Chapter 5

Nonlinear Model and Differential Calculus in Economic Theory

Topics:

- Quadratic theory
- Other nonlinear functions
- Slope and derivatives of a function
- Rule of differentiation
- Non differentiable functions
- Convexity and concavity
- Maxima-Minima
- Inflection point
- Examples in Economics
    - Derivative and marginality
    - Relations among the total, the average and the marginal functions
    - Elasticity, total revenue and marginal revenue

Comparative Statics and Derivative

VOCAB: "Difference Quotient" / "Derivative"/ "Differentiation"/ "Differential Calculus"

Comparative Statics is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables.

For example,

When $G = G_0$

$$\left.\begin{array}{l}(1.)\ Y = C + I + G\\(2.)\ C = a + bY_d\\(3.)\ Y_d = Y - T\\(4.)\ I = I_0\\(5.)\ G = G_0\end{array}\right\} \Rightarrow Y_E\big|_{G=G_0} = \frac{a - bT + I_0 + G_0}{1-b}$$

When $G = G_1$

$$\left.\begin{array}{l}(1.)\ Y = C + I + G\\(2.)\ C = a + bY_d\\(3.)\ Y_d = Y - T\\(4.)\ I = I_0\\(5.)\ G = G_1\end{array}\right\} \Rightarrow Y_E\big|_{G=G_1} = \frac{a - bT + I_0 + G_1}{1-b}$$

$$\begin{array}{lll}when & G_0 & \rightarrow & Y_E\big|_{G_0}\\ when & G_1 & \rightarrow & Y_E\big|_{G_1}\end{array}$$

$$\Delta G = G_1 - G_0 \quad \Rightarrow \quad \Delta Y = Y_E\big|_{G_1} - Y_E\big|_{G_0}$$

It should be clear that the problem under consideration is essentially one of finding *a rate of change: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogenous variable.*

The notion of rate of change is directly concerned with the mathematical concept of *derivative*, in *differential calculus*. (studies the rates at which quantities change)

Looking at a function: $y = f(x)$

$$\begin{aligned} &when \quad x_1 \quad \rightarrow \quad y_1 = f(x_1) \\ &when \quad x_2 \quad \rightarrow \quad y_2 = f(x_2) \end{aligned}$$

$$\Delta x = x_2 - x_1 \quad \Rightarrow \quad \Delta y = y_2 - y_1$$

We are interested in $\frac{\Delta y}{\Delta x}$ which is called "difference quotient"

When $\Delta x \rightarrow 0$, $\frac{\Delta y}{\Delta x} \longrightarrow \frac{dy}{dx}$ i.e. $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = f'(x)$

, which is called "derivative of f at x"

Notation for derivative of f at x: $f'(x)$, $\frac{df(x)}{dx}$, $\frac{dy}{dx}$, $D_x f(x)$, y'.

A *derivative* is a function. The word *derivative* means a derived function, from the original function $y = f(x)$, which is also called a *primitive function*.

When we have a Primitive function $y = f(x)$ and try to find Derived function $\frac{dy}{dx}$,

we call this process "differentiation".

Derivative and differential Calculus will be used in finding maximum, minimum points and in optimization problem.

The Slope of a Curve and The Derivative of a function

The concept of the **slope** of a curve is the geometric counterpart of the concept of the **derivative**. Both concepts deal with the **marginal** notion used in economics.

The slope of a total cost function: $C = F(q)$ measures the change in total cost resulting from a unit increase in output, i.e. the marginal cost(MC). That is,

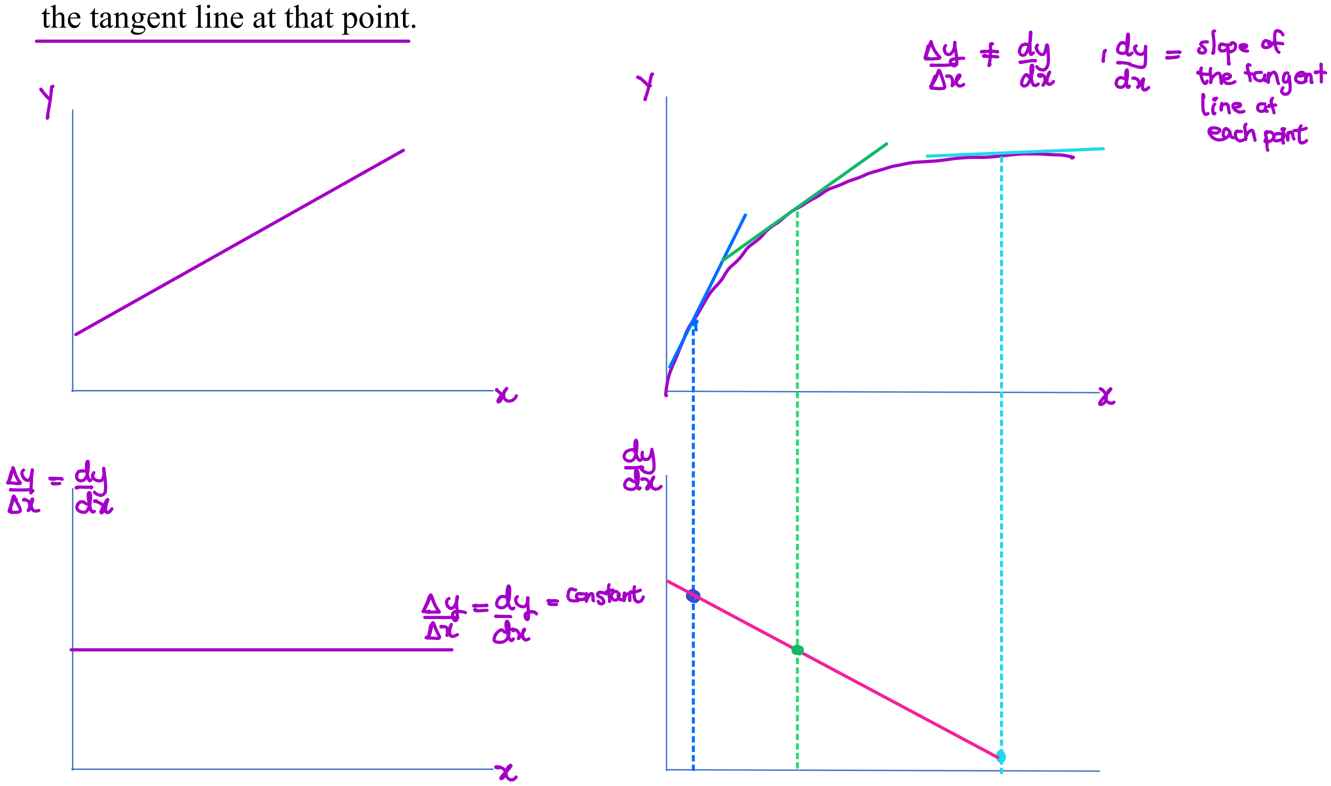
MC = slope of total cost function = $\frac{\Delta C}{\Delta q}$ or $\frac{dC}{dq}$

The slope of a utility function: $u = U(x)$ measures the change in utility resulting from a unit increase in consumption, i.e. the marginal utility(MU). That is,

MU = slope of utility function = $\frac{\Delta u}{\Delta x}$ or $\frac{dU}{dx}$

⊙ If a function is linear, the slope is constant and is equal for every points on the linear curve.

