B.E. International Program
Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics
Semester 1/2015

Homework 4 - Suggested Answers

Due 17 November 2015

Question 1: optimal factor inputs decision I

Suppose that the output Q of a firm depends on two inputs: x and y. The output level is determined by the production function

$$Q = f(x, y) = 8x + 12y - x^2 - 2y^2$$

Suppose that the output price is \$3 per unit, and the input prices for x and y are \$6 and \$12 per unit, respectively.

a. Determine whether the firm's production function is convex or concave by the derivative conditions.

Ans. $f_x = 8 - 2x$; $f_y = 12 - 4y$

$f_{xx} = -2$; $f_{xy} = f_{yx} = 0$; $f_{yy} = -4$

$|H_1| = -2 < 0$ and $|H| = 8 > 0$

Hence, the production function f(x,y) is a strictly concave function.

b. Find the levels of x^* and y^* that maximize the firm's profit, and verify the answer by using the second-order sufficient conditions.

Ans.

$$Max_{x,y}\ \pi(x, y) = 3(8x + 12y - x^2 - 2y^2) - 6x - 12y$$

FONC:

$\pi_x = 0 \Rightarrow \pi_x = 24 - 6x - 6 = 0 \Rightarrow x^* = 3$

$\pi_y = 0 \Rightarrow \pi_y = 36 - 12y - 12 = 0 \Rightarrow y^* = 2$

SOSC:

$\pi_{xx} = -6 < 0$; $\pi_{yy} = -12 < 0$; $\pi_{xy} = \pi_{yx} = 0$

$\pi_{xx}\pi_{yy} - \pi_{xy}^2 = 72 - 0 > 0$

$\Rightarrow$ the SOSC for a maximum profit is satisfied.

Hence, $(x^*, y^*) = (3, 2)$ maximizes the firm's profit.

Question 2: optimal factor inputs decision II

Suppose that the output Q of a firm depends on two inputs of the quantities K and L. The output level is determined by the production function

$$Q = 32K + 24L - 4K^2 - 2KL - 2L^2$$

a. Is the firm's production function strictly concave? Explain.

Ans.

$Q_K = 32 - 8K - 2L$

$Q_L = 24 - 2K - 4L$

$Q_{KK} = -8$; $Q_{LL} = -4$; $Q_{KL} = Q_{LK} = -2$

$|H_1| = -4 < 0$ and $|H| = Q_{KK}Q_{LL} - (Q_{KL})^2 = 32 - 4 = 28 > 0$

Hence, the production function Q(K,L) is a strictly concave function.

b. Write down the firm's profit function when the price of Q is \$1 and the per-unit factor prices of K and L are r and w, respectively, where both r and w are positive numbers. Find the levels of K^* and L^* that maximize the firm's profits.

Ans.

$\pi(K, L) = 1(32K + 24L - 4K^2 - 2KL - 2L^2) - rK - wL$

$Max_{K,L}\ \pi(K,L)$

FONC:

$$\pi_K = 32 - 8K - 2L - r = 0 \qquad \text{--(1)}$$

$$\pi_L = 24 - 2K - 4L - w = 0 \qquad \text{--(2)}$$

From (1)&(2),

$$K^* = \frac{40-2r+w}{14}\ ;\ L^* = \frac{64-4w+r}{14}$$

c. Verify that the second-order sufficient conditions for maximum profits are satisfied.

Ans.

SOSC:

$$\pi_{KK} = -8 < 0\ ;\ \pi_{LL} = -4 < 0\ ;\ \pi_{KL} = -2$$

$$\pi_{KK}\pi_{LL} - (\pi_{KL})^2 = 32 - (-2)^2 = 28 > 0$$

Hence, the SOSC for a maximum profit is satisfied.

d. Determine the effect of an increase in r on the firm's use of each input. (i.e. determine $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial r}$).

Ans.

$$\frac{\partial K^*}{\partial r} = \frac{\partial}{\partial r}\left(\frac{40-2r+w}{14}\right) = -\frac{1}{7} < 0\ ;\ \frac{\partial L^*}{\partial r} = \frac{\partial}{\partial r}\left(\frac{64-4w+r}{14}\right) = \frac{1}{14} > 0$$

Question 3: Multi-product problem

The demand for a monopolist's two products are determined by the equations

$$p_1 = 30 - q_1 \qquad \text{and} \quad p_2 = 28 - 2q_2$$

where p_1 and p_2 are prices per unit of the two goods, and q_1 and q_2 are the corresponding quantities. The costs of producing and selling q_1 units of the first good and q_2 units of the second good are

$$C(q_1, q_2) = 2q_1^2 + 4q_1q_2 + q_2^2 \ .$$

a) Find the monopolist's profit $\pi(q_1, q_2)$ from producing and selling q_1 units of the first good and q_2 units of the second good.

