EE422 Mathematical Economics 2

Lecture 6
Continuous Time: First-Order Differential Equations

Read Chapter 5 Sydaeter et al.

or Chapter 14-15 Chiang.

Contents of the Second Part

- Ch 5 Differential equations: first-order equations in one variable
- Ch 6 Differential equations: second-order equations and systems in the plane
- Ch 8 Calculus of Variation
- Ch 9 Control Theory: Basic Techniques
- All from Sydsaeter, K. et al. (2005)

Ch.5. First-Order Equations in One variable

- 5.1 Introduction
- 5.2 The direction is given: find the path
- 5.3 Separable Equations
- 5.4 First-Order Linear Equations
- 5.5 Exact Equations and Integrating Factors
- 5.6 Transformation of Variables (Bernoulli)
- 5.7 Qualitative Theory and Stability of solution (phase diagram, convergence, divergence)
- 5.8 Existence and Uniqueness

5.1 Introduction

- In a differential equation,
  - the unknown is a function of time or time path *x(t)* and
  - the equation includes derivatives of the function, *dx/dt or* $\dot{x}$
  - In economics, we assume time is an independent argument or independent variable.
- We focus on ordinary differential equation , that is, a function of only one independent variable, t.
- In this chapter, we first look at the first order derivatives of the unknown functions of one variable. $\dot{x}, \dot{x}^2, \dot{x}^3\ only\ one\, dot.$
- If we have two independent variables, this is called partial differential equations.

CLASSIFICATION BY TYPE If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).** For example,

A DE can contain more than one dependent variable

$$\frac{dy}{dx} + 5y = e^x, \qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \qquad \text{and} \qquad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \qquad (2)$$

unknown function or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

CLASSIFICATION BY ORDER The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

second order, first order

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

5.1 Introduction

$Let\ x(t)\, \text{simply}\ x, \text{and}\ \dot{x} = dx \,/\, dt.$

$\text{Here}\ t\ \text{denotes the real-valued time arguement.}$

$\text{Examples of first-order diff. eq.}$

$\dot{x} = ax;\ \text{natural growth}$

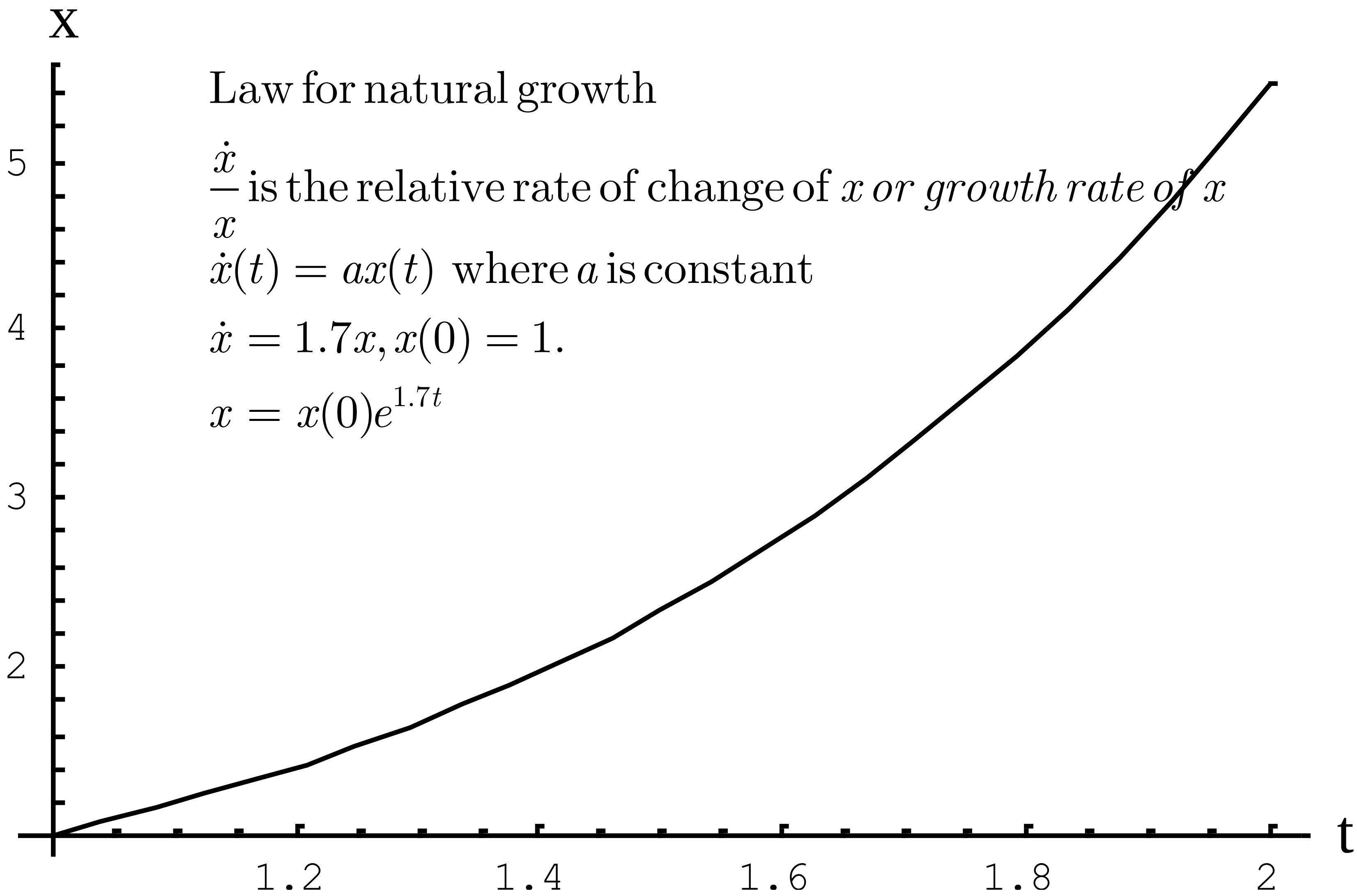
$\dot{x} + ax = b;\ \text{growth towards a limit}$

$\dot{x} + ax = bx^2.\ \ \text{logistic growth}$

$or\ \ (a)\ \dot{x} = x + t;\ \ b\ \ \dot{k} = sf(x) - \lambda k.$

- Solving (*a*) means finding all functions x(t) that satisfies (*a*): for every value of t, *dx/dt* of *x(t)* is equal to *x(t)* + *t*.

5.1 Introduction



Law for natural growth

$\frac{\dot{x}}{x}$ is the relative rate of change of x *or growth rate of* x

$\dot{x}(t) = ax(t)$ where a is constant

$\dot{x} = 1.7x, x(0) = 1.$

$x = x(0)e^{1.7t}$

POPULATION DYNAMICS One of the earliest attempts to model human **population growth** by means of mathematics was by the English economist Thomas Malthus in 1798. Basically, the idea behind the Malthusian model is the assumption that the rate at which the population of a country grows at a certain time is proportional* to the total population of the country at that time. In other words, the more people there are at time t, the more there are going to be in the future. In mathematical terms, if $P(t)$ denotes the total population at time t, then this assumption can be expressed as

$$\frac{dP}{dt} \propto P \qquad \text{or} \qquad \frac{dP}{dt} = kP, \tag{1}$$

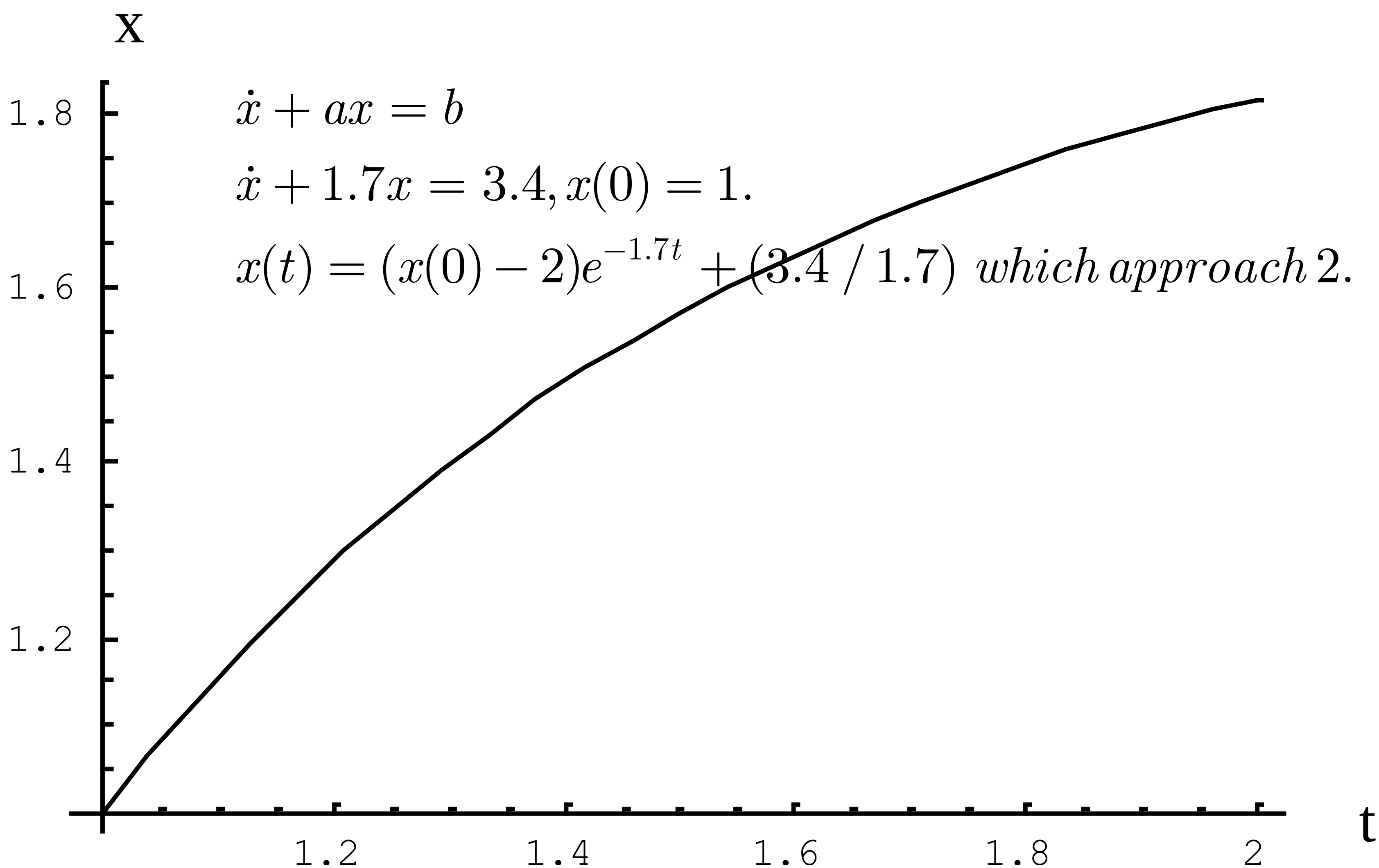
where k is a constant of proportionality. This simple model, which fails to take into account many factors that can influence human populations to either grow or decline (immigration and emigration, for example), nevertheless turned out to be fairly accurate in predicting the population of the United States during the years 1790–1860. Populations that grow at a rate described by (1) are rare; nevertheless, (1) is still used to model *growth of small populations over short intervals of time* (bacteria growing in a petri dish, for example).