⊙ If a function is nonlinear function, the slope is not constant. Slope for each point on the curve might not be equal. Slope at each point on a nonlinear function is the slope of the tangent line at that point.



Differentiation / derivation / process of obtaining the derivative

From definition of derivative, we can find the derivative from a primitive function as the following.

For example, let $f(x) = 2x^2 + 4$ find $f'(x)$:

(1.) $f'(x) = 4x$

(2.) Use derivative definition: $f'(x) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$

Let x increases from x_1 to $x_2 = x_1 + h$, so $\Delta x = x_2 - x_1 = h$ with

$$f(x_1) = 2x_1^2 + 4 \text{ และ } f(x_2) = 2x_2^2 + 4 = 2(x_1 + h)^2 + 4 = 2x_1^2 + 4x_1h + 2h^2 + 4$$

Therefore,

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{h}$$
$$= \frac{(2x_1^2 + 4x_1 h + 2h^2 + 4) - (2x_1^2 + 4)}{h}$$
$$= 4x_1 + 2h$$

x_1 is actually can be any x $\frac{\Delta y}{\Delta x} = 4x + 2h$

$$f'(x) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$$
$$= \lim_{h \to 0}(4x + 2h)$$
$$= 4x \qquad \#$$

Differentiability of a function

"A function is differentiable at point x_0 if it is smooth and continuous at point x_0."

✪ Continuity

🔔 Continuity is a necessary condition for a function to be differentiable.

The function f is continuous at a if
1. We can find $f(a)$, i.e. $x = a$ must be in the domain of the function f.
2. We can find $\lim_{x \to a} f(x)$
3. $\lim_{x \to a} f(x) = f(a)$

Example of a function that is not continuous at $x = 0.6$

$f(x)$

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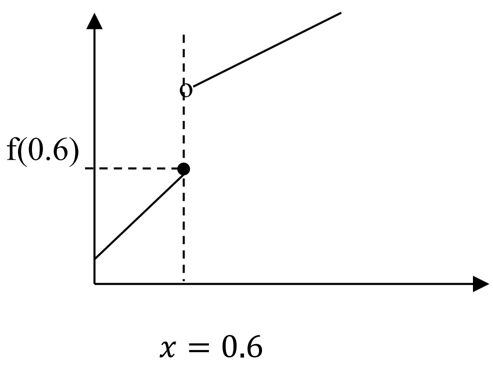
1. $f(0.6)$ can be found.
2. $\lim_{x \to 0.6^-} f(x) \neq \lim_{x \to 0.6^+} f(x)$

$\therefore f(x)$ is not continuous at $x = 0.6$

not continuous function



H.w.: Is this function continuous $f(x)=\begin{cases} x^2 & \text{when } x<2 \\ x+1 & \text{when } x\geq 2 \end{cases}$?

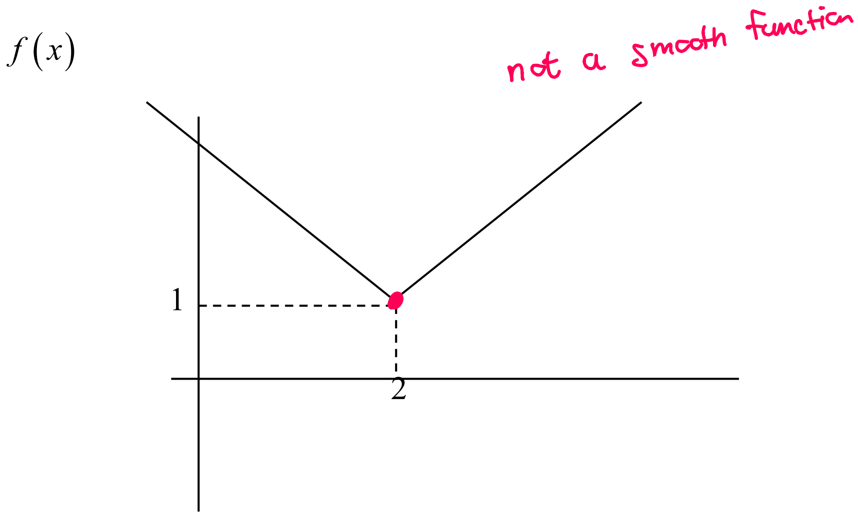
✪ Smooth Function(has no kink)

🔔 Smooth function is a sufficient condition for differentiability.

The differentiability condition is:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

Example of a function that has kink: $f(x)=|x-2|+1$



Continuity: $f(2)=\lim_{x\to 2} f(x)=1$, $f(x)$ is continuous at $x=2$

But if we try to find derivative:

Function f has derivative at a if:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

If we let $x=a+h$

when $h\to 0$, $x\to a$ and we can rewrite $f'(a)$ as:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad , \quad x\neq a$$

Derivative of $f(x)$ at $x=2$ is:

$$f'(2)=\lim_{x\to 2}\frac{|x-2|+1-1}{x-2}=\lim_{x\to 2}\frac{|x-2|}{x-2}$$

$$\text{where } \lim_{x\to 2^-}\frac{|x-2|}{x-2}=\frac{-(x-2)}{x-2}=-1$$

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2}=\frac{x-2}{x-2}=1$$

$$\therefore \lim_{x\to 2^-}\frac{|x-2|}{x-2}\neq \lim_{x\to 2^+}\frac{|x-2|}{x-2}$$

Therefore, $f'(2)=\lim_{x\to 2}\frac{|x-2|}{x-2}$ cannot be found.

Even $f(x)$ is continuous at $x=2$ but we cannot find $f'(2)$ because $f(x)$ has a kink at $x = 2$.

Notation:

$f\in c^{(0)}$ or $f\in c$: means f is continuous.

$f\in c^{(1)}$ or $f\in c'$: means f is continuously differentiable (A function f with a continuous derivative function, i.e. the everywhere-smooth function)

Rule of Differentiation

1. If $f(x) = c$, c is a constant $f'(x) = 0$

2. If $f(x) = cg(x)$, c is a constant $f'(x) = cg'(x)$

3. If $f(x) = x^n$, n is any real number $f'(x) = nx^{n-1}$

4. If $f(x) = U(x) \pm V(x)$, $f'(x) = U'(x) \pm V'(x)$

5. If $f(x) = U(x)V(x)$, $f'(x) = U(x)V'(x) + V(x)U'(x)$

6. If $f(x) = \frac{U(x)}{V(x)}$, $f'(x) = \frac{V(x)U'(x)-U(x)V'(x)}{[V(x)]^2}$

7. [chain rule] If we have a differentiable function $z = U(y)$ and another differentiable function $y = V(x)$, then the derivative of z with respect to x is equal to the derivative of z with respect to y, times the derivative of y with respect to x .

$$\frac{dz}{dx}=\frac{dz}{dy}\frac{dy}{dx}=U'(y)V'(x)$$

Change in x determines change in y via function V, and change in y determines change in z via function U.