Ans. $\pi(q_1, q_2) = (30 - q_1)q_1 + (28 - 2q_2)q_2 - (2q_1^2 + 4q_1q_2 + q_2^2)$

b) Find the values q_1 and q_2 that maximize $\pi(q_1, q_2)$. Show the sufficient conditions for profit maximization.

$$Max_{q_1,q_2}\, \pi(q_1, q_2) = (30 - q_1)q_1 + (28 - 2q_2)q_2 - (2q_1^2 + 4q_1q_2 + q_2^2)$$

FONC:

$$\pi_1 = 30 - 2q_1 - (4q_1 + 4q_2) = 0 \quad \text{--(1)}$$

$$\pi_2 = 28 - 4q_2 - (4q_1 + 2q_2) = 0 \quad \text{--(2)}$$

$$\Rightarrow \qquad q_1^* = 3.4 \ \text{ and } \ q_2^* = 2.4$$

SOSC:

$$\pi_{11} = -6 < 0\,; \ \pi_{22} = -6 < 0$$

$$\pi_{11}\pi_{22} - (\pi_{12})^2 = (-6)(-6) - (-4)^2 = 36 - 16 = 20 > 0$$

Thus, the sosc for a maximum is satisfied.

Question 4

Consider the function f defined for all (x,y) such that

$$f(x,\ y; a) = \tfrac{1}{2}x^2 - x + ay(x-1) - \tfrac{1}{3}y^3 + a^2y^2 \ ,$$

where a is a constant.

a. Prove that $(x^*,\ y^*) = (1 - a^3, a^2)$ is a stationary point of $f(x,\ y; a)$.

FOC: $[x]:\ \ x - 1 + ay = 0$

$[y]:\ a(x-1) - y^2 + 2a^2y = 0$

To prove that $(x^*, y^*) = (1-a^3, a^2)$ is "a stationary point, we must show that (x^*, y^*) satisfies the first-order conditions. Evidently, if you plug the given $(x^*, y^*) = (1-a^3, a^2)$ into the two equations above, both conditions hold. Thus, $(1-a^3, a^2)$ is a stationary point.

b. Given that $G(a) = f(x^*, y^*; a)$, show the derivative of G with respect to a.

$$\begin{aligned} G(a) &= f(x^*, y^*; a) \\ &= \tfrac{1}{2}(1-a^3)^2 - (1-a^3) + a(a^2)((1-a^3)-1) - \tfrac{1}{3}(a^2)^3 + a^2(a^2)^2 \\ &= \tfrac{1}{2}(1-a^3)^2 - (1-a^3) - \tfrac{1}{3}(a^6) \end{aligned}$$

$$\tfrac{dG}{da} = (1-a^3)(-3a^2) + 3a^2 - 2a^5 = a^5$$

c. Calculate $\frac{\partial f(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1-a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?

$$\frac{\partial f(x,y;a)}{\partial a} = y(x-1) + 2a\,y^2$$

Plug $(x^*, y^*) = (1-a^3, a^2)$ into the above equation, we yield that:

$$a^2(1-a^3-1) + 2a\,(a^2)^2 = a^5$$

The equivalent results as obtained from (b) and (c) are formally known under *the Envelop theorem*. Heuristic proof is given in this example. (See the rigorous proof in Chiang.) In general, the theorem states that the order in which you differentiate the function with respect to "a" doesn't matter. At the optimal solution, both methods *must* result in the same answer.

d. Where in the xy-plan is f convex.

Consider the second-order derivative matrix:

$$H = \begin{bmatrix} 1 & a \\ a & -2y + 2a^2 \end{bmatrix};$$

$|H1| = 1$, and thus is guaranteed to be greater than zero. We only need to establish a condition that ensures |H2| is positive. From the calculation, we know that $|H2| = -2y + a^2 > 0$. This term is evidently positive when $y < \frac{a^2}{2}$.

Question 5: Price discrimination

Consider a monopolist producer of a product whose technology can be given by a constant marginal cost function, i..e MC = 4. The demand curves for the two market segments of this product are given below.

Segment A: P = 100 - 2Q
Segment B: P = 50 - .5Q

a.) If a monopolist can practice third-degree price discrimination, what price will they set in the two markets?
(Hint: your work should start from defining the objective function, and state down all the first-order conditions. You need to check for the sufficient condition to warrant that your answer is truly a maximum point.)