5.1 Introduction

$$\dot{x} + ax = b$$

$$\dot{x} + 1.7x = 3.4, x(0) = 1.$$

$$x(t) = (x(0) - 2)e^{-1.7t} + (3.4\,/\,1.7)\ \textit{which approach}\ 2.$$

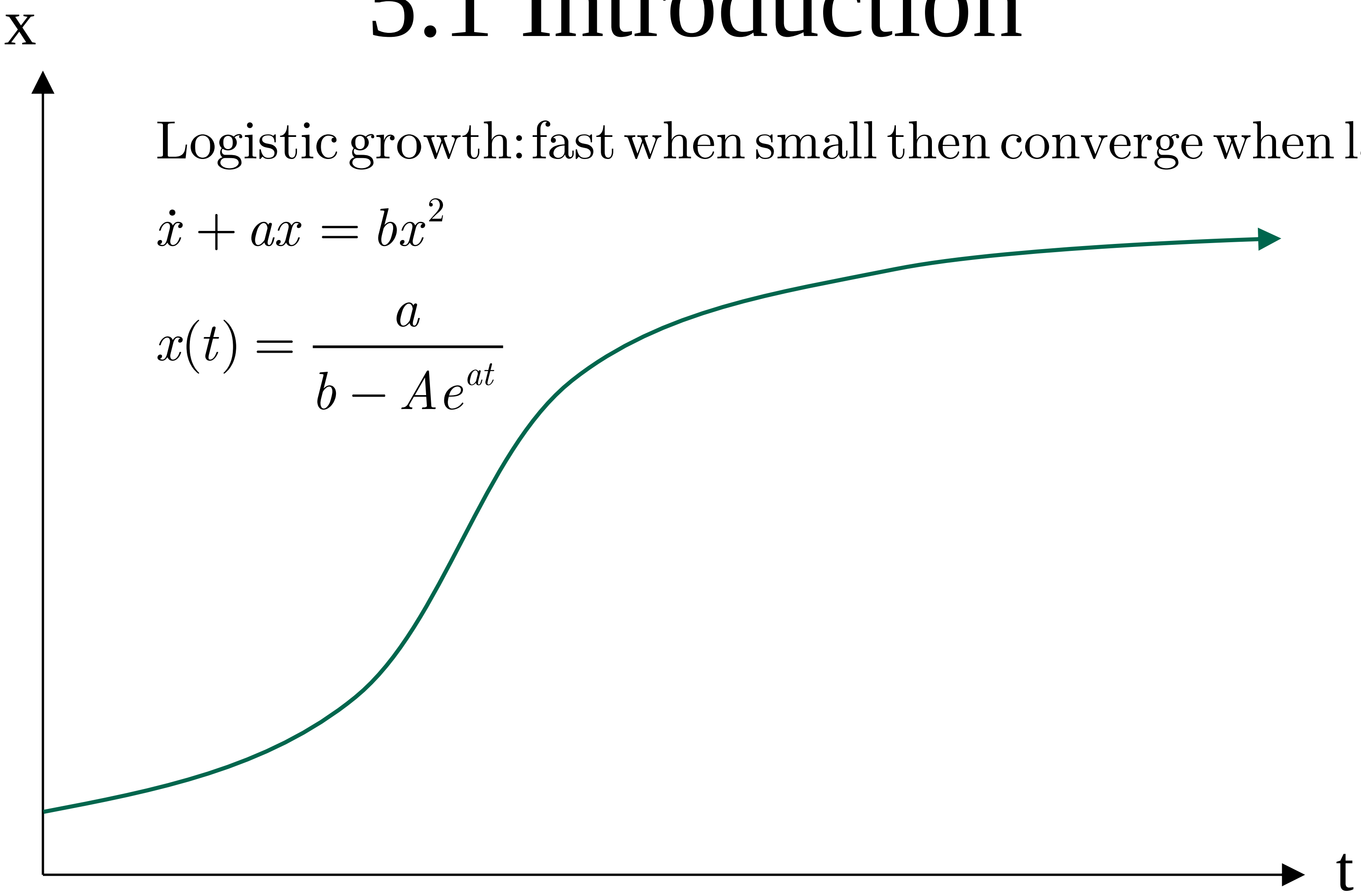


5.1 Introduction

Logistic growth: fast when small then converge when large

$$\dot{x} + ax = bx^2$$

$$x(t) = \frac{a}{b - Ae^{at}}$$



SPREAD OF A DISEASE A contagious disease—for example, a flu virus—is spread throughout a community by people coming into contact with other people. Let $x(t)$ denote the number of people who have contracted the disease and $y(t)$ denote the number of people who have not yet been exposed. It seems reasonable to assume that the rate dx/dt at which the disease spreads is proportional to the number of encounters, or *interactions,* between these two groups of people. If we assume that the number of interactions is jointly proportional to $x(t)$ and $y(t)$—that is, proportional to the product xy—then

$$\frac{dx}{dt} = kxy, \tag{4}$$

where k is the usual constant of proportionality. Suppose a small community has a fixed population of n people. If one infected person is introduced into this community, then it could be argued that $x(t)$ and $y(t)$ are related by $x + y = n + 1$. Using this last equation to eliminate y in (4) gives us the model

$$\frac{dx}{dt} = kx(n + 1 - x). \tag{5}$$

An obvious initial condition accompanying equation (5) is $x(0) = 1$.

NEWTON'S LAW OF COOLING/WARMING According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and dT/dt the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m \qquad \text{or} \qquad \frac{dT}{dt} = k(T - T_m), \tag{3}$$

where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it stands to reason that $k < 0$.

5.1 Introduction

$$\dot{x} = ax \Rightarrow \frac{\dot{x}}{x} = a$$

$$\Rightarrow \int \frac{1}{x}\dot{x}\,dt = \int a\,dt$$

$$\Rightarrow \ln|x| + c_1 = at + c_2 \;\; or$$

$$\ln|x| = at + C,\; now\,let\,anti\log$$

$$\Rightarrow |x| = e^{at+C} = e^{at}e^{C} = e^{at}A$$

$$To\,find\,A,\; let\,at\;\; t = 0, x = x(0), so\,A = x(0)$$

$$thus, x = x(0)e^{at}$$

5.1 Introduction

-A first-order differential equation is written

$$\boxed{\dot{x} = F(t, x) \qquad\qquad (*)}$$

where $x(t)$ is the unknown function and F is a function of two variables.

-Finding the general solution means finding a formula which specifies all functions $x(t)$ satisfying $(*)$

–The graph of solution is called an integral curve.

$$Ex.\ consider\ \dot{x} = x + t.$$

$$Let\ check\ the\ following\ candidates$$

$$(1)\, x = -t - 1 : \dot{x} = -1,$$

$$check\ x + t = -t - 1 + t = -1$$

$$(2)\, x = e^{t} - t - 1 : \dot{x} = e^{t} - 1,$$

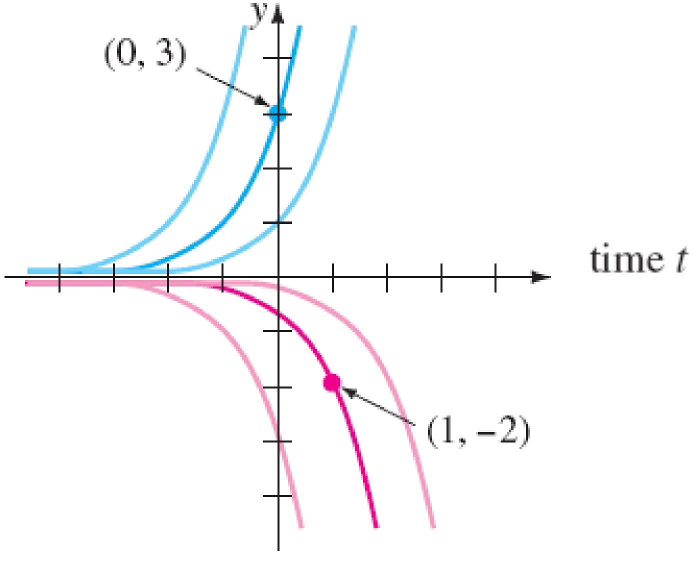
$$check\ x + t = e^{t} - t - 1 + t = e^{t} - 1$$

$$(3)\ x = Ce^{t} - t - 1, similarly$$

- From (3), we found that there are infinitely solutions for C.
- The set of all solutions of a diff equation is called <u>its general solution</u>.
- While any specific function in (1) and (2) is called <u>a particular solution</u>.
- We can plot solutions on the t-x plane.
- All the differential equations in this chapter will be of this form. The presence of x on the RHS means that the general solution cannot be found by integration alone. And there is no general formula for solving (*).
- We introduce various technique for particular kinds of differential equations.

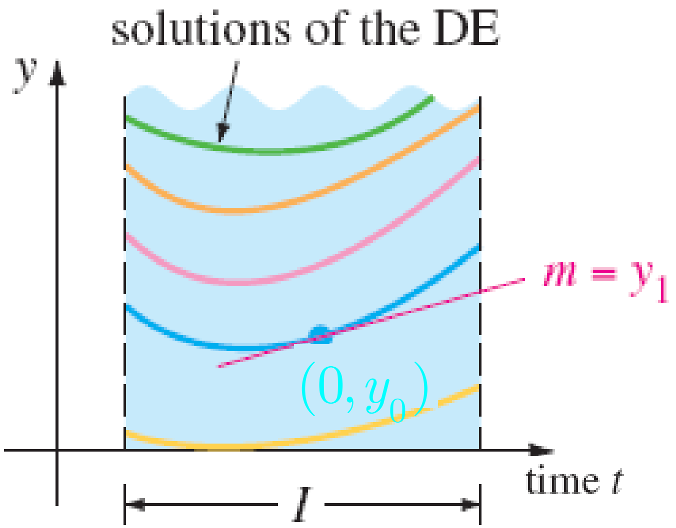
- Ex 2. From the previous example, find the solution that passes through the point $(t, x) = (0, 1)$
- We must have $x(t=0)=1$ or $Ce^{0} - 0 - 1 = 1$
- Thus we have $C = 2$. The required solution is $x(t) = 2e^{t} - t - 1$.
- If $t = 0$ denotes the initial time, then $x(0) = 1$ is called "an initial condition".
- A differential equation and an initial condition together define an "initial-value problem".
- And the solution to this problem is a particular solution of the system that pass through (0,1).

- Ex 3. From $\dot{y} = y$, find the solutions that passes through the point (t, y) = (0, 3) and (1,-2)
- We must have $y(t) = Ce^t$.
- For (0,3), $3=Ce^0 \Rightarrow C=3$. The required solution is $y(t) = 3e^t$.
- For (1,-2), $-2=Ce \Rightarrow C=(-2)/e$. The solution is $y(t)=(-2)e^{t-1}$.



- Ex 4. $\ddot{y} = f(t, y, \dot{y})\, with\, y(0) = y_0\; and\; \dot{y}(0) = y_1$

- The solution must pass $(0, y_0)$ and the slope of this point is y_1.

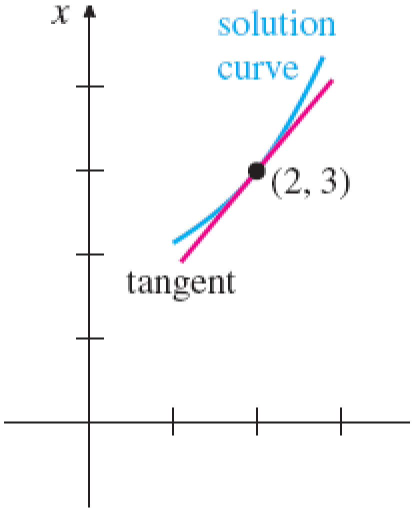


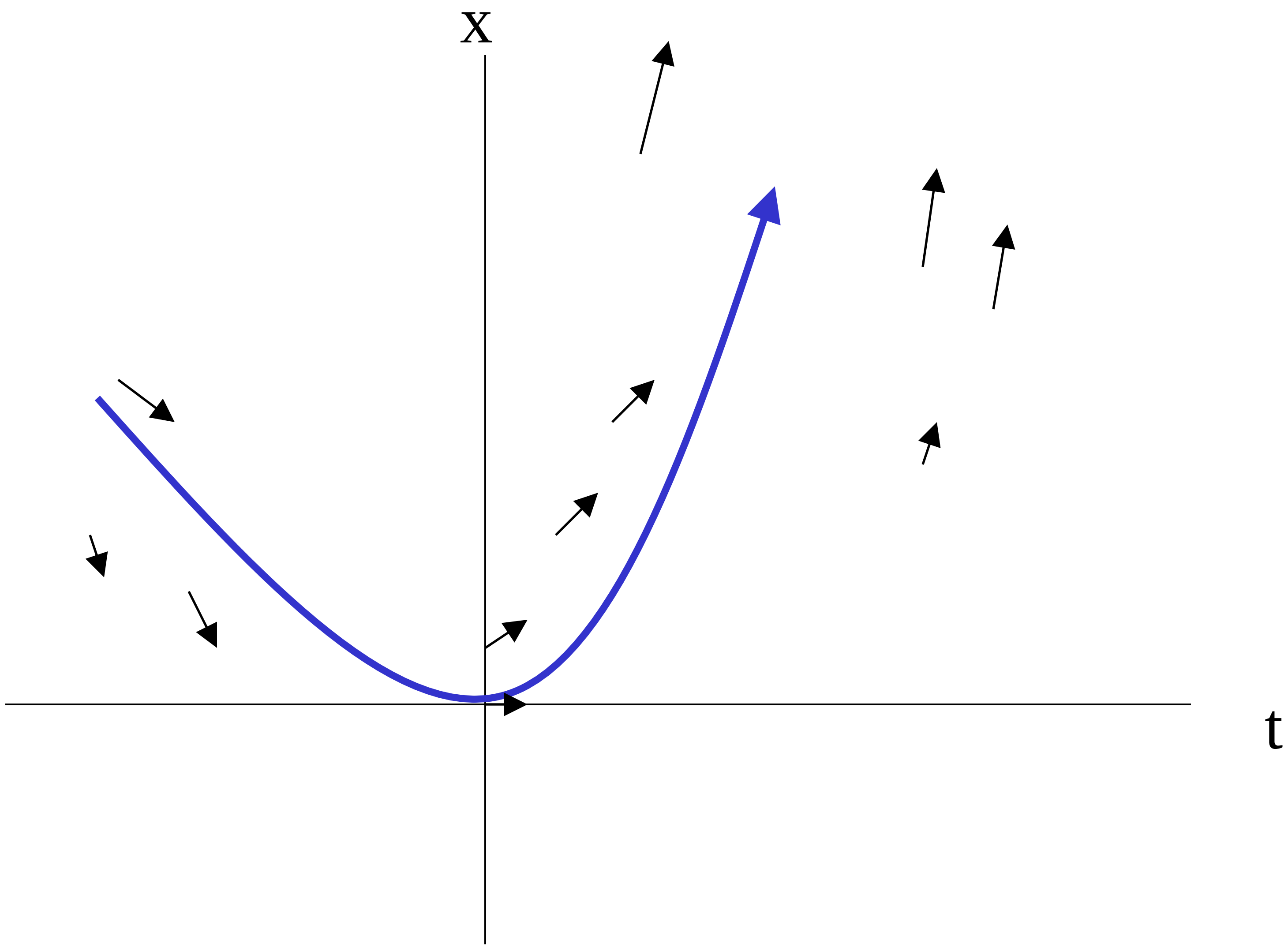
5.2 Find the Path

- Only few equations could be solved for explicit solutions
- And sometimes we don't need them
- We may interest in properties of solution or behavior of them. (how solutions behave)
- We call it qualitative theory: including existence, uniqueness, and stability of equilibria.