8. [Derivatives of Inverse Function] Let $y = f(x)$, we have $\frac{dy}{dx}$. The inverse function of $f, x = f^{-1}(y)$, its derivative is $\frac{dx}{dy}$, and

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

9. [Derivatives of log function]
If $y = \log_a x$, $\frac{dy}{dx} = \frac{1}{xlna}$.
If $y = ln\,x$, $\frac{dy}{dx} = \frac{1}{xlne} = \frac{1}{x}$.
If $y = ln\,V(x)$, $\frac{dy}{dx} = \frac{V'(x)}{V(x)}$.

10. [Derivatives of exponential function]

If $y = a^x$, where $a > 0, a \neq 1$, $\frac{dy}{dx} = a^x lna$

If $y = e^x$, $\frac{dy}{dx} = e^x$

If $y = e^{V(x)}$, $\frac{dy}{dx} = e^{V(x)}V'(x)$

H.w.:

(a.) find $f'(x)$ for the following functions:

f(x) = $\sqrt{2}x^{\frac{-2}{3}}$

f(x) = $2x^3 + 3x^2 - 5x + 1$

$f(x) = (2x+3)(3x^2)$

$f(x) = \frac{ax^2 + b}{cx}$

$f(x) = (x - 4x^2)^3$

$f(x) = (1 - x^2)\sqrt{1 - 2x}$

(b.)

If $y = f(x) = 5x + 25$, find $\frac{dx}{dy}$

If $y = f(x) = x^5 + x$, find $\frac{dx}{dy}$

(c.)

Find $f'(x)$ if $f(x) = e^{3x-1}$

Find $f'(x)$ if $f(x) = \ln(1 + 2x + x^2)$

Find $f'(x)$ if $f(x) = (1+x)(1+e^{x^2})(3-x)^{\frac{1}{2}}$

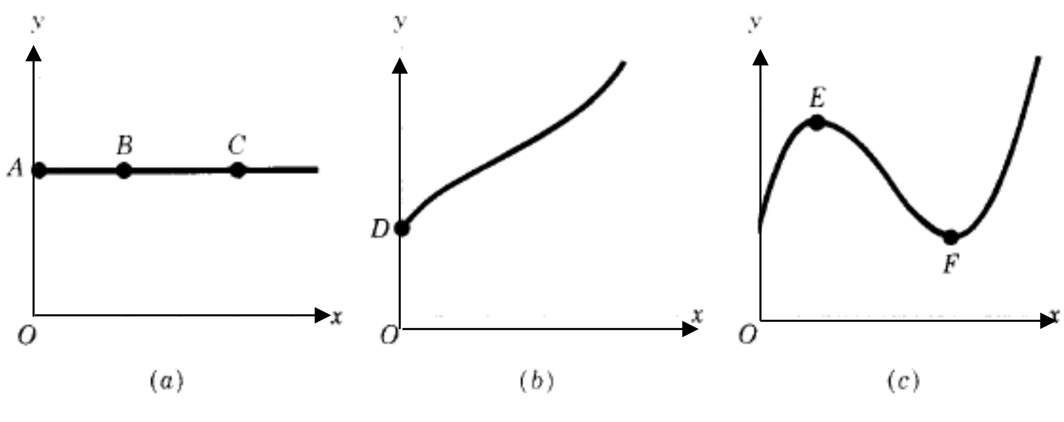
Find $f'(x)$ if $f(x) = (x^2 - 1)\sqrt{\frac{2-3x^2}{1-2x^3}}$

(d.) Let $y = x^4 - x^{\frac{4}{3}} + 6x^{\frac{1}{3}}$, find the second derivative of $f(x)$ with respect to x, $f''(x)$.

Maxima and Minima, Convexity and Concavity

Global vs. Local Extremum Concept

Consider (a), (b) , (c) and (d)



Graph(a.): Constant Function

No maximum / minimum of y

$\frac{dy}{dx} = 0$

Graph (b.): Strictly Increasing Function

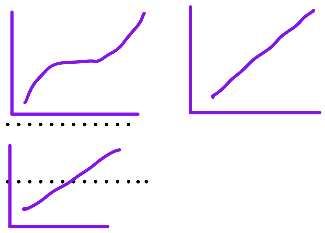
When $x \uparrow$, $y \uparrow$, no maximum value of y,

minimum of y at point D

$\frac{dy}{dx} > 0$

Note:

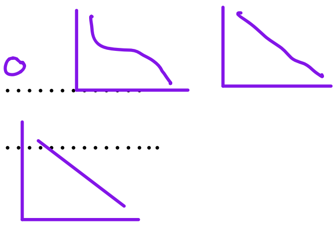
Monotonic Increasing Function



if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$, $\frac{dy}{dx} \geq 0$

Strictly monotone increasing → $\frac{dy}{dx} > 0$

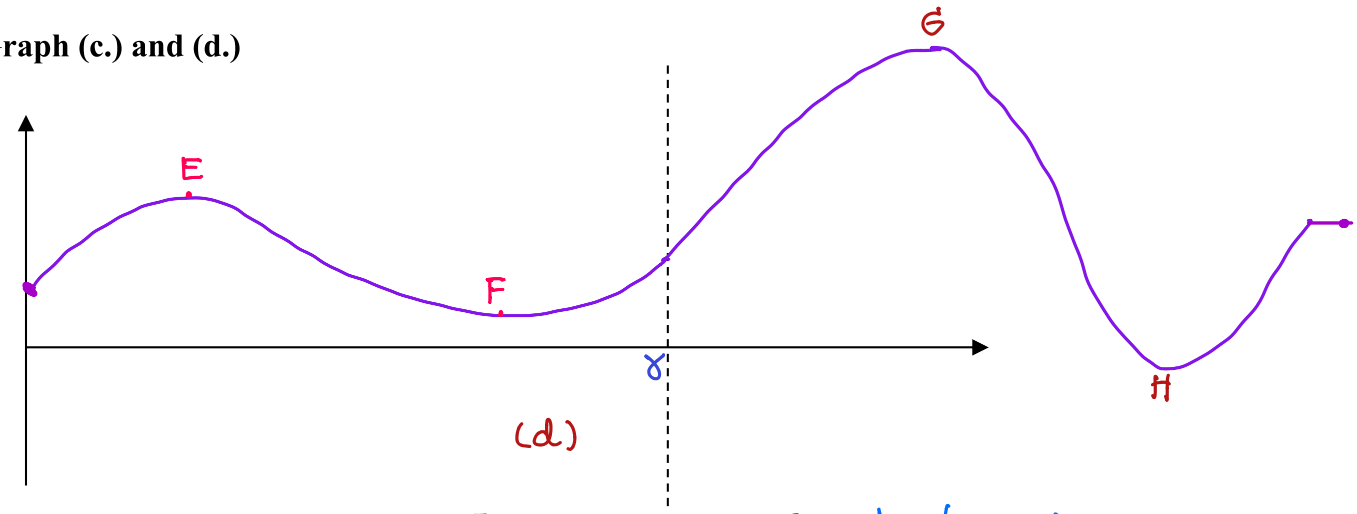
Monotonic Decreasing Function



if $x_1 \leq x_2$ then $f(x_1) \geq f(x_2)$, $\frac{dy}{dx} \leq 0$

Strictly monotone decreasing → $\frac{dy}{dx} < 0$

Graph (c.) and (d.)



