
           w |   .7962188   .0399439    19.93   0.000     .7119444    .8804931
          p1 |   .0898847   .0906479     0.99   0.335    -.1013658    .2811351
       _cons |    16.2366   1.302698    12.46   0.000     13.48815    18.98506
------------------------------------------------------------------------------

. estimates store ols

. ivregress 2sls c p1 (p w=t wg g yr p1 x1 k1)

Instrumental variables (2SLS) regression               Number of obs =      21
                                                       Wald chi2(3)  =  837.28
                                                       Prob > chi2   =  0.0000
                                                       R-squared     =  0.9767
                                                       Root MSE      =  1.0218

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |   .0173022   .1180494     0.15   0.883    -.2140704    .2486748
           w |   .8101827   .0402497    20.13   0.000     .7312947    .8890707
          p1 |   .2162338    .107268     2.02   0.044     .0059925    .4264752
       _cons |   16.55476   1.320793    12.53   0.000     13.96605    19.14346
------------------------------------------------------------------------------
Instrumented:  p w
Instruments:   p1 t wg g yr x1 k1

. estimates store twostage

. hausman twostage ols

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |    twostage       ols         Difference          S.E.
-------------+----------------------------------------------------------------
           p |    .0173022     .1929343       -.1756322        .0749424
           w |    .8101827     .7962188        .0139639         .004952
          p1 |    .2162338     .0898847        .1263492         .057353
------------------------------------------------------------------------------
                           b = consistent under Ho and Ha; obtained from ivregress
            B = inconsistent under Ha, efficient under Ho; obtained from regress

    Test:  Ho:  difference in coefficients not systematic

                  chi2(3) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =        8.81
                Prob>chi2 =      0.0319
                (V_b-V_B is not positive definite)
```