Setting $MR_A = MR_B = MC$ and solve for each Q in the two market segments.
For A, we know that 100 - 4Q = 4 ⇒ Q = 24 ⇒ P = 52.
For B, we know that 50 - Q = 4 ⇒ Q= 46 ⇒ P = 27.
Total Q combined to the two market is 24 + 46 = 70.

To check for the sufficient condition, we construct the Hessian matrix of firm's profit function. You should yield that,

$$H = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix};$$

Clearly, |H1| = - 4 < 0 and |H2| = 4 > 0. Thus, profit function is concave at the point (24,46). This confirms that our solution to the FOCs is a maximizer.

Here is I calculate the level of profit. Because marginal cost is constant equal to 4, thus total cost function should take the form of C(Q) = 4Q + F where F is the level of fixed cost. Using this result, we know that the level of total profit if equal to,

$$\pi(\text{third degree}) = 24 * 52 + 46 * 27 - 4(70) - F = 2490 - 280 - F = 2210 - F$$

b) Now suppose the monopolist *cannot* price discriminate. Instead, they must charge a single price in both markets. What price will they charge?
(Hint: you should start from discussing implication of demand curve faced by the monopolist, when the monopolist cannot distinguish type of the buyer. Then, start the optimization problem using that assumption.)

When monopolist fails to do the price discrimination, the monopolist will be producing as if it is facing a single market demand equation. (Failure to do the discrimination can arise if the monopolist cannot prevent the resell.) Using our technique discussed before, market demand equation can be given by;

$$P = \begin{cases} 100 - 2Q \ ; & 0 < Q \leq 25 . \\ 60 - 0.4Q \ ; & Q > 25. \end{cases}$$

Check for MR = MC for each part of the market demand.

Case [1] Suppose Q is set under $0 < Q \leq 25$.

$$100 - 4Q = 4 \Rightarrow Q = 24 \text{ and } P = 52$$

This case is possible because the answer yields us the level output Q that is consistent to the domain of the market demand function. This is a possible solution where monopolist only supply to a particular segment in the market, and exclude some other segments from the market. We don't know which one is better. So, we need to check for the level of maximized profit in this case.
Total profit is equal to: $\pi = 52 * 24 - 4 * 24 - F = 48 * 24 - F = 1152 - F$

Case [2] Alternatively, suppose that Q is set under $Q > 25$.

$60 - 0.8Q = 4 \Rightarrow Q = 70$ and $P = 32$.

This is the solution. Q = 70 is greater than 25 which is needed to justify the second part of the demand function.

Total profit is equal to: $\pi = 32 * 70 - 4 * 70 - F = 28 * 70 - F = 1960 - F$

Profit under case [2] is greater than that arises under case [1]. Thus, if the monopolist fails to do the discrimination, monopolist will choose to supply to both segments of the market. That is Q = 70 and P = 32.

Note further that comparing your answers between (a) and (b), total profit will drop if the monopolist cannot do the price discrimination.

Question 6

Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium

Each of the two markets is to be sold for equilibrium separately. (Setting demand and supply equal to each other, for each of the two markets.)

This yields us $Q_A = 3$ and $P_A = 4$ for the market A. For market B, we yield that $Q_B = 6$ and $P_B = 14$

b. Suppose the government imposes unit tax on both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.

For simplicity, we assume that tax is being imposed on consumers. (For the purpose of using the results obtained in "b" and working on the next problem "c", this assumption doesn't matter.)

So, $p_A{}^d = p_A{}^s + t_A$ and $p_B{}^d = p_B{}^s + t_B$. We yield that, after-tax equilibrium output would be $Q_A = \frac{9-t_A}{3}$ and $Q_B = \frac{18-t_B}{3}$. You can solve for the rest buy plugging these two Q back into the supply equation of each of the two markets. Doing so would give you producer's price, which then can be used to calculate for the consumers' price. To do so, just add producer's price with the unit tax imposed in each market.

c. How much revenue can the government collect from the taxation?

$$TR = t_A Q_A + t_B Q_B = t_A(\tfrac{9-t_A}{3}) + t_B(\tfrac{18-t_B}{3})$$

d. Determine the level of t_A and t_B that maximizes government's revenue.

Using the first order conditions approach, we yield that

$[t_A] : 9 - (2)t_A = 0$ ==> $t_A = 4.5$

$[t_B] : 18 - (2)t_B = 0$ ==> $t_B = 9$