For the part of domain, $[0, \gamma]$, E is local maximum.
F is local minimum

local max, local min can be global max, min

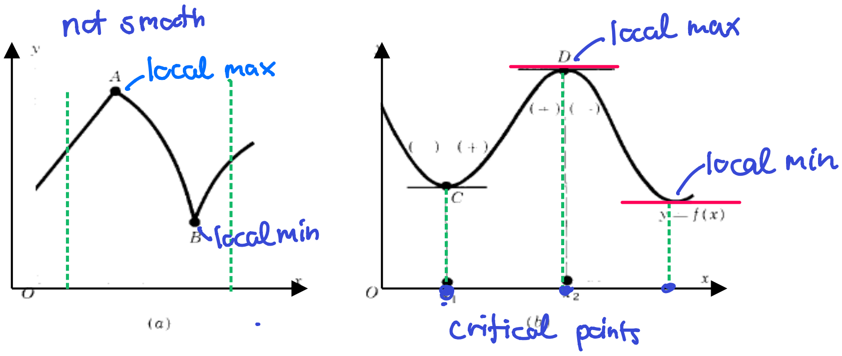
In this graph, G is global max
H is global min.

How to test if a critical point is a local max, a local min, or inflection point

✪ First-Derivative Test ✪

If y = f(x) has Local Max/ Local Min) at $x = x_0$, possible cases for the first derivative are:

1. $f'(x_0)$ *does not exist*
2. $f'(x_0) = 0$

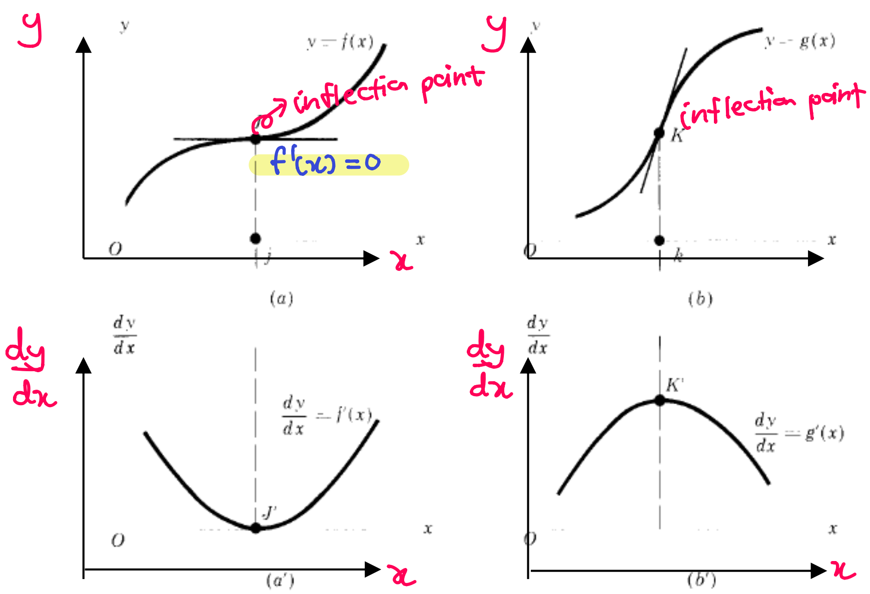


If $f(x)$ **is a smooth function, the necessary condition** for a point to be a local max or local min is:

$$f'(x) = 0$$

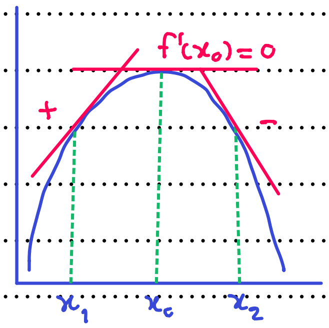
We call the order pair (x, y) that satisfies this necessary condition a stationary point, in which x is "critical point or critical value", and y is "stationary value".

The condition $f'(x) = 0$ is a necessary condition, but not a sufficient condition) for a point to be a local maximum or local minimum.



If the first derivative of f(x) at $x = x_0$ is 0 ($f'(x_0) = 0$), then $f(x_0)$ will be

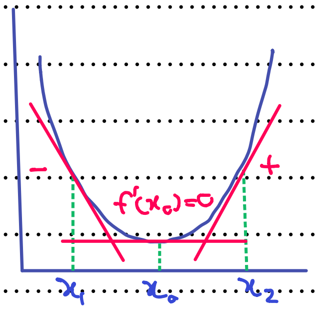
1. **Local maximum,** when the first derivative changes from being positive to negative when $x < x_0$ and $x > x_0$



$x_1 < x_0 \rightarrow f'(x_1) > 0$

$x_2 > x_0 \rightarrow f'(x_2) < 0$

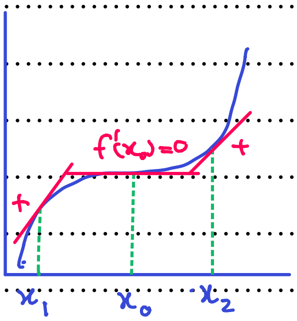
2. **Local minimum,** when the first derivative changes from being negative to positive when $x < x_0$ and $x > x_0$



$x_1 < x_0 \rightarrow f'(x_1) < 0$

$x_2 > x_0 \rightarrow f'(x_2) > 0$

3. **Inflection point,** when the first derivative doesn't change sign when $x < x_0$ and $x > x_0$



$x_1 < x_0 \rightarrow f'(x_1) > 0$ $(<)$

$x_2 > x_0 \rightarrow f'(x_2) > 0$ $(<)$

H.w.: Find local maxima or minima of the following functions:

$y = f(x) = x^3 - 12x^2 + 36x + 8$

Average-Cost Function $AC = f(Q) = Q^2 - 5Q + 8$

✪ Second-Derivative Test ✪

Interpretation of the Second Derivative

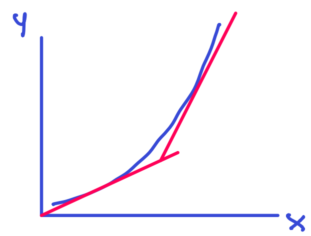
f′(x) ……slope of graph $y = f(x)$……

f″(x) ……rate of change of slope of graph $y = f(x)$……
slope of graph $y' = f'(x)$

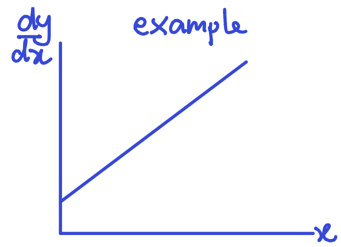
$f'(x) > 0$, $f'(x) < 0$ } f(x) will be } strictly increase / strictly decrease

$f''(x) > 0$, $f''(x) < 0$ } slope of……f(x) will be } strictly increasing / strictly decrease

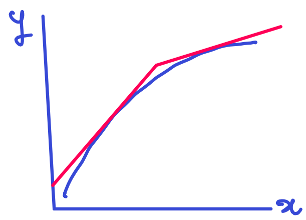
If $f'(x) > 0$ and $f''(x) > 0$, convex f



