Note Title 8/30/2016

Last time

2.1.4 Exact DE

2.1.5 Separable DE

2.1.6 Bernoulli's Equation

} possible non-linear DE

$$y'(t) + u(t).y(t) = w(t)\,[y(t)]^r \qquad (1) \quad \checkmark\checkmark$$

$r \neq 1$ and $r \neq 0$

Let $z(t) = [y(t)]^{1-r} \iff y(t) = [z(t)]^{\frac{1}{1-r}}$

(1) becomes

$$\frac{1}{1-r} z'(t) + u(t)\, z(t) = w(t) \qquad (2)$$

or $\quad z'(t) + (1-r)\, u(t)\, z(t) = (1-r)\, w(t)$

Ex $\quad y'(t) + t.y(t) = 3t\,[y(t)]^2$

$r = 2$

$$z(t) = [y(t)]^{1-r} = [y(t)]^{1-2} = y^{-1}$$

$$y(t) = z(t)^{-1}$$

From (2)

$$\frac{1}{1-r} z'(t) + u(t)\, z(t) = w(t)$$

$u(t) = t \qquad w(t) = 3t$

$$\frac{1}{1-2} z'(t) + t\, z(t) = 3t$$

$$(-1)\; z'(t) + t\, z(t) = 3t$$

$$z'(t) - t\, z(t) = -3t$$

Next ? Find $y(t) = [z(t)]^{-1}$

$$\boxed{z(t) = e^{-\int u(t)dt}\left[A + \int e^{\int u(t)dt} w(t)\, dt\right]}$$

OR

$$z'(t) = (z-3)\,t \qquad \text{separable}$$

step i $\quad g(z) = z-3$

$$f(t) = t$$

step ii $\quad \frac{1}{z-3} z'(t) = t$

step iii $\quad \int \frac{1}{z-3}\, dz = \int t\, dt$

step iv $\quad \ln|z-3| + c_1 = \frac{t^2}{2} + c_2$

$$\ln|z-3| = \frac{t^2}{2} + c \quad ; \; c = c_2 - c_1$$

$$e^{\ln|z-3|} = e^{\frac{t^2}{2} + c}$$

$$z(t) = A e^{\frac{t^2}{2}} + 3 \; ; \quad A = e^{c}$$

$$y(t) = [Ae^{t^2/2} + 3]^{-1}$$

What if you use formula.

Ex $$y'(t) + \frac{1}{t} y(t) = [y(t)]^3$$

$r \neq 1$; $r = 3$

$$\frac{1}{1-r} z'(t) + u(t) z(t) = w(t)$$

$$z'(t) - 2t^{-1} z(t) = -2$$

$u(t) = -2t^{-1}$ $\quad$ $w(t) = -2$

$$y(t) = (At^2 + 2t)^{-\frac{1}{2}}$$

Ex $$y'(t) = -ty + t^3 y^3$$

so far, quantitative analysis

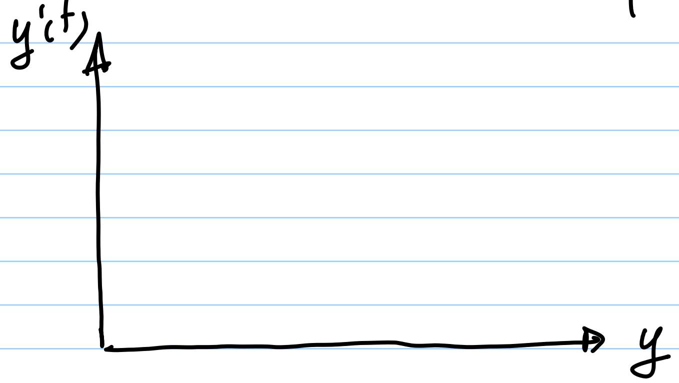
2.17 The Qualitative - Graphical Analysis