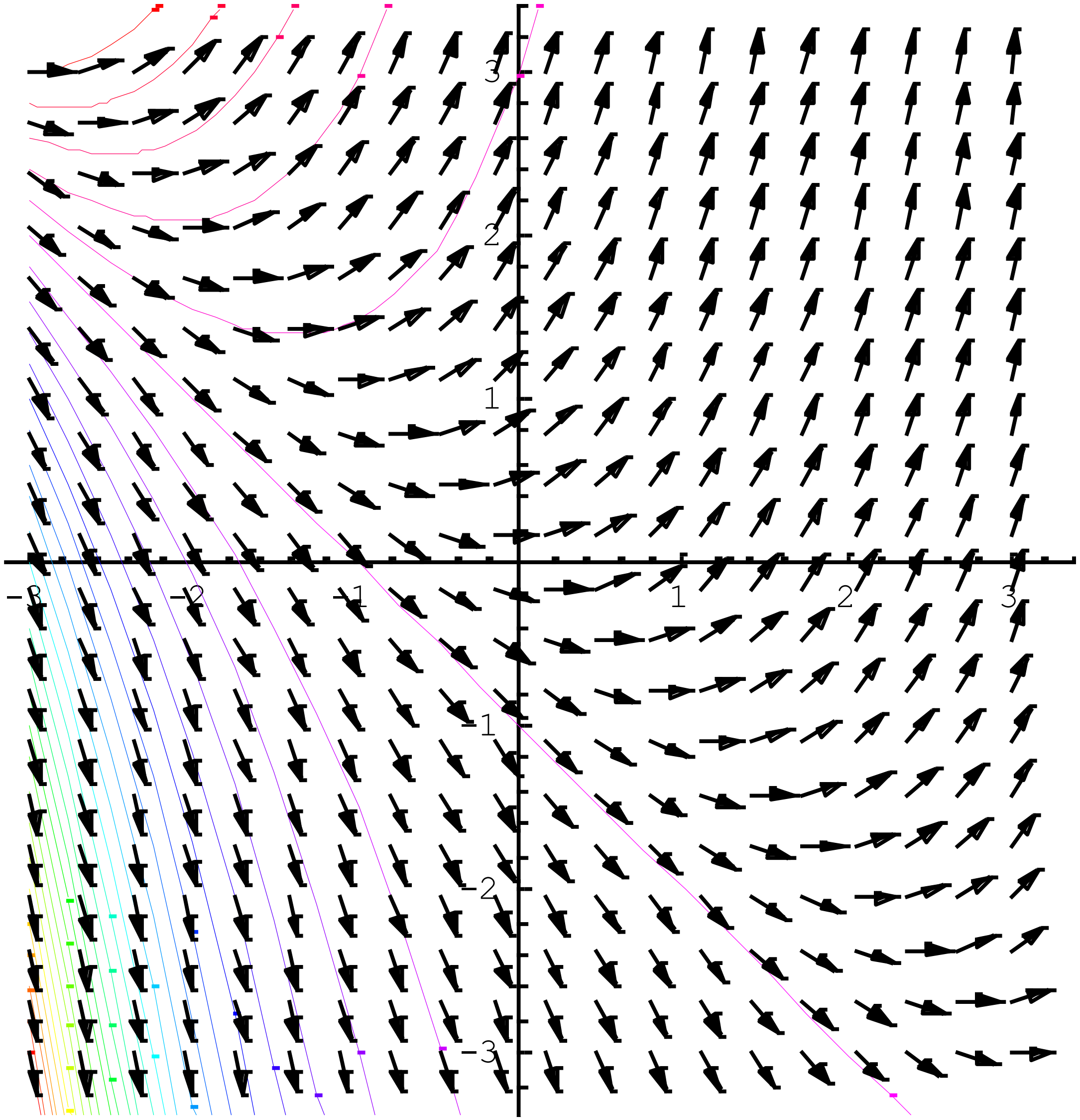
5.2 Find the Path

- We could draw a direction diagram and sketch for the integral curve (or solution curve) that must follow those directions.
- Consider xdot = x + t : RHS tells you slopes of tangent lines at points on its graph
- Thus, we know that a point
  - (t, x) = (0, 0), slope is 0+0= 0;
  - (t, x) = (1, 2), slope is 2+1= 3;
  - (t, x) = (2, 3), slope is 2+3= 5.



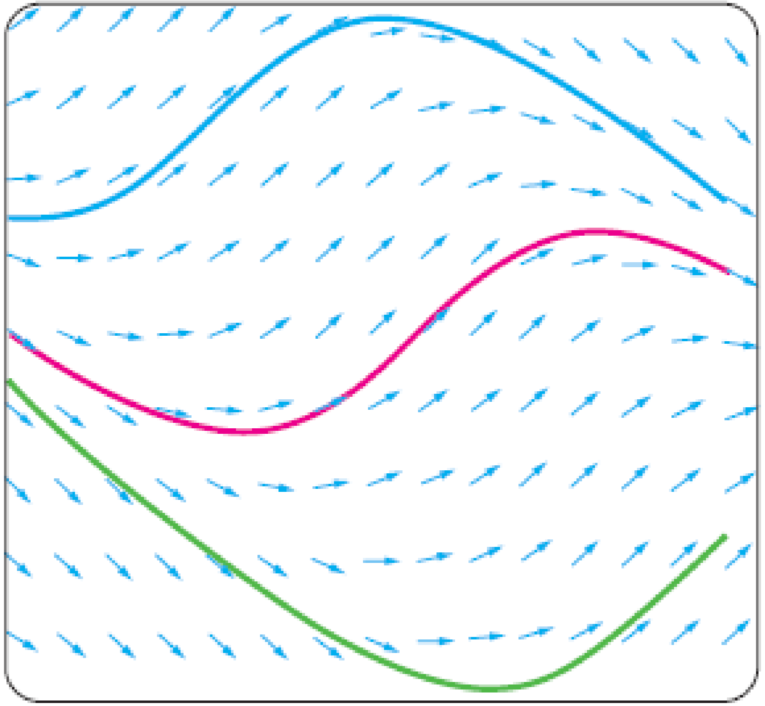


The integral curve or solution curve through (0,0)



The slope field (or direction field) of xdot=x+t.

$$Ex.\, A\, direction\ \ field\, of\ \ \dot{x} = \sin(x+t)$$



Solutions following flow of a direction field.

5.3 Seperable Equations

$$\dot{x} = F(t,x) = f(t)g(x)................(1)$$

a product of 2 functions, one on t, another on x.

•This diff eq. is seperable (or have seperated vars).

This can be solved in terms of integrals known functions.

•Example. $\dot{x} = -2tx^2$

Steps for solving separable dif Equations:

Separate, Integrate, and Evaluate.

$$Solve\ \ \dot{x} = f(t)g(x)...............(1)$$

$$A.\ Write\,(1)\,as \qquad \frac{dx}{dt} = f(t)g(x)........(*)$$

$$B.\,Seperate\,\text{variables}: \frac{dx}{g(x)} = f(t)dt\,.$$

$$(x\,on\,the\ \ left,\ t\,on\ \ the\,right)$$

$$C.\,Integrate: \qquad \int \frac{dx}{g(x)} = \int f(t)dt$$

$$D.\,\text{Evaluate}\ \ C\,\text{and}\,\text{solve}\,\text{for}\,x.$$

Example. Solve $\dfrac{dx}{dt} = f(t)g(x) = -2tx^2$ and find the integral curve that passes $(t, x) = (1, -1)$.

$$A.\, write\, (1)\, as\ \frac{dx}{dt} = (-2t)(x^2) \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

$$B. Separate \text{ variables} : -\frac{dx}{x^2} = 2t\, dt\, .$$

$$C. Integrate: -\int \frac{1}{x^2}\, dx = \int 2t\, dt$$

$$D. Evaluate\ C\ and\ solve\ for\ x : \frac{1}{x} = t^2 + C$$

thus, the general solution is $x = 1 / [t^2 + C]$

$for\ x = -1\ at\ t = 1,\ C = -2.$ So, the integral curve or the particular solution is $x = 1 / [t^2 - 2]$.

$$Example.\ Solve\ \frac{dx}{dt} = \frac{t^3}{x^6+1}.$$

$$Here\ f(t) = t^3,\ and\ \ g(x) = 1\,/\,[x^6+1].$$

$$Separate:\quad [x^6+1]dx = t^3\,dt\,.$$

$$Integrate:\ \int [x^6+1]dx = \int t^3\,dt$$

Evaluate and solve for x : $\dfrac{1}{7}x^7 + x = \frac{t^4}{4} + C$

Note that $g(x)$ *is* never 0 for any x,

So, no particular solution.

$Example.\ (Economic\ Growth)$

$(a) X = AK^{1-\alpha}L^{\alpha} \quad (b)\, \dot{K} = sX \ (c)\ L = L_0 e^{\lambda t}.$

assume all positive parameters,and $0 < \alpha < 1.$

$Find\ K = K(t), assume\ K(0) = K_0.$

$\dot{K} = sX = sAK^{1-\alpha}L^{\alpha} = sAK^{1-\alpha}L_0^{\alpha}e^{\alpha\lambda t}.$

This is a seperable diff eq.

Here $f(t) = sAL_0^{\alpha}e^{\alpha\lambda t},\ and\ \ g(K) = K^{1-\alpha}.$

$$Here\, f(t) = sAL_0^{\alpha} e^{\alpha\lambda t},\ and\ \ g(K) = K^{1-\alpha}.$$

$$Separate: \quad \frac{1}{[K^{1-\alpha}]} dK = sAL_0^{\alpha} e^{\alpha\lambda t}\, dt\,.$$

$$Integrate: \ \int [K^{-1+\alpha}] dK = \int sAL_0^{\alpha} e^{\alpha\lambda t}\, dt$$

$$Evaluate\, and\, solve\, for\, K: \frac{1}{\alpha} K^{\alpha} = \tfrac{1}{\alpha\lambda} sAL_0^{\alpha} e^{\alpha\lambda t} + C$$

$$For\, K = K_0\, at\ \ t = 0, we\, get\ \ C = K_0^{\alpha} - (s/\lambda) AL_0^{\alpha},$$

$$thus\, K = [K_0^{\alpha} + (s/\lambda) AL_0^{\alpha} (e^{\alpha\lambda t} - 1)]^{1/\alpha}.$$

$$Example.(Compound\ Interest)\quad \dot{w} = r(t)w, separable\, eq.$$

$$\text{Sep+int.} => \int \frac{1}{w} dw = \int r(t)\, dt$$

$$\ln w + C_1 = R(t)\quad where\, R(t) = \int r(t)\, dt$$

$$Thus,\quad w(t) = e^{R(t)+C_1} = e^{C_1} e^{R(t)} = Ce^{R(t)}$$

$$\text{Let}\, w(0)\, at\;\; t = 0,\; w(0) = Ce^{R(0)}\, or\, C = w(0)e^{-R(0)}$$

$$So\;\; w(t) = w(0)e^{-R(0)}e^{R(t)} = w(0)e^{R(t)-R(0)}$$

$$But\, R(t) - R(0) = \int_0^t r(s)\, ds$$

$$Thus, w(t) = w(0)e^{\int_0^t r(s)ds} = w(0) \exp \int_0^t r(s)\, ds.$$

5.4 First-Order Linear Equations:

$$\dot{x} + a(t)x = b(t)$$

It is linear for LHS is a linear fn of the dependent variable, $ie.$

x and $\dot{x}$; or terms such as $\ln x$ or $(\dot{x})^2$ does not appear.

However, $a(t)$ and $b(t)$ are not required to be constant.

$Ex.(a)\dot{x} = x + t;\;\; b\;\; \dot{x} + 2tx = 4t.$

5.4 The Simplest Case

Consider the following where a, b constants

$$\boxed{\dot{x} + ax = b, \;\; where\, a \neq 0.}$$

Let multiply by e^{at}, called an integrating factor

$$\dot{x}e^{at} + axe^{at} = be^{at} \quad \text{becomes}$$

$$\frac{d}{dt}\left(xe^{at}\right) = be^{at} \;\; \text{take indefinite integral,}$$

$$xe^{at} = \int be^{at}dt = \frac{b}{a}e^{at} + C.$$

$multiply\, by\, e^{-at}\, gives\, the\, solution\ for\ \ x,$

$$\boxed{x = Ce^{-at} + (b/a).......(*)}$$

$This\, is\, the\, general\, solution.$

$Note\ that\, if\ \ (a)\, C = 0,\ x(t) = b/a\ \ or\, stationary$

$state\ for\, the\, equation.\ This\, can\, be\, also\, obtained\, by\ \ let\, \dot{x} = 0.$

$(b)\, If\ a\, is\, positive,\ x \to b/a\ \ \ as\ \ t \to \infty. The\, equation\, is\, said\, to$

$be\, stable\ for\, all\, solutions\, converges\, to\, an\, equilibrium\, as\, t \to \infty.$

$$(c) if\ at\, t = 0, x = x(0), then\, x(t) = (x(0) - \frac{b}{a})e^{-at} + \frac{b}{a}.$$

Examples

$$1. \quad \dot{x} + 2x = 8$$

$$from\, the\, solution\ form,\ we\, have$$

$$x = Ce^{-2t} + 4, as\, the\, general\, solution.$$

$$The\, equilibrium\, state\, is\, x = 4.$$

$$and\, because\, a = 2 > 0$$

$$so, as\ t \to \infty, x = 4,$$

$$the\, equation\, is\, stable.$$

2. (price adjustment mechanism)

$$\text{Let}\, D\;\;P\;\; = a - bP;\ \text{demand}$$

$$S(P) = \alpha + \beta P;\ \ \text{supply}$$

$$Assume\, P = P(t)\, and\, \dot{P} = \lambda[\, D(P) - S(P)]$$

$$\text{Substitute and get}\ \ \dot{P} = \lambda[a - bP - \alpha - \beta P]$$

$$\text{rearrange as}\ \ \dot{P} + \lambda(b + \beta)P = \lambda(a - \alpha).$$

$$\text{From the solution form,}$$

$$P = Ce^{-\lambda(b+\beta)t} + (a - \alpha) \,/\, (b + \beta).$$

$\mathrm{So, the\ equilibrium\ \ price} = (a - \alpha)\ /\ (b + \beta).$

$For\ \lambda(b + \beta)\ \mathrm{is\ positive, the\ equation\ is\ stable.}$

$If\ t = 0, P(t) = P_0. \quad We\, have$

$$P = [P_0 - \frac{(a - \alpha)}{(b + \beta).}]e^{-\lambda(b+\beta)t} + \frac{(a - \alpha)}{(b + \beta)}.$$

- 3

Population Growth Beginning in the next section we will see that differential equations can be used to describe or *model* many different physical systems. In this problem suppose that a model of the growing population of a small community is given by the initial-value problem

$$\frac{dP}{dt} = 0.15P(t) + 20, \quad P(0) = 100,$$

where P is the number of individuals in the community and time t is measured in years. How fast—that is, at what *rate*—is the population increasing at $t = 0$? How fast is the population increasing when the population is 500?