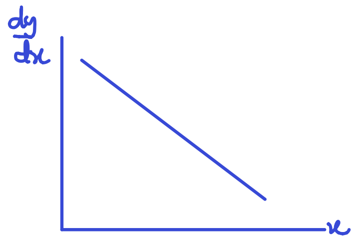
as $x\uparrow$, slope $\uparrow$, i.e. $y\uparrow$ at increasing rate



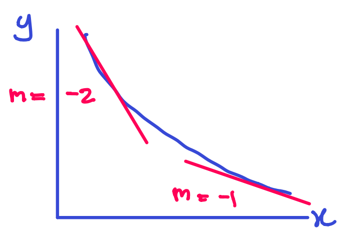
If $f'(x) > 0$ and $f''(x) < 0$, concave f



as $x\uparrow$, slope $\downarrow$
$y\uparrow$ at decreasing rate

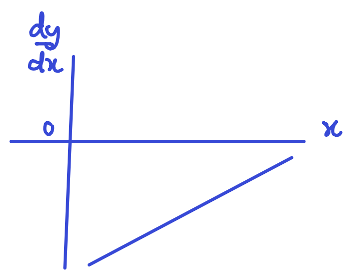


If $f'(x) < 0$ and $f''(x) > 0$, convex f

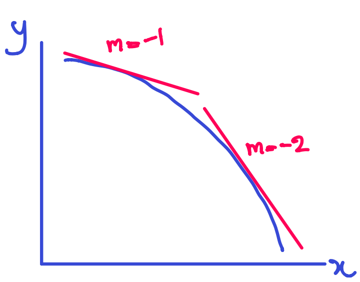


as $x\uparrow$, magnitude of slope $\downarrow$ (less negative) slope $\uparrow$

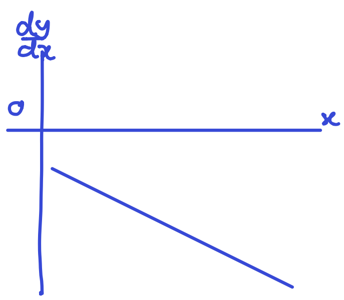
$\frac{\Delta m}{\Delta x} = \frac{-1-(-2)}{1} = 1 > 0$

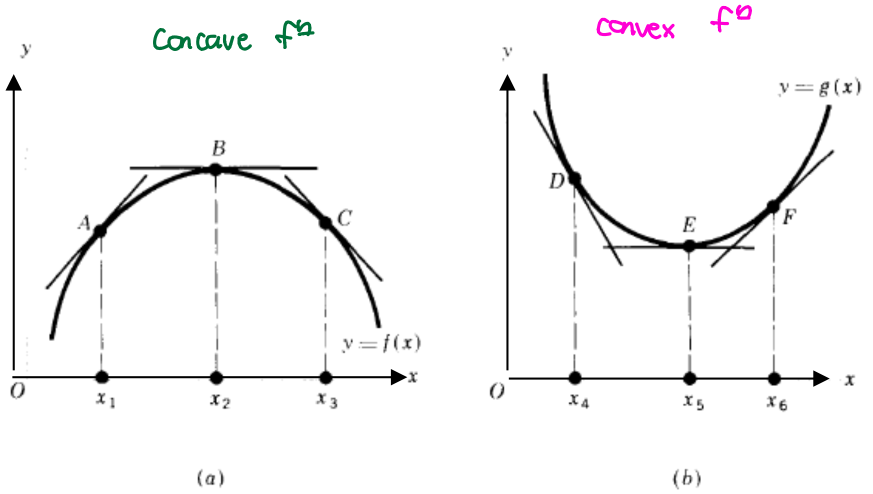


If $f'(x) < 0$ and $f''(x) < 0$, concave f



as $x\uparrow$, magnitude of slope $\uparrow$ (more negative) slope $\downarrow$





From graph (a.) From A → B → C

$x_1 \to x_2 \to x_3$, $f'(x_1) > f(x_2) > f'(x_3)$

$f'(x) \downarrow$ slope keeps decreasing

$f''(x) < 0$

and $f''(x_2) < 0$

From graph (b.)

$x_4 \to x_5 \to x_6$, $f'(x_4) < f'(x_5) < f'(x_6)$

$f'(x) \uparrow$ slope keeps increasing

$f''(x) > 0$

$f''(x_5) > 0$

If $f'(x_0) = 0$, $f(x_0)$ will be

→ a local maximum when $f''(x_0) < 0$

→ a local minimum when $f''(x_0) > 0$

Curvature of a Graph

A graph can be:

→ Strictly Convex/ Convex

→ Strictly Concave/Concave

$f : \mathbb{R} \to \mathbb{R}$ is (strictly) convex if $\forall x, y \in \mathbb{R}$, $\lambda \in (0,1)$

$f(\lambda x + (1-\lambda) y) \le (<) \; \lambda f(x) + (1-\lambda) f(y)$

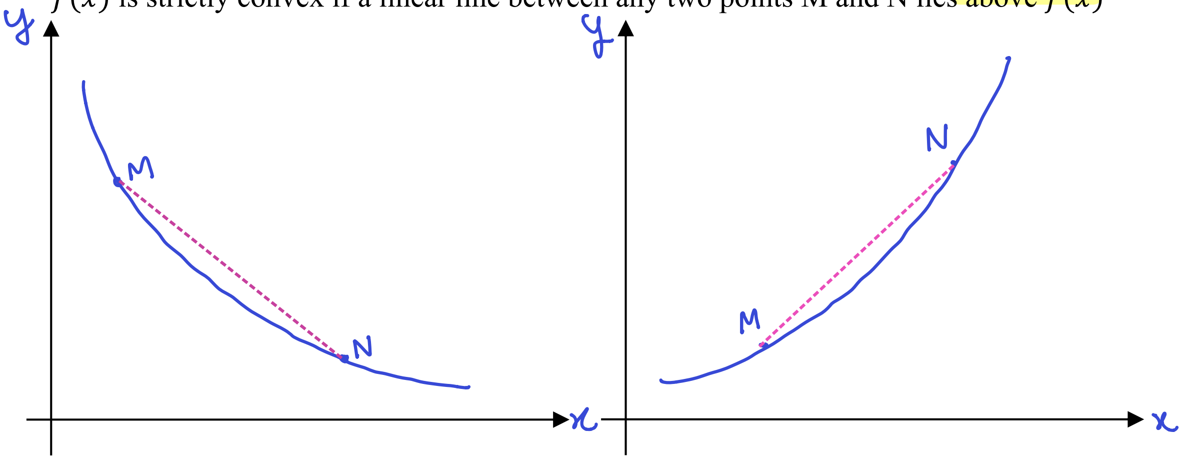
f^n of lin com $\le$ lin com of f^n

$f : \mathbb{R} \to \mathbb{R}$ is (strictly) concave if $\forall x, y \in \mathbb{R}$, $\lambda \in (0,1)$