General Form $$\frac{dy}{dt} = F(y, t)$$

An Autonomous function. $$\frac{dy}{dt} = f(y)$$

Note $y = f(x)$ we can plot y against x

$y'(t) = f(y)$ — we can plot $y'(t)$ against y → phase diagram



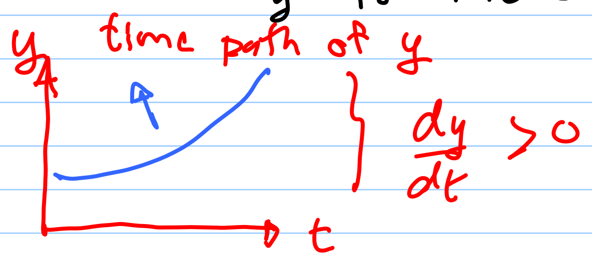
E Two general remarks

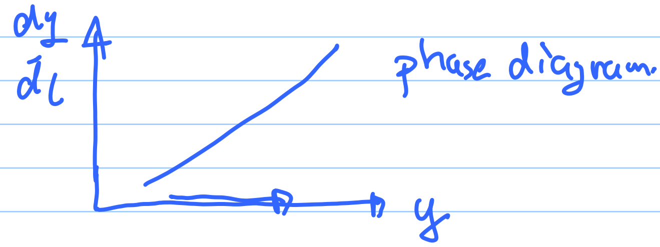
(R1) Anywhere ABOVE the horizontal axis $\Rightarrow \dfrac{dy}{dt} > 0$

overtime y is increasing

y is increasing with t

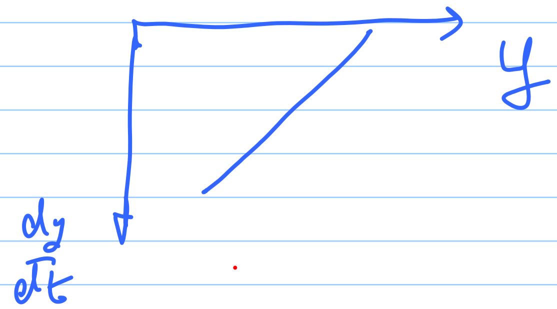
$\Rightarrow$





Anywhere BELOW the horizontal axis $\Rightarrow \dfrac{dy}{dt} < 0$

y is decreasing with t



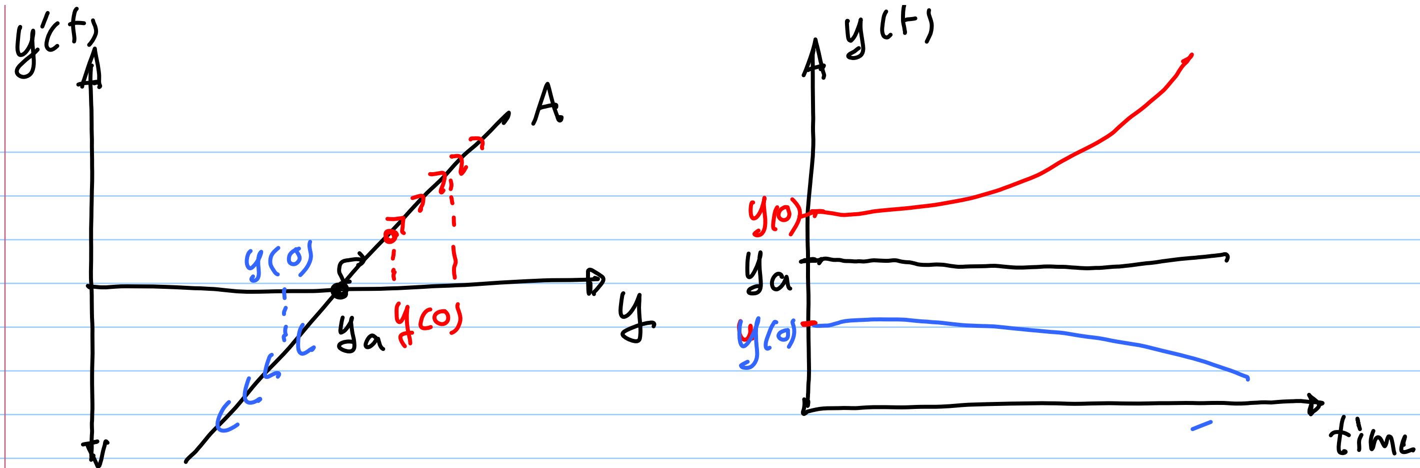
(R2) An equilibrium level of $y(t)$ $\Rightarrow$ in the intertemporal sense $\dfrac{dy}{dt} = 0$