EXAMPLE 4 Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

$$\frac{dT}{dt} = k(T - T_m), \tag{2}$$

where k is a constant of proportionality, $T(t)$ is the temperature of the object for $t > 0$, and T_m is the ambient temperature—that is, the temperature of the medium around the object. In Example 4 we assume that T_m is constant.

SOLUTION In (2) we make the identification $T_m = 70$. We must then solve the initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 300 \tag{3}$$

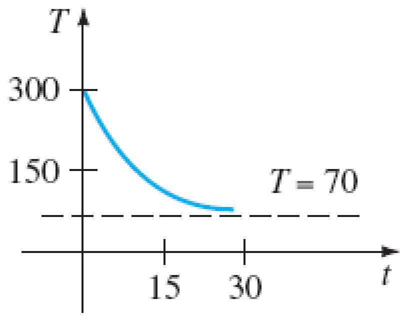
and determine the value of k so that $T(3) = 200$.

Equation (3) is both linear and separable. If we separate variables,

$$\frac{dT}{T - 70} = k\,dt,$$

yields $\ln|T - 70| = kt + c_1$, and so $T = 70 + c_2e^{kt}$. When $t = 0$, $T = 300$, so $300 = 70 + c_2$ gives $c_2 = 230$; therefore $T = 70 + 230e^{kt}$. Finally, the measurement $T(3) = 200$ leads to $e^{3k} = \frac{13}{23}$, or $k = \frac{1}{3}\ln\frac{13}{23} = -0.19018$. Thus

$$T(t) = 70 + 230e^{-0.19018t}. \tag{4}$$



Population Model In one model of the changing population $P(t)$ of a community, it is assumed that

$$\frac{dP}{dt} = \frac{dB}{dt} - \frac{dD}{dt},$$

where dB/dt and dD/dt are the birth and death rates, respectively.

(a) Solve for $P(t)$ if $dB/dt = k_1 P$ and $dD/dt = k_2 P$.

(b) Analyze the cases $k_1 > k_2$, $k_1 = k_2$, and $k_1 < k_2$.

Homework

Constant-Harvest Model A model that describes the population of a fishery in which harvesting takes place at a constant rate is given by

$$\frac{dP}{dt} = kP - h,$$

where k and h are positive constants.

(a) Solve the DE subject to $P(0) = P_0$.

(b) Describe the behavior of the population $P(t)$ for increasing time in the three cases $P_0 > h/k$, $P_0 = h/k$, and $0 < P_0 < h/k$.

(c) Use the results from part (b) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time $T > 0$ such that $P(T) = 0$. If the population goes extinct, then find T.

5.4.2 Variable Right Hand Side

Consider the following RHS case

$$\boxed{\dot{x} + ax = b(t),}$$

multiply by e^{at} *gives*

$$\dot{x}e^{at} + axe^{at} = b(t)e^{at},\ or\ \frac{d}{dt}(xe^{at}) = b(t)e^{at}$$

Take indefinite integral, we get

$$xe^{at} = \int b(t)e^{at}dt + C. \quad Multiply\ by\ e^{-at}$$

$$x = Ce^{-at} + e^{-at}\int e^{at}b(t)\,dt.$$

5.4.2 Variable Right Hand Side

$$\dot{x} + ax = b(t) \Leftrightarrow$$

$$x = Ce^{-at} + e^{-at} \int e^{at} b(t)\, dt.$$

5.4.2 Variable Right Hand Side

$$Ex.\quad \dot{x} + 5x = 4e^{3t}.\ Apply\, the\, formula$$

$$x = Ce^{-5t} + e^{-5t}\int e^{5t}4e^{3t}\,dt$$

$$= Ce^{-5t} + e^{-5t}4\int e^{8t}\,dt$$

$$= Ce^{-5t} + e^{-5t}\,\tfrac{1}{2}\int 8e^{8t}\,dt$$

$$= Ce^{-5t} + e^{3t}\,\tfrac{1}{2}.$$

Some integrating rules,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x dx = e^x + C,$$

$$\int \frac{1}{x} dx = \ln x + C,$$

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$

$$\int e^{at}dt = \frac{1}{a}\int e^{at}adt = \frac{e^{at}}{a}$$

$$\int e^{t^2}4tdt = 2\int e^{t^2}2tdt = 2e^{t^2}$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\int udv = uv - \int vdu.$$

integration by substitution or by change of var.

$\int f(x)dx = \int f(g(u))g'(u)du \;\; where\, x = g(u).$

$thus \;\; dx = g'(u)du.\; or$

$\int f(t)dt = \int f(g(u))g'(u)du \;\; where\, t = g(u)$

$and \;\, dt = g'(u)du.$

$ex. \int \dfrac{\ln x}{x} dx; set\, t = \ln x; dt = \frac{1}{x} dx$

$= \int tdt = \frac{1}{2} t^2 + C = \frac{1}{2} (\ln x)^2 + C.$

Definite integration

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

$$\frac{d}{dx}\int_a^{x} f(t)dt = f(x),\ and$$

$$\frac{d}{dx}\int_{x}^{b} f(t)dt = -f(x)$$

Differentiation under the integral sign.

$$Let\, I(y) = \int\limits_a^b f(x,y)dx, then\, I'(y) = \int\limits_a^b \frac{\partial f(x,y)}{\partial y} dx.\ \ Or$$

$$\frac{d}{dy}\int_a^b f(x,y)dx = \int\limits_a^b \frac{\partial f(x,y)}{\partial y} dx.$$

$$Ex. P = \int_0^T e^{-rt} g(t)dt.\ Notice\, that\, P\, depends\, on\, r.$$

$$\frac{dP}{dr} = \int_0^T \tfrac{\partial}{\partial r}(e^{-rt} g(t))dt = \int_0^T \tfrac{\partial}{\partial r}(e^{-rt})g(t)dt$$

$$= -\int_0^T e^{-rt} tg(t)dt$$

Leibniz's formula

$$F(x) = \int_c^d f(x,t)dt \Rightarrow F'(x) = \int_c^d \frac{\partial f(x,t)}{\partial x} dt.$$

l imits of integration are independent of x.

$$ex. K = \int_0^T f(t)e^{-rt}dt.\ Find\, dK / dr.$$

$$\frac{dK}{dr} = \int_0^T f(t)(-t)e^{-rt}dt.$$

Leibniz's formula: general case.

Now, when x *changes, both* $v(x)$ *and* $u(x)$ *will change, also* $f(x,t)$ *will change for each* t.

What is the total effect on $F(x)$ *from a change in* x?

$$F(x) = \int_{u(x)}^{v(x)} f(x,t)\,dt \Rightarrow$$

$$F'(x) = f(x, v(x))v'(x) - f(x, u(x))u'(x)$$

$$+ \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}\,dt.$$

5.4.3 General Case: Variable coefficient and Variable term

$$\dot{x} + a(t)x = b(t)$$

$First\ \ multiply\, by\ \ e^{A(t)}\ \ get$

$$\dot{x}e^{A(t)} + a(t)xe^{A(t)} = b(t)e^{A(t)},$$

$and\, we\, need\, to\, find\, A(t)\, such\ \ that\, LHS\, equals$

$$\frac{d}{dt}(xe^{A(t)})$$

$$we\, know\, that\ \frac{d}{dt}(xe^{A(t)}) = \dot{x}e^{A(t)} + x\,\dot{A}(t)e^{A(t)}.$$

5.4.3 General Case

$$\text{If we choose } \dot{A}(t) = a(t) \text{ by choosing } A(t) = \int a(t)dt,$$

$$\text{then } \frac{d}{dt}(xe^{A(t)}) = b(t)e^{A(t)}.$$

$$\text{Integrate both sides get } \quad xe^{A(t)} = \int b(t)e^{A(t)}dt + C$$

$$\text{and multiply by } \quad e^{-A(t)}, \quad get$$

$$x = Ce^{-A(t)} + e^{-A(t)} \int e^{A(t)} b(t)\, dt,$$

$$where\, A(t) = \int a(t)\, dt.$$

5.4.3 General Case

To summarize, we show that

$$\dot{x} + a(t)x = b(t) \Leftrightarrow$$
$$x = Ce^{-A(t)} + e^{-A(t)} \int e^{A(t)} b(t)\,dt,$$
$$where\, A(t) = \int a(t)\,dt.$$

Notice that we replace a in 5.4.2 by A.

General solutions have 2 parts. First part is a solution to homogenous equation, second is a particular part of non-homogenous equation.

$$Ex.\ \dot{x} + a(t)x = b(t) => \dot{x} + 2tx = 4t$$

$$We\, know\, that\, a(t) = 2t,\ b(t) = 4t,$$

$$and \int a(t)dt = t^2 + C_1,\ we\, can\, choose\, C_1 = 0$$

$$or\, any\, value, so\, that \int a(t)dt = t^2\ [= A(t)].$$

$$The\, solution\, form\, is$$

$$x = Ce^{-t^2} + e^{-t^2} \int e^{t^2} 4t\, dt = Ce^{-t^2} + e^{-t^2} [2e^{t^2}]$$

$$= Ce^{-t^2} + 2.$$

$$If\ given\, that\, (t, x) = (0, -2), we\, can\, find\, C$$

$$from - 2 = Ce^0 + 2, thus\, C = -4$$

$$Ex.\ \dot{x} + 2tx = t$$

$$Use\,the\,formula: a(t) = 2t,\ b(t) = t,$$

$$and \int a(t)dt = t^2 + C_1.\ \ We\,can\,choose\,C_1 = 0$$

$$or\,any\,value, so\,that \int a(t)dt = t^2\ [= A(t)].$$

$$The\,solution\,form\,is$$

$$x = Ce^{-t^2} + e^{-t^2} \int e^{t^2} t\,dt$$

$$= Ce^{-t^2} + e^{-t^2} [\tfrac{1}{2} e^{t^2}]$$

$$= Ce^{-t^2} + 1/2.$$

5.4.4 General Case when $x(t_0)=x_0$ is given

If the value of $x(t)$ is known for $t = t_o$,

then the constant C can be determined.

$$\dot{x} + a(t)x = b(t) \quad and \quad x(t_0) = x_0 \; : see \;\; p.201$$

You wanna show this

$$x = Ce^{-A(t)} + e^{-A(t)} \int e^{A(t)} b(t)\, dt,$$

$$where\, A(t) = \int a(t)\, dt.$$

becomes

5.4.4 General Case when $x(t_0)=x_0$ is given

$$x = x_0 e^{-\int_{t_0}^{t} a(\xi)\,d\xi} + \int_{t_0}^{t} b(s) e^{-\int_{s}^{t} a(\xi)\,d\xi}\, ds,$$

$$where\ \ A(t) - A(s) = \int_{s}^{t} a(\xi)\, d\xi$$

$Ex.\ (weatlh\ accumulation)$

$Show\ \ PDV\ of\ asset\ at\ time\ t, y\ is\ deposit, c\ withdrawal$

$$\dot{w} = r(t)w + y(t) - c(t) \quad and \quad w(0) = w_0 :$$

$$here\ a(t) \equiv -r(t); b(t) \equiv y(t) - c(t);$$

$$w = w_0 e^{\int_0^t r(s)ds} + \int_0^t [y(\tau) - c(\tau)] e^{\int_\tau^t r(s)ds} d\tau.$$

Let find PDV

$$\text{since } \int_\tau^t r(s)ds = \int_0^t r(s)ds - \int_0^\tau r(s)ds$$

$$multiply\ \ by\ \ e^{-\int_0^t r(s)ds}$$

$$\underbrace{we^{-\int_0^t r(s)ds}}_{\substack{PDV \\ of\ asset \\ at\,time\,t}}$$

$$= w_0 + \int_0^t [y(\tau) - c(\tau)] \underbrace{e^{-\int_0^{\tau} r(s)ds}}_{\substack{discount \\ factor\,applied \\ to\,asset\,at\,time\,\tau}} d\tau$$

5.5 Exact Differential Equations

$$Consider\ \frac{d}{dt}h(t,x) = 0.$$

$We\,want\,to\,find\ h(t,x).$ First,

$$dh(t,x) = \frac{\partial h(t,x)}{\partial t}dt + \frac{\partial h(t,x)}{\partial x}dx = 0$$

$$find\quad \frac{d}{dt}h(t,x) = \frac{\partial h(t,x)}{\partial t} + \frac{\partial h(t,x)}{\partial x}\dot{x}$$

$$Let\ \frac{\partial h(t,x)}{\partial t} \equiv f(t,x);\ \frac{\partial h(t,x)}{\partial x} \equiv g(t,x)$$

$$so,\ \frac{d}{dt}h(t,x) = f(t,x) + g(t,x)\dot{x} = 0.$$

5.5 Exact Differential Equations

$$f(t,x) + g(t,x)\dot{x} = 0 \quad \ldots\ldots.(1)$$

$$where\ f\ and\ g\ are\ c^1\ functions.$$

$$Suppose\ we\ have\ h(t,x)\ such\ that$$

$$\frac{\partial h(t,x)}{\partial t} = f(t,x)\ \ and\ \ \frac{\partial h(t,x)}{\partial x} = g(t,x)$$

$$\text{since}\ \ \frac{d}{dt}h(t,x) = \frac{\partial h(t,x)}{\partial t} + \frac{\partial h(t,x)}{\partial x}\dot{x}$$

$$thus\ (1)\ is\ \frac{d}{dt}h(t,x) = 0$$

$Thus, the\, solution\, to\, (1)\, is$

$x = x(t)\, such\, that\;\; h(t,x) = C.$

$Key\;\; is\, to\, find\, the\, right\;\; h(t,x)$

$P.204\, shows\;\; the\; formula\; for$

$$h(t,x) = \int_{t_0}^{t} f(\tau,x)\,d\tau + \int_{x_0}^{x} g(t_0,\xi)\,d\xi$$

Note that the equation we try to solve is called exact if a necessary and sufficient condition for $h(t,x)$ is satisfied:

$$\boxed{f_x^{'}(t,x) = g_t^{'}(t,x),}$$

since both equals $\dfrac{\partial^2 h(t,x)}{\partial t \partial x}$.

