

Solution: Quiz 5

1. Find the solution set for

$$\left| \frac{1+x}{x-|x|} \right| \leq \frac{1}{|x|}.$$

Solution: Notice that

$$\left| \frac{1+x}{x-|x|} \right| \leq \frac{1}{|x|} \Leftrightarrow -\frac{1}{|x|} \leq \frac{1+x}{x-|x|} \leq \frac{1}{|x|}.$$

So we will consider 2 cases based on the definition of absolute value: $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0. \end{cases}$

(I) For $x < 0$ or $x \in (-\infty, 0)$,

$$\begin{aligned} -\frac{1}{|x|} &\leq \frac{1+x}{x-|x|} \leq \frac{1}{|x|} \\ -\frac{1}{-x} &\leq \frac{1+x}{x-(-x)} \leq \frac{1}{-x} \\ \frac{1}{x} &\leq \frac{1+x}{2x} \leq -\frac{1}{x} \end{aligned}$$

$$\begin{aligned} \text{Multiplying by } x < 0 \quad 1 &\geq \frac{1+x}{2} \geq -1 \\ 2 &\geq 1+x \geq -2 \\ 2-1 &\geq x \geq -2-1 \Rightarrow x \in [-3, 1] \end{aligned}$$

and the solution set in this case is $[-3, 1] \cap (-\infty, 0) = [-3, 0)$.

(II) For $x \geq 0$, we have $|x| = x$ implying $x - |x| = 0$. However the term $x - |x|$ is the denominator and cannot be zero. I.e. $x - |x| \neq 0 \Leftrightarrow |x| \neq x \Leftrightarrow x \not\geq 0 \Leftrightarrow x \notin [0, \infty)$. So there is no solution in this case.

From (I) and (II), the solution set only comes from (I) and it is given by $[-3, 0)$. ■