$y(t)$ is stationary over time,

If the equilibrium exists

© Phase Diagram and Time Path

<1> Phase line A



If $y(0) = y_a$ $\Rightarrow$ the equilibrium (y_a) can be attained

If not, $y(0) > y_a \Rightarrow y(t)$ is ever increasing

$y(0) < y_a \Rightarrow y(t)$ is ever decreasing

The equilibrium y_a is Dynamically <u>UNSTABLE</u>

$$\frac{dy}{dt} + a\,y = b \quad ; \quad a, b \Rightarrow \text{constants}$$

$$\frac{dy}{dt} = (-a)y + b$$

Slope is $(-a)$ $\Rightarrow$ Phase line A

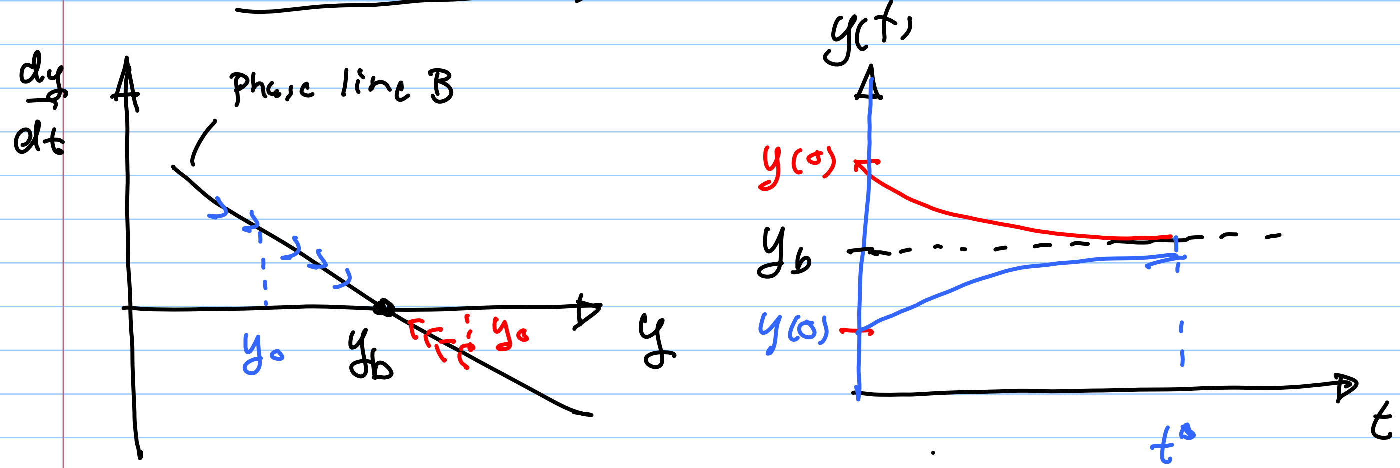
Slope > 0

$(-a) > 0$

$a < 0$

$$y(t) = \underbrace{\left[y(0) - \frac{b}{a}\right]e^{-at}}_{y_c = \text{deviation from the equilibrium}} + \underbrace{\frac{b}{a}}_{y_p \Rightarrow \text{equilibrium level}}$$