$Example.\ 1+tx^2+t^2x\dot{x}=0$

$notice\ that\ it\ is\ not\ linear\ or\ separable.$

$f(t,x)=1+tx^2; and\ g(t,x)=t^2x;$

$and\ f_x^{'}(t,x)=g_t^{'}(t,x)=2tx \ldots thus\ exact.$

$$Guess\ h(t,x)=t+\frac{t^2}{2}x^2;$$

$h_t^{'}=1+tx^2=f(t,x);\ h_x^{'}=t^2x=g(t,x).$

$$Thus, the\ solution\ is\ x(t)\ that\ t+\frac{t^2}{2}x^2=C.$$

$or\ apply\ the\ formula\ to\ find\ h(t,x)$

$$h(t,x) = \int_{t_0}^{t} f(\tau,x)d\tau + \int_{x_0}^{x} g(t_0,\xi)d\xi$$

$$h(t,x) = \int_{t_o}^{t} (1+\tau x^2)d\tau + \int_{x_0}^{x} t_0^2 \xi d\xi$$

$$= [\tau + \tfrac{1}{2}\tau^2 x^2]_{t_0}^{t} + t_0^2 \left.\frac{\xi^2}{2}\right|_{x_0}^{x}$$

$$= t + \tfrac{1}{2}t^2x^2 - (t_0 + \tfrac{1}{2}t_0^2 x_0^2).$$

$$Ex2.\ 2xtdx + x^2dt = 0,\ rewrite\,as$$

$$2xt\dot{x} + x^2 = 0,\ or\ \ x^2 + 2xt\dot{x} = 0.$$

$$Here\,f(x,t) = x^2;\ g(x,t) = 2xt$$

$$and\ \ f_x^{'}(x,t) = g_t^{'}(x,t) = 2x\ldots.exact$$

$$we\,can\,guess\ \ h(t,x) = x^2t$$

$$The\,solution\,is\,x(t)\,that\,x^2t = C$$

$$or\,x(t) = Ct^{-1/2}$$

$$Formula\, h(t,x) = \int_{t_0}^{t} f(\tau, x)d\tau + \int_{x_0}^{x} g(t_0, \xi)d\xi$$

$$h(t,x) = \int_{t_o}^{t} x^2 d\tau + \int_{x_0}^{x} 2t_0\xi d\xi$$

$$= x^2\tau\Big|_{t_0}^{t} + t_0\xi^2\Big|_{x_0}^{x}$$

$$= x^2 t - x^2 t_0 + t_0 x^2 - t x_0^2 = x^2 t - t x_o^2.$$

$[Modified\, Ex2]$ $2t\dot{x} + x = 0.$ Hint: use x as an integrating factor to get the exact condition.

- When the equation is not exact, we can made them by multiplying it with a suitable function.
- This function is called an <u>integrating factor</u>.
- In general it is hard to find such factor, even when one exists.
- There are some special cases we can find them
- P. 205 gives a formula for an integrating factor for those cases.

$$Ex3.\ \ 1-(t+2x)\dot{x}=0$$

$$here\, f(t,x)=1,\ and\, g(t,x)=-t-2x$$

$$but\, this\, is\, not\, exact.\ \ If\, we\, multiply\, by\, e^{-x}$$

$$e^{-x}-e^{-x}(t+2x)\dot{x}=0$$

$$here\, f(t,x)=e^{-x},\ and\, g(t,x)=-e^{-x}(t+2x)$$

$$check\, f_x^{'}(t,x)=-e^{-x};\ g_t^{'}=-e^{-x}....exact.$$

$$solution\, is\, x(t)\ \ where$$

$$h(t,x)=te^{-x}+2xe^{-x}+2e^{-x}=C$$

$$check\, h_t^{'}=e^{-x};h_x^{'}=-te^{-x}-2xe^{-x}=-e^{-x}(t+2x)$$

5.6 Transformation of Variables: Bernoulli's Equations

- So far, we can find explicit solutions by using our formula when the equation is linear in *x* or separable, or exact.
- Sometimes, transformation the problem or variables that looks difficult into a familiar <u>linear</u> equation make us solve difficulty.
- One example is Bernoulli's equation

$$\boxed{\dot{x} + a(t)x = b(t)x^r}$$

$If\ r = 0,\ the\ \ equation\, is\, linear.$

$If\ r = 1,\ the\, equation\, is\, separable. That\, is$

$$\dot{x} = (b(t) - a(t))x.$$

$Suppose\ \ that\, r \neq 1, let\ \ divide\,(1)\, by\ \ x^r\ get$

$x^{-r}\dot{x} + a(t)x^{1-r} = b(t).$

$Let\, z \equiv x^{1-r}, so\, \dot{z} = (1-r)x^{-r}\dot{x},$

$use\, this\, above, we\, get$

$$(1-r)^{-1}\dot{z} + a(t)z = b(t).$$

$$We\ get\ the\ linear\ dif\ equation\ in\ z$$

$$Notice\ that\ (1-r)^{-1}\dot{z} + a(t)z = b(t).$$

$$we\ can\ find\ z(t)\ by\ our\ formula.$$

$$Once\ z(t)\ found, use\ x(t) = z(t)^{1/(1-r)}.$$

$Ex.1\quad Solve\ \dot{x} = -tx + t^3x^3 \ldots\ldots\ldots\ldots\ldots(1)$

$This\, is\, a\, Bernoulli\, eq.\, with\, r = 3.\ Dividing\, it\, by\, x^3 gets$

$$x^{-3}\dot{x} + tx^{-2} = t^3 \quad \ldots\ldots\ldots\ldots\ldots(1')$$

$Let\, z \equiv x^{1-r} = x^{1-3} = x^{-2}, \dot{z} = -2x^{-3}\dot{x}; (1')\, becomes$

$$\frac{\dot{z}}{-2} + tz = t^3;\ multiply\, by -2\, get$$

$$\dot{z} - 2tz = -2t^3$$

$with\ \ a(t) = -2t;\quad b(t) = -2t^3.$

$and\, A(t) = \int a(t)dt = \int -2t\, dt = -t^2.$

$$Use\ formula\quad \dot{x} + a(t)x = b(t) \Leftrightarrow$$

$$x = Ce^{-A(t)} + e^{-A(t)} \int e^{A(t)} b(t)\, dt,$$

$$where\ A(t) = \int a(t)\, dt.$$

$$z = Ce^{t^2} - 2e^{t^2} \int e^{-t^2} t^3\, dt \quad ....(2).$$

The last term of (2) can be solved using

integration by parts: $\int ue^u du$

$$Let\ u = -t^2,\ then\ e^u = e^{-t^2}\ and\ du = -2t\, dt,\ thus$$

$$\int e^{-t^2} t^3\, dt = \int e^u (-t^2)(-t) dt = \int e^u u(\tfrac{1}{2}) du$$

$$\int e^{-t^2} t^3\, dt = (\tfrac{1}{2}) \int u e^u\, du = (\tfrac{1}{2}) u e^u - (\tfrac{1}{2}) \int e^u du$$

$$= (\tfrac{1}{2}) u e^u - (\tfrac{1}{2}) e^u = -(\tfrac{1}{2}) t^2 e^{-t^2} - (\tfrac{1}{2}) e^{-t^2}$$

$$z = Ce^{t^2} - 2e^{t^2} [-(\tfrac{1}{2}) t^2 e^{-t^2} - (\tfrac{1}{2}) e^{-t^2}].$$

$$= Ce^{t^2} + t^2 + 1.$$

$Thus,$

$$x = \frac{1}{\sqrt{z}}$$

$$Ex\,2.\ \dot{x} + tx = 3tx^2. \Rightarrow x^{-2}\dot{x} + tx^{-1} = 3t$$

$$Let\,z = x^{1-r} = x^{1-2} = x^{-1}$$

$$\Rightarrow \dot{z} = -x^{-2}\dot{x}\ \ use\,this\,we\,get - \dot{z} + tz = 3t$$

$$or\ \ \dot{z} - tz = -3t$$

$$[x = Ce^{-A(t)} + e^{-A(t)}\int e^{A(t)}b(t)\,dt; A(t) = \int a(t)\,dt]$$

$$A(t) = \int -tdt = -0.5t^2; \int e^{A(t)}b(t)\,dt = \int e^{-0.5t^2}(-3t)dt$$

$$= 3\int e^{-0.5t^2}(-t)dt = 3e^{-0.5t^2}$$

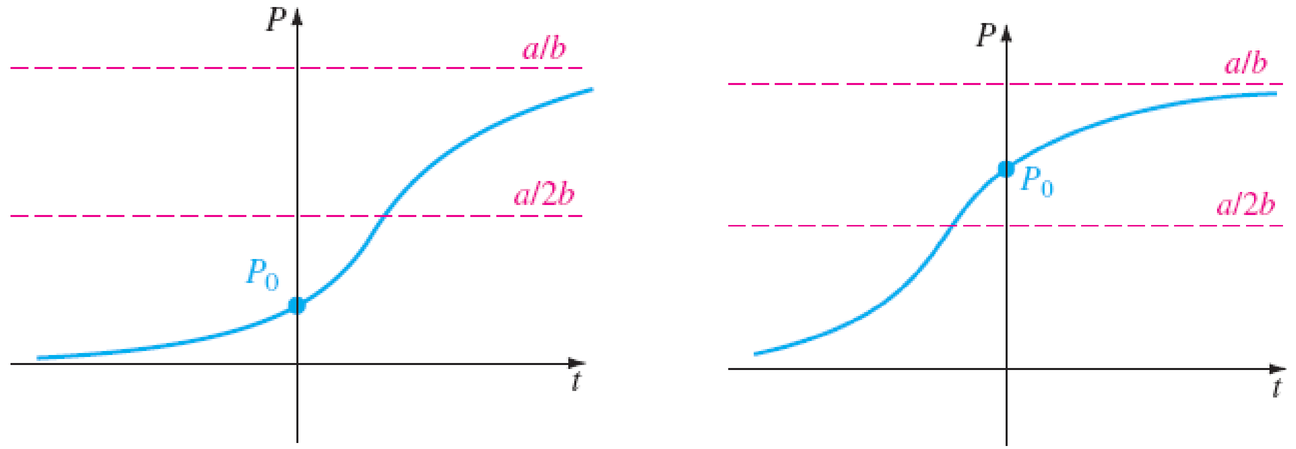
$$z(t) = Ce^{0.5t^2} + 3,\ \ thus\ \ x(t) = 1/[Ce^{0.5t^2} + 3].$$

POPULATION DYNAMICS

$$\frac{dP}{dt} = P(a - bP).$$

One of the equations he studied was (4), where $a > 0$ and $b > 0$. Equation (4) came to be known as the **logistic equation,** and its solution is called the **logistic function.** The graph of a logistic function is called a **logistic curve.**

If we rewrite (4) as $dP/dt = aP - bP^2$, the nonlinear term $-bP^2$, $b > 0$, can be interpreted as an "inhibition" or "competition" term. Also, in most applications the positive constant a is much larger than the constant b.