$f(\lambda x + (1-\lambda) y) \ge (>) \; \lambda f(x) + (1-\lambda) f(y)$

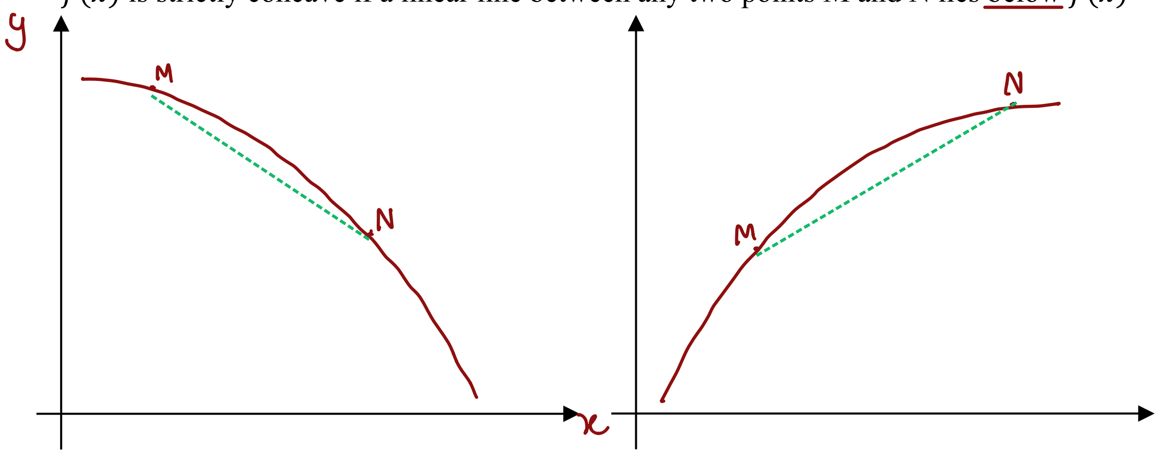
f^n of lin com $\ge$ lin com of f^n

$f(x)$ is strictly convex if a linear line between any two points M and N lies above $f(x)$



$f''(x)$ ....> 0.......... for strictly convex function

$f(x)$ is strictly concave if a linear line between any two points M and N lies below $f(x)$



$f''(x)$ ....< 0.......... for strictly concave function

$y = x^2 \to 2x = 0 \to 2$

$y = x^4 \to 4x^3 = 0 \to 12x^2 \Rightarrow 0$

SUMMARY

at x_0, a critical value

Condition	Maximum	Minimum
First-order necessary	$f'(x_0) = 0$	$f'(x_0) = 0$
Second-order necessary There are max/min that have $f'(x_0)=0$ e.g. $y=x^4$, $y=-x^4$ We can do 1st derivative test to differentiate from inflection point	$f''(x_0) \leq 0$	$f''(x_0) \geq 0$
Second-order sufficient	$f''(x_0) < 0$	$f''(x_0) > 0$

H.W.:

(1.) $f(x) = \frac{1}{8}\left(x^4 - 8x^2\right)$

(2.) $f(x) = \frac{x^4}{4} - \frac{3}{2}x^2$

(3.) $f(x) = x^4$

Application of differential calculus in economics

Demand **(1.)** $Q^d = a + bP$

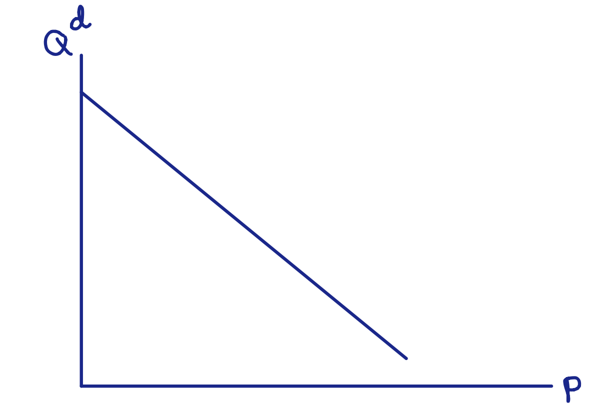
I. $\dfrac{dQ^d}{dP}$

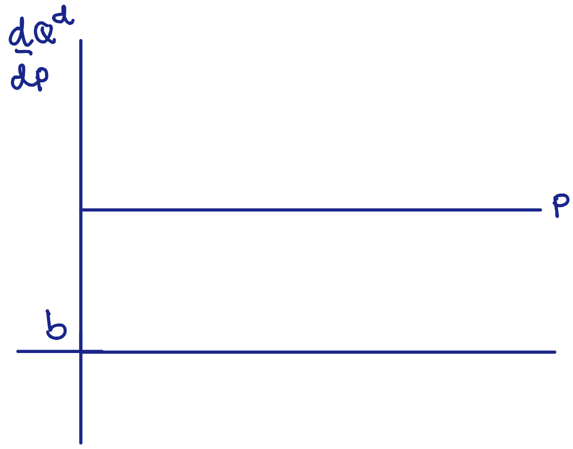
$\dfrac{dQ^d}{dp} = b < 0$

II. $\dfrac{d^2Q^d}{dP^2}$

$\dfrac{d^2Q^d}{dp^2} = 0$

The graphs are:



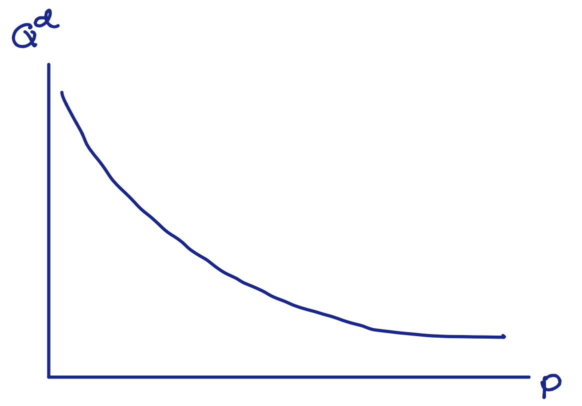


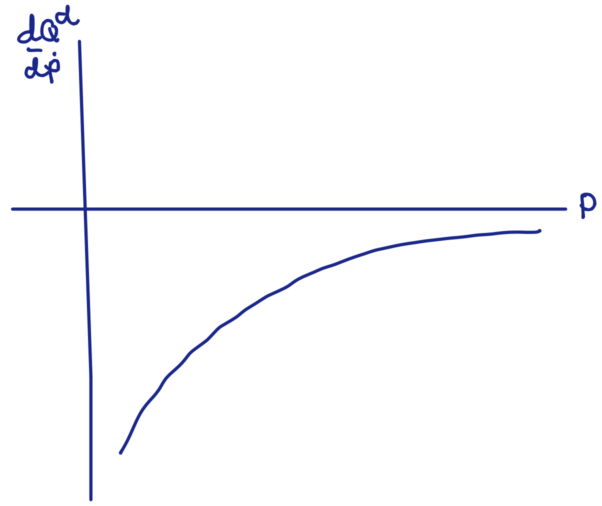
(2.) $Q^d = \dfrac{a}{P}$

$\dfrac{dQ^d}{dp} = -\dfrac{a}{p^2} < 0, \quad a > 0$

$\dfrac{d^2Q^d}{dp^2} = \dfrac{2a}{p^3} > 0$ slope is increasing.

$\dfrac{d^3Q^d}{dp^3} = -\dfrac{6a}{p^4} < 0$ slope is concave.