(2) Phase Line B



The equilibrium y_b is Dynamically STABLE

$$\frac{dy}{dt} = -ay + b$$

slope $= (-a)$ $\rightarrow$ slope < 0

$-a < 0$

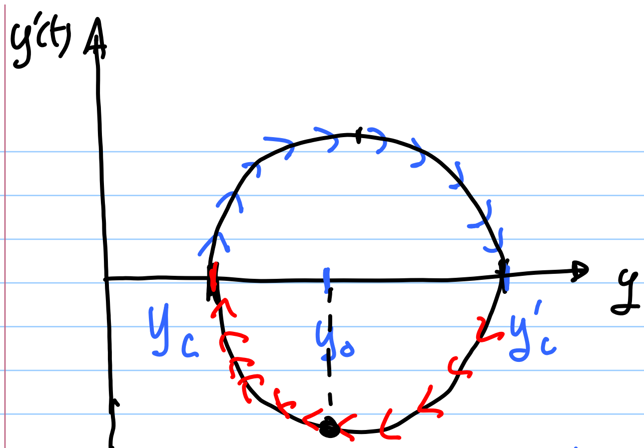
$\boxed{a > 0}$

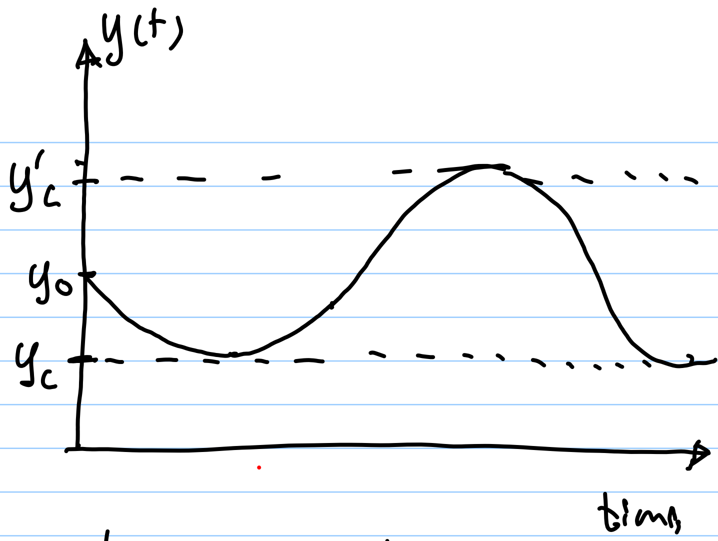
$$y(t) = \left[y(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a}$$

As $t \to \infty$, $e^{-at} \to 0$

$$\Rightarrow y(t) \to \frac{b}{a} \left(= y_p \right)$$

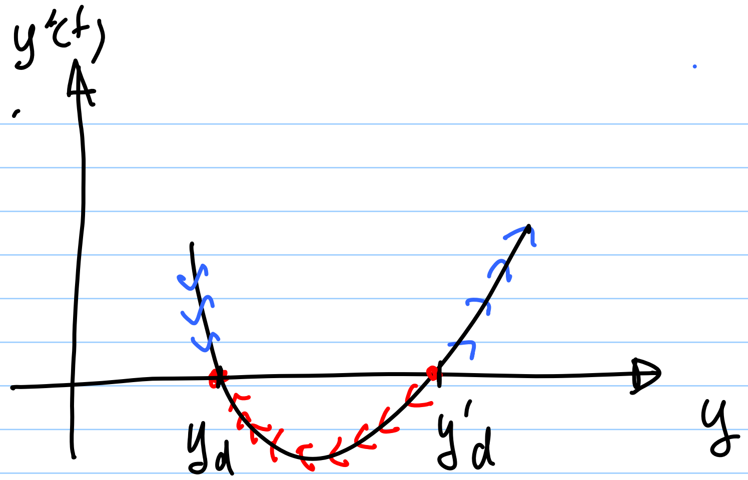
3) Phase line C





So y_c and y'_c do not represent the equilibrium in the intertemporal sense.

Ex 3



y_d is a locally asymptotically stable equilibrium

y'_d is an unstable equilibrium

@ The Solow Growth Model

The production f^n : $Q = f(K, L)$; $K, L > 0$ — (1)

Q = output K = capital

L = Labor

Note: 3 variables are real variables

$\Rightarrow$ $f_K > 0$ $f_L > 0$

$[MP_K > 0]$ $[MP_L > 0]$

$\Rightarrow$ $f_{KK} < 0$ $f_{LL} < 0$

$\Rightarrow$ Assuming Linearly homogeneous CRTS f^n

$$Q = L \cdot f\left(\frac{K}{L}, 1\right)$$

$$= L\,\phi(k)$$

$k = \frac{K}{L}$ = capital - labor ratio

= capital per capita.