SOLUTION OF THE LOGISTIC EQUATION One method of solving (4) is separation of variables. Decomposing the left side of $dP/P(a - bP) = dt$ into partial fractions and integrating gives

$$\left(\frac{1/a}{P} + \frac{b/a}{a - bP}\right)dP = dt$$

$$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a - bP| = t + c$$

$$\ln\left|\frac{P}{a - bP}\right| = at + ac$$

$$\frac{P}{a - bP} = c_1 e^{at}.$$

It follows from the last equation that

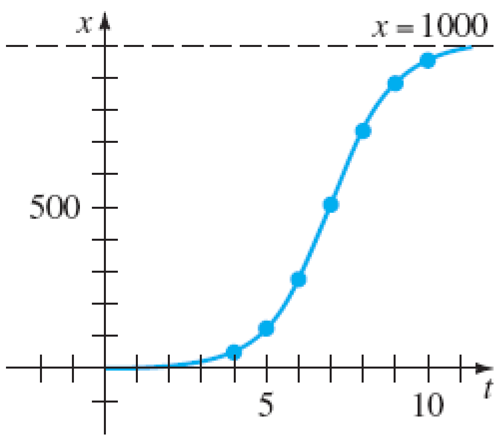
$$P(t) = \frac{ac_1 e^{at}}{1 + bc_1 e^{at}} = \frac{ac_1}{bc_1 + e^{-at}}.$$

If $P(0) = P_0$, $P_0 \neq a/b$, we find $c_1 = P_0/(a - bP_0)$, and so after substituting and simplifying, the solution becomes

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}. \tag{5}$$

EXAMPLE 1 Logistic Growth

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.



t (days)	x (number infected)
4	50 (observed)
5	124
6	276
7	507
8	735
9	882
10	953

(b)

SOLUTION Assuming that no one leaves the campus throughout the duration of the disease, we must solve the initial-value problem

$$\frac{dx}{dt} = kx(1000 - x), \quad x(0) = 1.$$

By making the identification $a = 1000k$ and $b = k$, we have immediately from (5) that

$$x(t) = \frac{1000k}{k + 999ke^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}.$$

Now, using the information $x(4) = 50$, we determine k from

$$50 = \frac{1000}{1 + 999e^{-4000k}}.$$

We find $-1000k = \frac{1}{4} \ln \frac{19}{999} = -0.9906$. Thus

$$x(t) = \frac{1000}{1 + 999e^{-0.9906t}}.$$

Finally, $$x(6) = \frac{1000}{1 + 999e^{-5.9436}} = 276 \text{ students}.$$

MODIFICATIONS OF THE LOGISTIC EQUATION There are many variations of the logistic equation. For example, the differential equations

$$\frac{dP}{dt} = P(a - bP) - h \qquad \text{and} \qquad \frac{dP}{dt} = P(a - bP) + h \tag{6}$$

could serve, in turn, as models for the population in a fishery where fish are **harvested** or are **restocked** at rate h. When $h > 0$ is a constant, the DEs in (6) can be readily analyzed qualitatively or solved analytically by separation of variables. The equations in (6) could also serve as models of the human population decreased by emigration or increased by immigration, respectively. The rate h in (6) could be a function of time t or could be population dependent; for example, harvesting might be done periodically over time or might be done at a rate proportional to the population P at time t. In the latter instance, the model would look like $P' = P(a - bP) - cP$, $c > 0$. The human population of a community might change because of immigration in such a manner that the contribution due to immigration was large when the population P of the community was itself small but small when P was large; a reasonable model for the population of the community would then be $P' = P(a - bP) + ce^{-kP}, c > 0, k > 0$. See Problem 22 in Exercises 3.2. Another equation of the form given in (2),

$$\frac{dP}{dt} = P(a - b \ln P), \tag{7}$$

is a modification of the logistic equation known as the **Gompertz differential equation.** This DE is sometimes used as a model in the study of the growth or decline of populations, the growth of solid tumors, and certain kinds of actuarial predictions.

Homework

1. The number $N(t)$ of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem

$$\frac{dN}{dt} = N(1 - 0.0005N), \quad N(0) = 1.$$

(a) Use the phase portrait concept of Section 2.1 to predict how many supermarkets are expected to adopt the new procedure over a long period of time. By hand, sketch a solution curve of the given initial-value problem.

(b) Solve the initial-value problem and then use a graphing utility to verify the solution curve in part (a). How many companies are expected to adopt the new technology when $t = 10$?

2. The number $N(t)$ of people in a community who are exposed to a particular advertisement is governed by the logistic equation. Initially, $N(0) = 500$, and it is observed that $N(1) = 1000$. Solve for $N(t)$ if it is predicted that the limiting number of people in the community who will see the advertisement is 50,000.

5.7 Qualitative Theory and Stability

- Previously, we look at the explicit solutions and try to identify unspecified parameters.
- When we don't have one, we can look at the properties of the solution: stability, uniqueness, and existence.
- First, let consider stability of any solution.
- Many economic models have Autonomous equations

$$\dot{x} = F(t,x) = F(x) .... called\, autonomous.$$

- In which *t* , time an independent variable, does not appear on RHS. If not autonomous, the equation will depend on t. Idea: some laws do not change over time.
- Tools to use here is called "Phase Diagram: a picture in x-xdot plane."

Phase Diagram

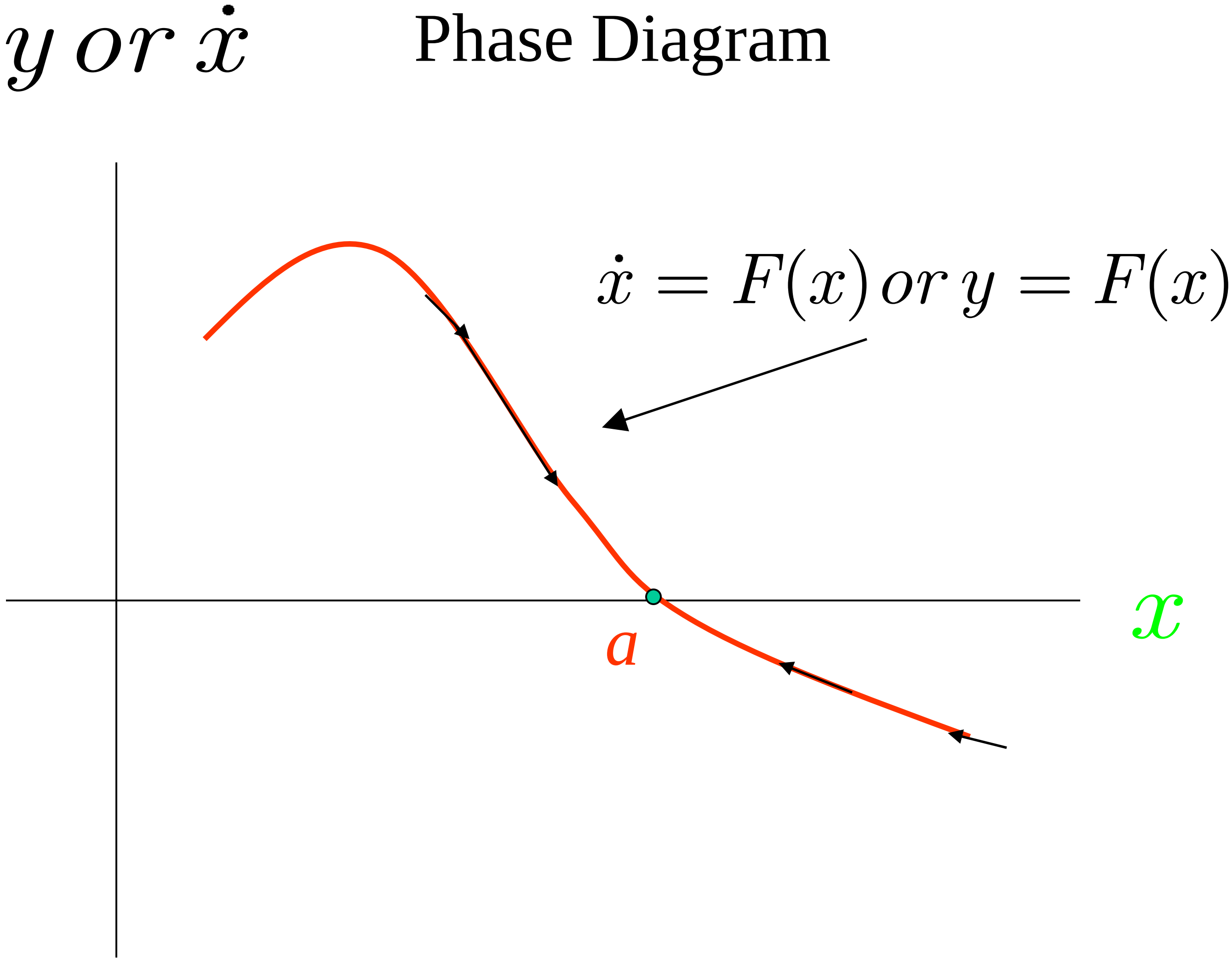


Figure 1. Example of Globally stable solution

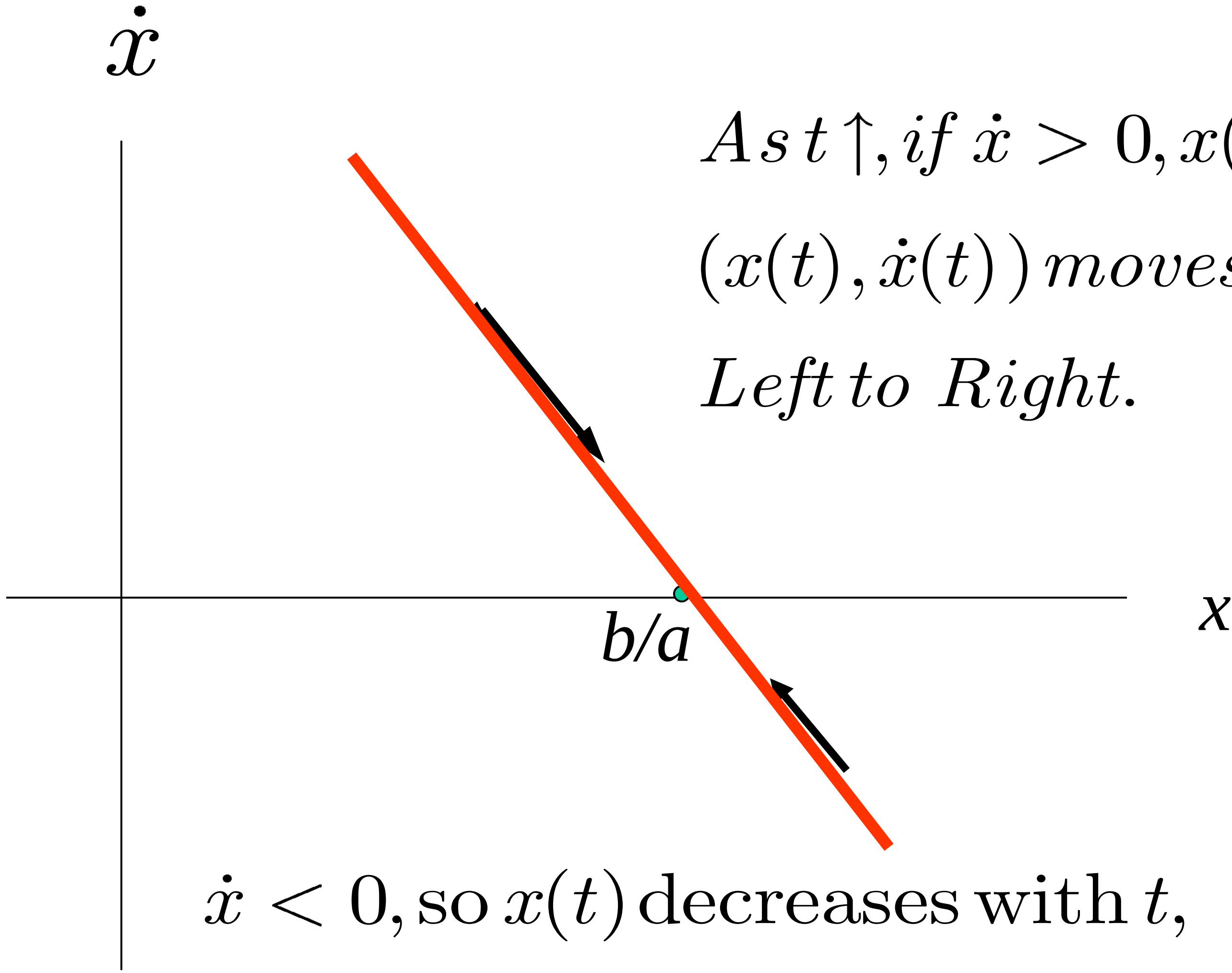
5.7 Qualitative Theory and Stability

- Solution to the autonomous equation is to find value of x such that $F(x)=0$. *(or xdot=0)*
- A point a is an <u>equilibrium</u> solution or stationary state if $F(a)=0$.
- In Figure 1, $x(t) \equiv a$ is the solution of the equation. Figure 1 has one equilibrium state, a.
- That is, if $x(t_o)=a$, for some t, then $x(t)=a$ for all t.

5.7 Qualitative Theory and Stability

- In Figure 1, the equilibrium is also globally asymptotically stable because if $x(t)$ is a solution with $x(t_0)=x_0$, then $x(t)$ will always converge to the point on the x-axis with $x=a$ for any starting point (t_0,x_0).
- Globally asymptotically stable implies that we can start from any starting point in domain.
- Now, let consider Figure 2 with the equation
- $xdot = b - ax$
- Find a point that $b-ax = 0$, we get $x^*=b/a$.
- Consider Figure 3 with the equation
- $xdot = 1-x$
- Find a point that $1-x = 0$, we get $x^*=1$.

Figure 2: $\dot{x} = F(x) = b - ax; F'(x) = -a; F(b/a) = 0$



$As\, t\uparrow, if\ \dot{x} > 0, x(t)\uparrow,$

$(x(t), \dot{x}(t))\, moves\ from$

$Left\ to\ Right.$

$\dot{x} < 0, \text{so } x(t) \text{ decreases with } t,$

so we move from right to left.