Total Utility

$$\frac{dTU}{dx} > 0$$

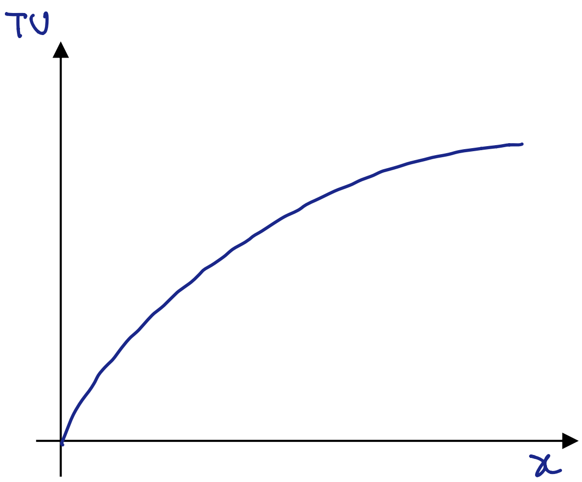
→ when consumpt^n ↑ , total utility ↑
→ marginal utility > 0
→ good
if it is bad, mu < 0

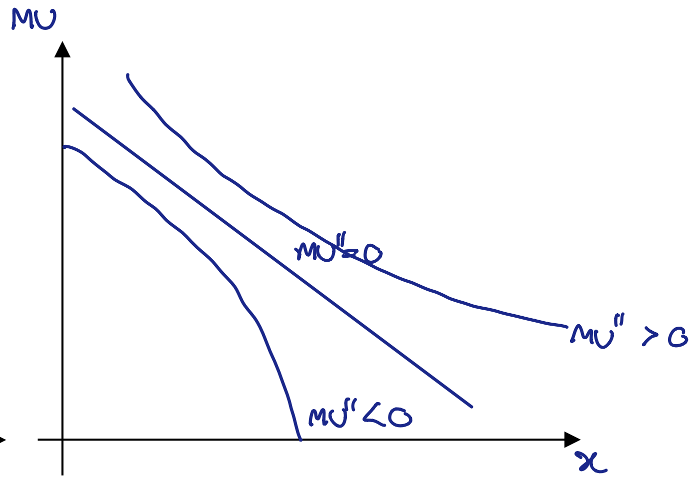
$$\frac{d^2TU}{dx^2} < 0$$

$\frac{d\ MU}{dx} < 0$

→ when consume ↑ , mu ↓
→ law of diminishing marginal utility
→ TU is concave

Total Utility





$\frac{dy}{dx}$ and marginality

$TC = 5Q^2 + 3Q + 100 \quad \rightarrow$ "Total cost"

$TR = 25Q \quad \rightarrow$ "Total revenue"

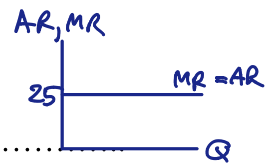
$U = 5x^{1/2} + 100 \quad \rightarrow$ "Utility function"

MC =

$= \frac{dTC}{dQ} = 10Q + 3$

MR=

$= \frac{dTR}{dQ} = 25$, Note $AR = \frac{TR}{Q} = 25$



MU=

$= \frac{dU}{dx} = \frac{5}{2x^{1/2}}$

$E_{y,x}$ = Elasticity of y with respect to x $= \frac{\% \Delta y}{\% \Delta x} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} = \frac{dy}{dx} \cdot \frac{x}{y}$

$\frac{dy}{dx}$ and price elasticity of demand: law of demand

Demand **(1.)** $Q^d = 250 - 10P$

$\frac{dQ^d}{dP} = -10$

$E_p = \frac{dQ^d}{dP} \cdot \frac{P}{Q^d}$

$= -10 \cdot \frac{P}{Q^d}$

if $P = 5$, $Q^d = 200$

$E_p = -10 \cdot \frac{5}{200} = -\frac{1}{4}$

(2.) $Q^d = 50P^{-1/4}$

$\frac{dQ^d}{dP} = -\frac{50}{4} P^{-5/4}$

$E_p = \frac{dQ^d}{dP} \cdot \frac{P}{Q^d}$

$= -\frac{50}{4} P^{-5/4} \cdot \frac{P}{Q^d}$

$= \frac{-50}{4} \cdot \frac{P^{-1/4}}{Q^d}$

if $P = 16$, $Q^d = 25$

$E_p = -\frac{1}{4}$

$\frac{dy}{dx}$ and income elasticity of demand: inferior vs. normal good

$Q^d = f(I)$

$E_I = \frac{dQ^d}{dI} \cdot \frac{I}{Q^d}$

$E_I < 0 \rightarrow I\uparrow, Q^d\downarrow \Rightarrow$ inferior good

$E_I > 0 \rightarrow I\uparrow, Q^d\uparrow \Rightarrow$ normal good → necessity goods : $0 < E_I \leq 1$; → luxury goods : $E_I > 1$

$\frac{dy}{dx}$ and cross price elasticity of demand: substitute vs. complementary goods

$Q^d_a = f(P_b)$

$E_c = \frac{dQ^d_a}{dP_b} \cdot \frac{P_b}{Q^d_a}$

$E_c < 0 \rightarrow P_b\uparrow, Q^d_a\downarrow \Rightarrow$ complementary goods

$E_c > 0 \rightarrow P_b\uparrow, Q^d_a\uparrow \Rightarrow$ substitute goods

$\frac{dy}{dx}$ and law of supply

$E_S = \frac{dQ^S}{dP} \cdot \frac{P}{Q^S} > 0$ law of supply

$\frac{dy}{dx}$ and output elasticity of labor

$TP = f(L)$

$E_L = \frac{dTP}{dL} \cdot \frac{L}{TP} = \frac{\frac{dTP}{dL}}{\frac{TP}{L}} = \frac{MP_L}{AP_L}$

Total Cost, Average Cost, and Marginal Cost

$TC = C(Q)$

$AC = \frac{TC}{Q}$

$MC = \frac{dTC}{dQ}$

$MC = \frac{d}{dQ}(AC(Q) \times Q)$; $TC(Q) = AC(Q) \times Q$

$MC = AC + Q\frac{dAC}{dQ}$

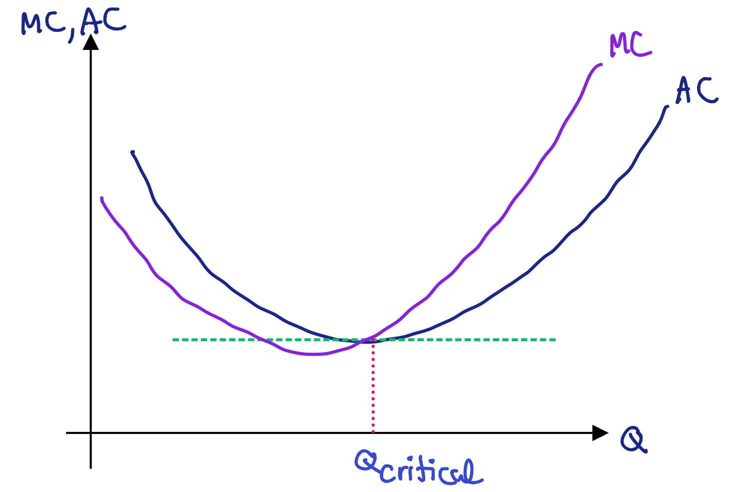
$MC - AC = Q\frac{dAC}{dQ}$ ———— ★

Relationship between MC and AC

① At minimum of AC → $\frac{dAC}{dQ} = 0$ → $MC = AC$

② $Q < Q_{critical}$ → $\frac{dAC}{dQ} < 0$ → $MC < AC$

③ $Q > Q_{critical}$ → $\frac{dAC}{dQ} > 0$ → $MC > AC$



How much does total cost increase if labor in production increases?