From Figure 2, we draw x(t) the integral curves

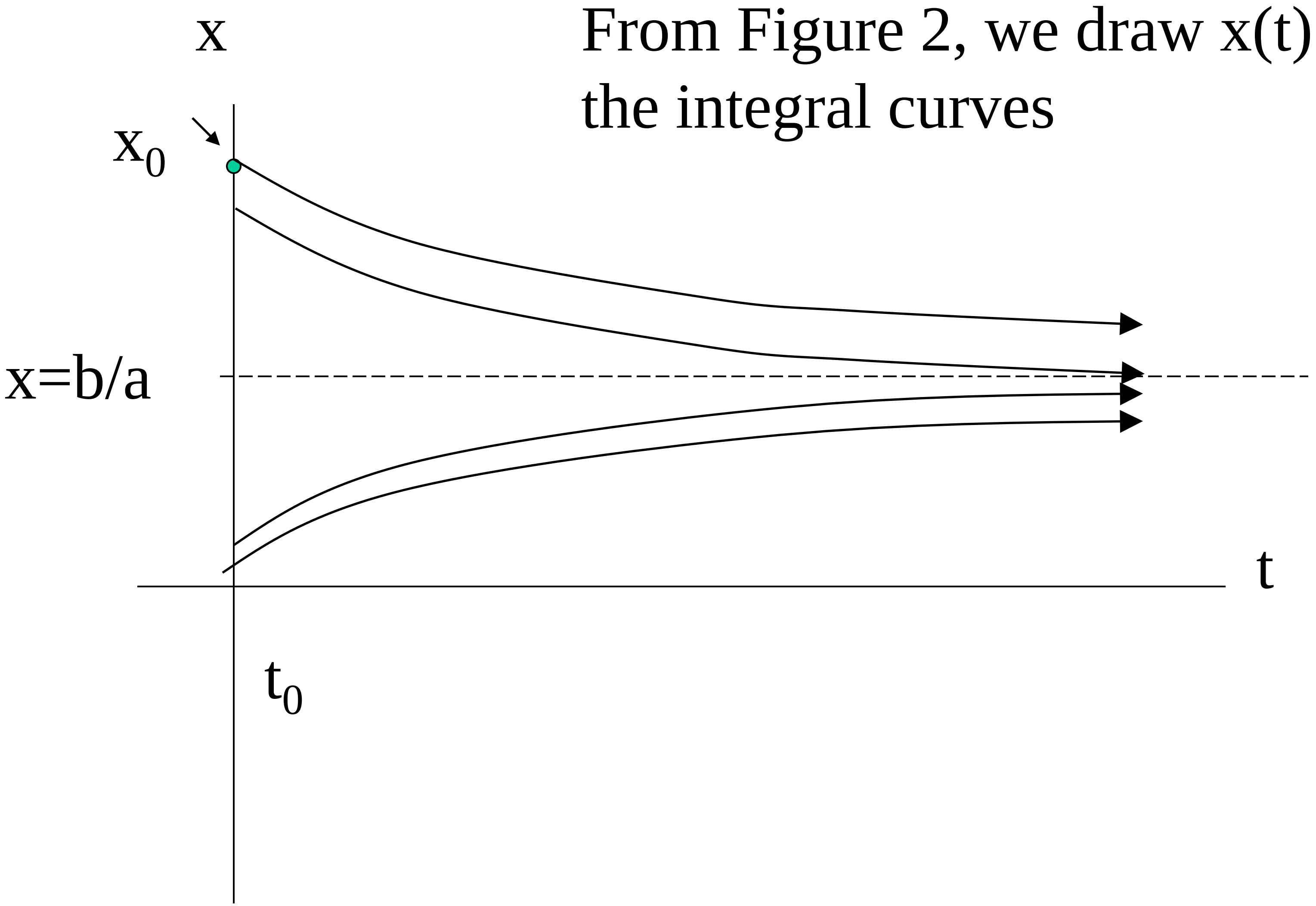
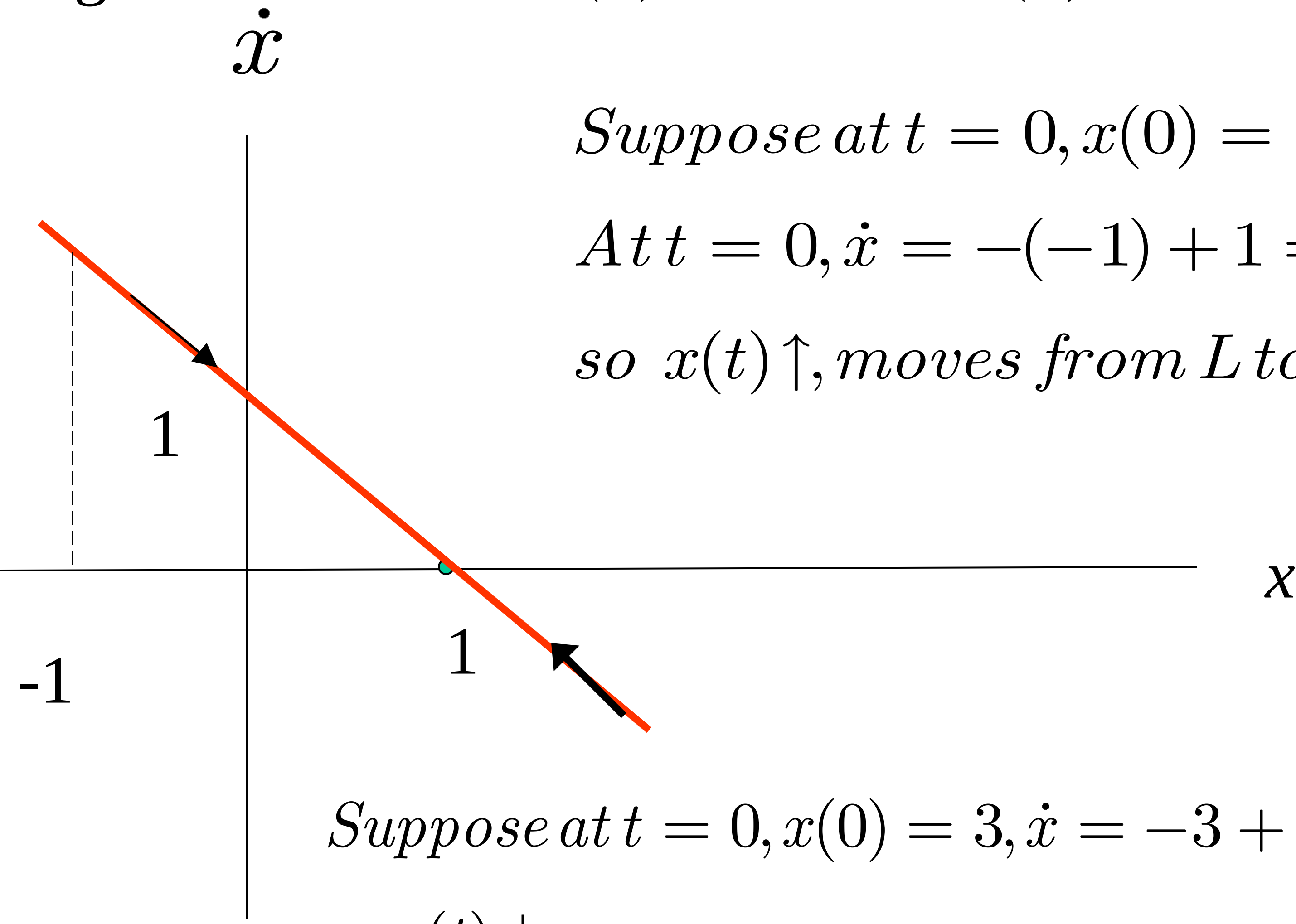


Figure 3: $\dot{x} = F(x) = 1 - x; F'(x) = -1; F(1) = 0$



$Suppose\, at\, t = 0, x(0) = -1$

$At\, t = 0, \dot{x} = -(-1) + 1 = 2 > 0,$

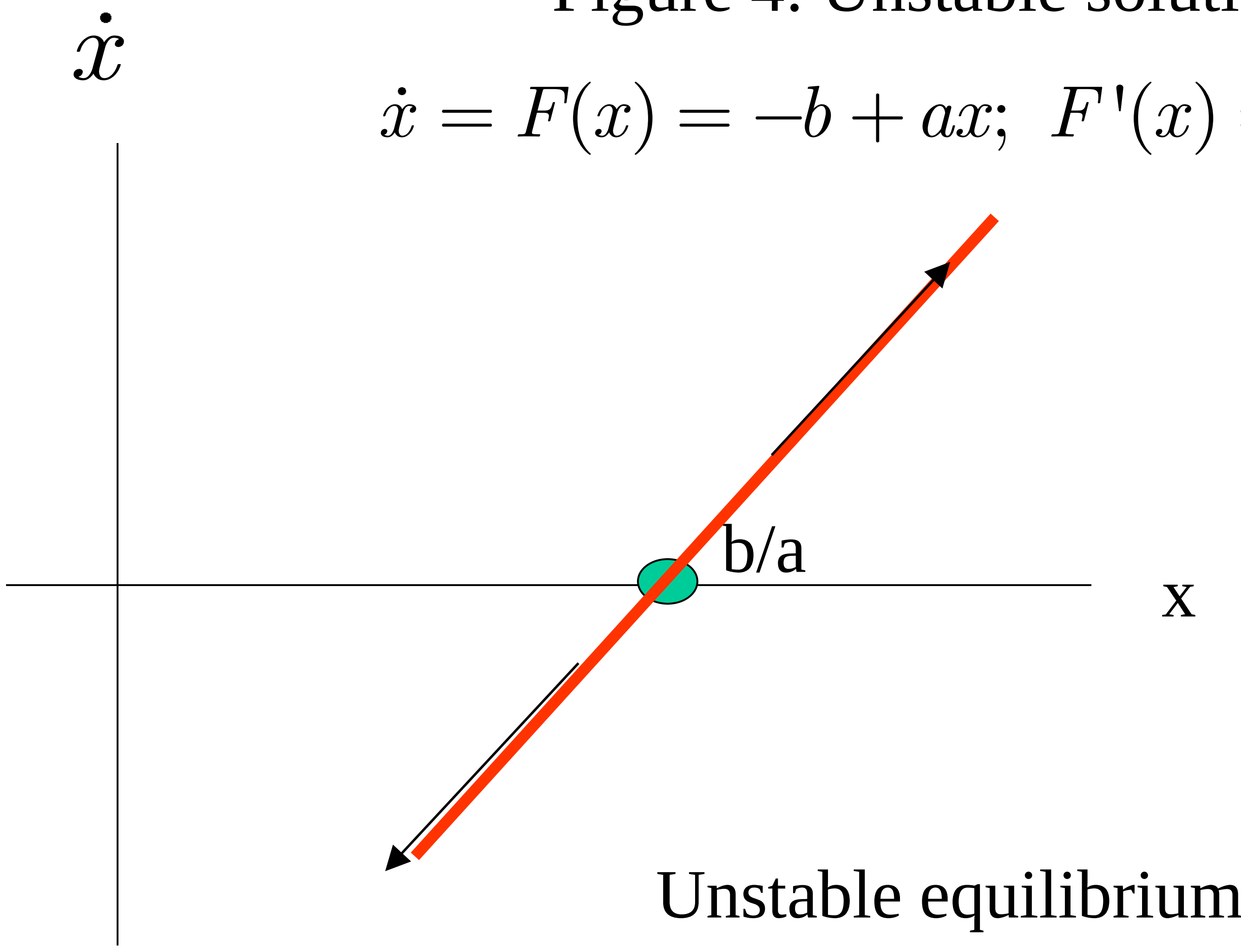
$so\;\; x(t) \uparrow, moves\; from\; L\; to\;\; R.$

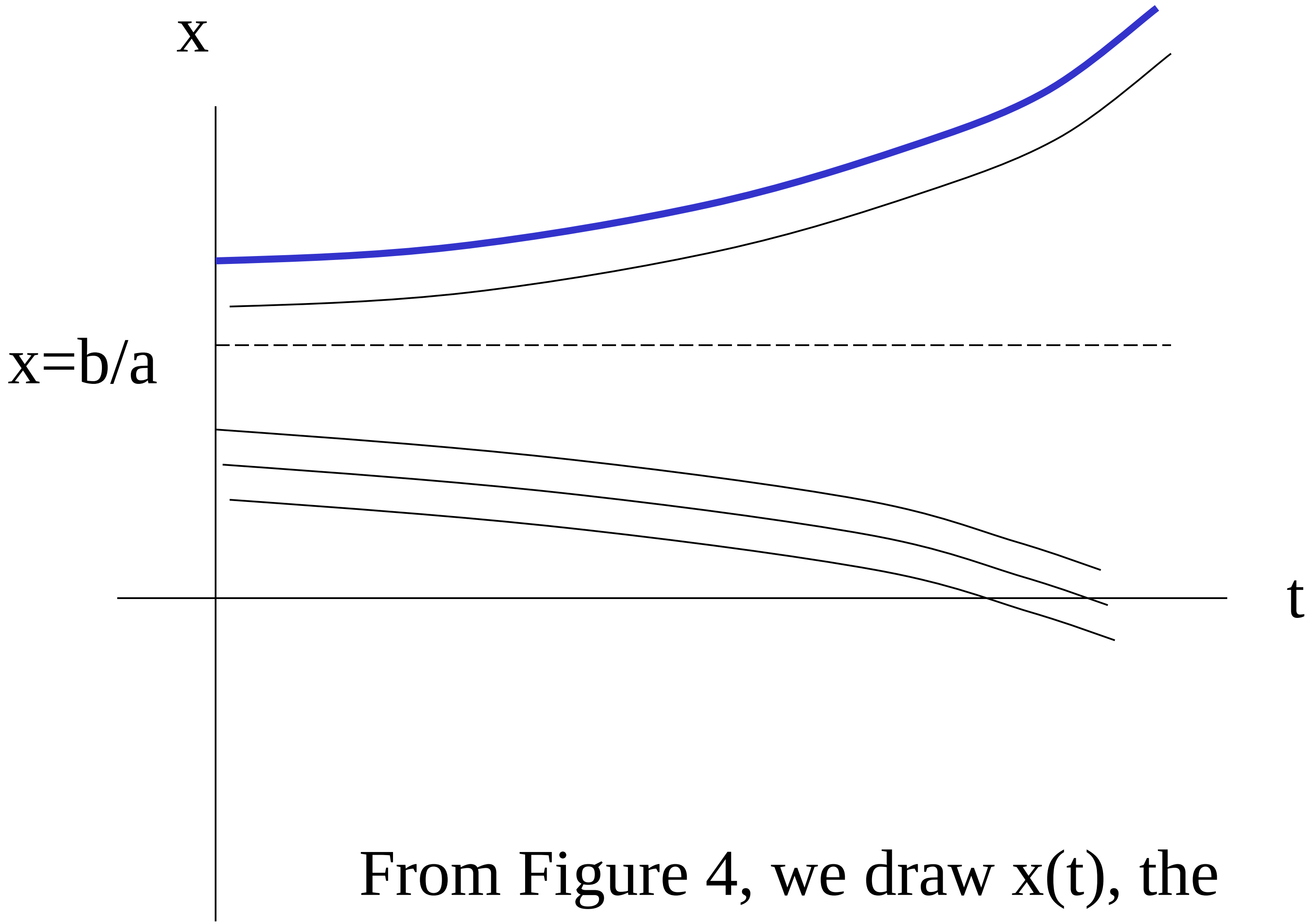
$Suppose\, at\, t = 0, x(0) = 3, \dot{x} = -3 + 1 = -2 < 0$

$so\, x(t) \downarrow$

Figure 4: Unstable solution

$$\dot{x} = F(x) = -b + ax; \quad F'(x) = a$$





From Figure 4, we draw x(t), the solution curve.