$TC(Q)$, $Q(L)$

$TC(Q(L))$ $\therefore$ $L\Delta \rightarrow Q\Delta \rightarrow TC\Delta$

$$\frac{dTC}{dL} = \frac{dTC}{dQ} \cdot \frac{dQ}{dL} = MC \times MP_L = MFC$$ = "Marginal Factor Cost"

TR, AR, MR

$Q^d = a - bP$ Demand f^n

$P = \frac{a}{b} - \frac{1}{b}Q^d$ Inverse demand f^n

In monopoly mkt.:

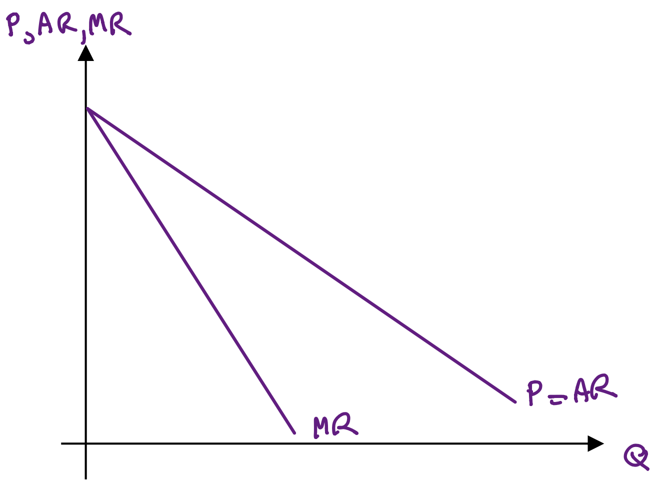
$TR = PQ = \left(\frac{a}{b} - \frac{1}{b}Q\right)Q = \frac{a}{b}Q - \frac{1}{b}Q^2$

$AR = \frac{TR}{Q} = \frac{a}{b} - \frac{1}{b}Q$; notice : $P = AR$

$MR = \frac{dTR}{dQ} = \frac{a}{b} - \frac{2}{b}Q$; notice : $|\text{slope } f_{MR}| = 2\,|\text{slope of } AR|$

same intercept

$MR < AR$



The relation between MR, AR, and Price elasticity of demand

$TR = PQ = AR \times Q$; $AR(Q)$

$MR = \frac{dTR}{dQ} = \frac{d}{dQ}(AR \times Q)$

$= AR + Q\frac{dAR}{dQ}$

$$MR = AR + Q\frac{dp}{dQ}$$

$$= AR + Q \cdot \frac{p}{p} \cdot \frac{dp}{dQ}$$

$$= AR + \frac{p}{\frac{dQ}{dp} \cdot \frac{p}{Q}}$$

$$MR = AR + \frac{p}{E_p}$$

$$MR = AR\left(1 + \frac{1}{E_p}\right) \quad (1) \#$$; Note: Since $E_p < 0$ ∴ $MR < AR$ at an Q

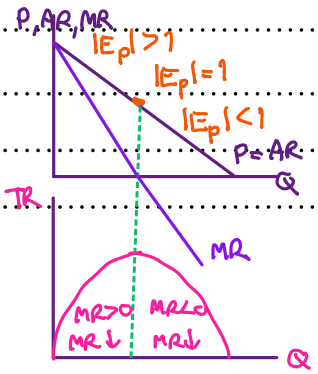
$$\frac{MR}{AR} - 1 = \frac{1}{E_p}$$

$$E_p = \frac{AR}{MR - AR} \#$$

$|E_p| > 1$ elastic $\rightarrow \left(1 + \frac{1}{E_p}\right) > 0 \rightarrow MR > 0$

$|E_p| = 1$ unit elastic $\rightarrow \left(1 + \frac{1}{E_p}\right) = 0 \rightarrow MR = 0$

$|E_p| < 1$ inelastic $\rightarrow \left(1 + \frac{1}{E_p}\right) < 0 \rightarrow MR < 0$



The elasticity of total revenue with respect to output

$$E_{TR,Q} = \frac{dTR}{dQ} \cdot \frac{Q}{TR}$$

$$= \frac{MR}{AR}$$

How much does total revenue increase if labor in production increases?

TR(Q)   Q(L)

TR(Q(L))   L Δ → Q Δ → TR Δ

$$\frac{d\,TR}{dL} = \frac{dTR}{dQ}\frac{dQ}{dL} = MR \times MP_L = MRP_L$$

marginal Revenue Product of labor

H.W. Let $P = 25 - 0.1Q$, $Q = 5L$, find MRP

TP, AP$_L$, MP$_L$

$TP = Q = f(L)$

$AP_L = \frac{TP}{L} = \frac{Q}{L}$

$MP_L = \frac{dTP}{dL} = \frac{dQ}{dL}$

$MP_L' = \frac{dMP_L}{dL} < 0 \rightarrow$ law of diminishing MP

Proof:

1.) At the maximum of AP_L, $AP_L = MP_L$

2.) When $MP_L > AP_L$, AP_L increases.

3.) When $MP_L < AP_L$, AP_L decreases.

$\frac{d\,AP_L}{dL} = \frac{d(TP/L)}{dL}$

$= \frac{L\frac{dTP}{dL} - TP}{L^2}$

$\frac{dAP_L}{dL} = \frac{MP_L}{L} - \frac{AP_L}{L}$ ☆

1.) at $AP_{L,max}$, $\frac{dAP_L}{dL} = 0$

From ☆

$\therefore \quad MP_L = AP_L$ #

2.) $AP_L \uparrow$, $\frac{dAP_L}{dL} > 0$

From ☆ $\frac{MP_L}{L} - \frac{AP_L}{L} > 0$

$MP_L > AP_L$ #

3) $AP_L \downarrow$, $\frac{dAP_L}{dL} < 0$

from ☆ $\frac{MP_L}{L} - \frac{AP_L}{L} < 0$

$MP_L < AP_L$ #

☞ MC and MP_L

$MC = \frac{dTC}{dQ}$

$= \frac{dTC}{dL} \cdot \frac{dL}{dQ}$

If we let $TC = wL$; $TFC = 0$

$\therefore \quad MC = \frac{w}{\frac{dQ}{dL}} = \frac{w}{MP_L}$ #

$MP_L \uparrow \rightarrow MC \downarrow$

$MP_L \downarrow \rightarrow MC \uparrow$

MP_L max $\rightarrow$ MC min

AVC and AP_L

$AVC = \frac{TVC}{Q}$

$= \frac{TVC}{L} \cdot \frac{L}{Q}$; $TVC = wL$, $TFC = 0$

$= w \cdot \frac{L}{Q}$

$AVC = \frac{w}{AP_L}$

$AP_L \uparrow\downarrow \rightarrow AVC \downarrow\uparrow$

AP_L max $\rightarrow$ AVC min