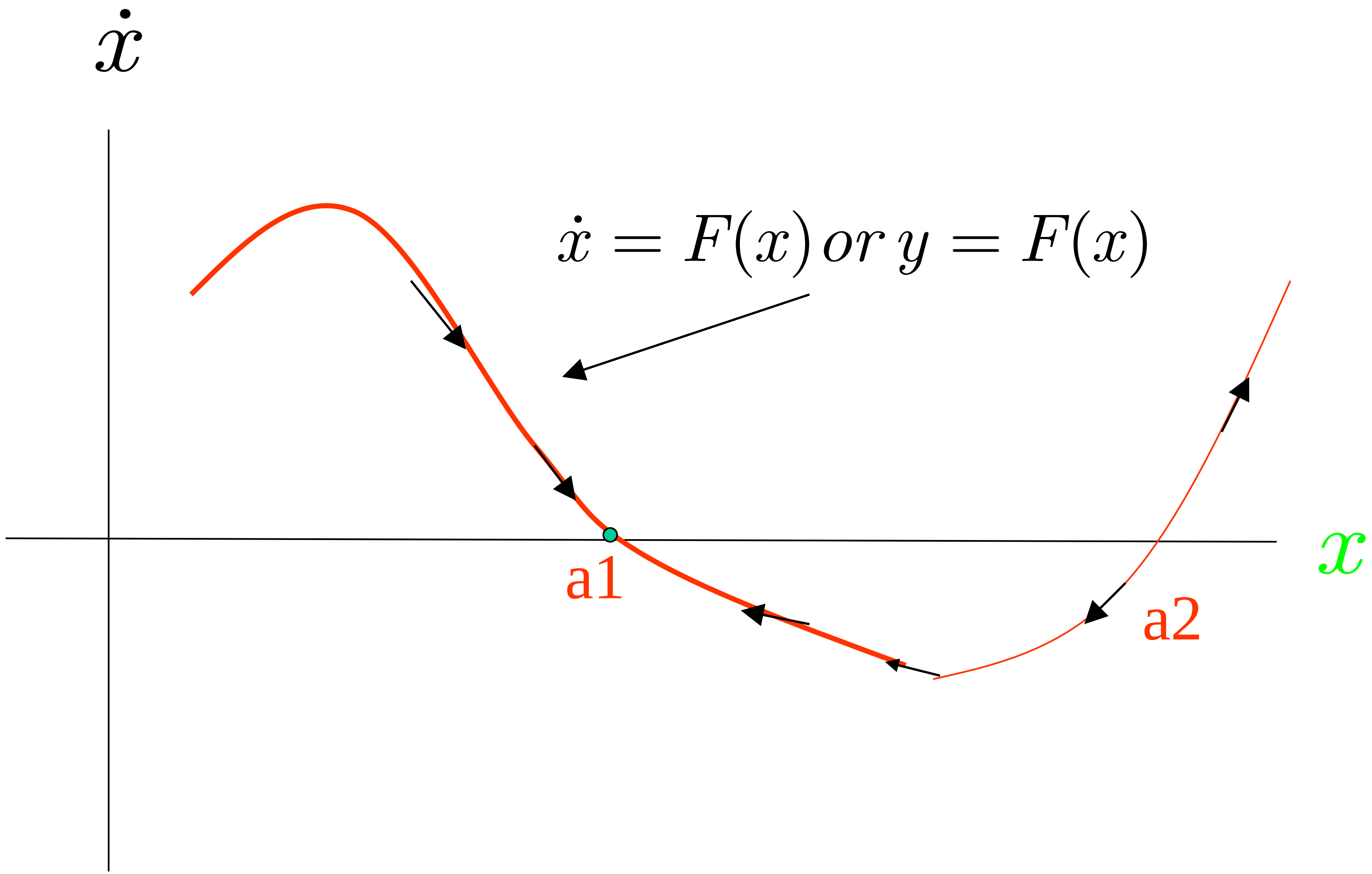


Figure 5. Example of locally stable solution

Stability

- We have the following result:

$$(a)\, F(a) = 0 \ \ and \ \ F'(a) < 0$$

$$\Rightarrow a\, is\, a\, locally\, asymtotically\, stable\, equilibrium.$$

$$(b)\, F(a) = 0 \ and \ \ F'(a) > 0$$

$$\Rightarrow a\, is\, a\, locally\, asymtotically\, unstable\, equilibrium.$$

$$(c)\ \ F(a) = 0\, and\, F'(a) = 0\,, inconclusive.$$

Example. Price adjustment process.

$$\dot{P} = H(D(P) - S(P)),\ H(0) = 0,\ H' > 0$$

Let P^e is an equilibrium price : $D(P^e) - S(P^e)$.

Let $F(P) = H(D(P) - S(P))$

We need $F'(P^e) < 0$ for stable equilibrium.

$$F'(P^e) = H'(.)[D'(P^e) - S'(P^e)] < 0$$

since $H' > 0$, we need $[D'(P^e) - S'(P^e)] < 0$.

means demand slope $<$ supply slope.

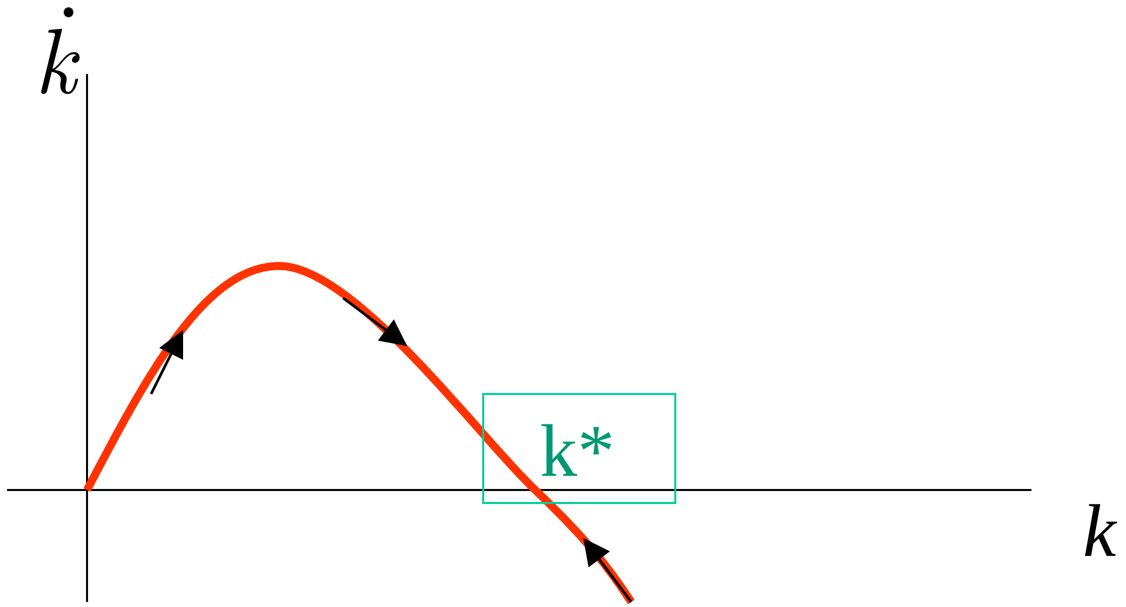
Generally, this is satisfied for $D' < 0, S' > 0$.

$$Example.\ Solow's\,Growth\,Model.$$

$$\dot{k} = sf(k) - \lambda k,\ s > 0,\ \ \lambda > 0.(per\,capita)$$

$$a\,unique\,equilibrium\,k^* : sf(k^*) = \lambda k^*$$

$$F(k) = sf(k) - \lambda k;\ \ F' = sf'(k^*) - \lambda < 0$$



$$F' = sf'(k^*) - \lambda < 0, stable\, equilibrium$$

$$\text{Let}\, f(k) = k^{\alpha}; so\, F' = s\alpha k^{\alpha - 1} - \lambda < 0.$$

$$\text{Suppose}\, \alpha = 0.4, \lambda = 2\%, s = 10\%$$

$$\text{we have}\, k^* = 14.62, f'(k^*) = -.012.$$

Note that if we start with

$$X(t) = F(K(t), L(t)), \quad \dot{K}(t) = sX(t)$$

$$\text{and}\, L(t) = L_0 e^{\lambda t}.$$

We can derive the capital accumulation in per-capita form by

$$let\, k(t) \equiv K(t)\,/\,L(t)\, and\, x(t) = f(k(t))$$

$$\frac{\dot{k}}{k} = \frac{d(\ln k)}{dt} = \frac{d(\ln K - \ln L)}{dt}$$

$$= \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{sF}{K} - \lambda$$

$$= \frac{sLf(k)}{K} - \lambda = \frac{sf(k)}{k} - \lambda.$$

$$\dot{k} = sf(k) - \lambda k.$$

$$Let\ f(k) = k^{\alpha}$$

$$\dot{k} = sk^{\alpha} - \lambda k \ \ or \quad \dot{k} + \lambda k = sk^{\alpha}$$

$$this\ is\ a\ Bernoulli\ equation\ in\ k\ with\ r = \alpha.$$

$$divide\ by\ k^{\alpha}\ : \qquad \dot{k}\ /\ k^{\alpha} + \lambda k^{1-\alpha} = s$$

$$Let\ z = k^{1-\alpha}, thus\ \dot{z} = (1-\alpha)k^{-\alpha}\dot{k} = (1-\alpha)\dot{k}\ /\ k^{\alpha}$$

$$we\ can\ obtain \qquad (1-\alpha)^{-1}\dot{z} + \lambda z = s$$

$$\dot{z} + \underbrace{(1-\alpha)\lambda}_{a} z = \underbrace{(1-\alpha)s}_{b}$$

$$use\ the\ formula\ yields$$

$$z(t) = [z(0) - \frac{s}{\lambda}]e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$$

$$\Rightarrow k^{1-\alpha} = [k(0)^{1-\alpha} - \frac{s}{\lambda}]e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}.$$

Since $(1-\alpha)$ *and* λ are both positive, so as $t \to \infty$, the first term will approach zero.

And the equilibrium capital-labor ratio is

$$k \to \left(\frac{s}{\lambda}\right)^{1/(1-\alpha)}$$

5.8 Existence and Uniqueness

- Good economic model must have a solution.
- If so, we want to know if it is a unique.

Theorem 5.8.1 (Local Existence and Uniqueness)

"For $\dot{x} = F(t,x)$, if F and F'_x are cont in open set A in the (t,x) plane. Let (t_0, x_0) in A, then there exists *one local* solution passing (t_0, x_0)." This ensure the existence of a solution only in a *small* neighborhood of t_0. Uniqueness requires if we found $x(t_0) = y(t_0)$, then $x(t) = y(t)$ for all t.

$Ex1.\ If\ F(t,x) = ax, or\ \dot{x} = ax.$

$For\ F\ and\ F'$ are continous anywhere, by Thm. there is a unique solution curve passing through (t_0, x_0). The solution is $x(t) = x(t_0)e^{a(t-t_0)}$.

$Ex2. F(t,x) = f(t)g(x)$: unique and existence is ensured if $f\ and\ g$ are continous and differentiable.

5.8 Existence and Uniqueness

- In economics, we may want to know whether there is a solution defined over the whole interval [0, inf] (say in Solow's growth model)or some defined interval. Then we use the Global existence Theorem. See Thm 5.8.3 page 218.
- However, normally we look at the behavior of a solution as t goes to infinity. Rarely we check for the existence !

Homework

- Section 5.1: 4
- Section 5.3: 1, 2, 3
- Section 5.4: 1, 4, 7
- Section 5.6: 1, 2
- Section 5.7: 1, 3