Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$(a^{-b})^{-\frac{1}{b}}$

From given information, transition equation

$$W_{t+1} = (W_t - C_t)(R_f + \omega_t(R_{rt} - R_f)) = S_t R_t$$

$$\max_{C_{T-1},\,\omega_{T-1}} E_{T-1}\left[\delta^{T-1}\left(\frac{C_{T-1}^{(1-\gamma)}}{1-\gamma}\right) + \delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

#FOC : diff. w.r.t. C_{T-1} : $\frac{\delta^{T-1}}{C_{T-1}^{*\gamma}} - E_{T-1}\left[\frac{\delta^T}{(S_{T-1}R_{T-1})^\gamma} R_{T-1}\right] = 0$

diff. w.r.t. ω_{T-1} : $\frac{\delta^T}{(S_{T-1}R_{T-1})^\gamma}(R_{r,T-1} - R_f) = 0$

C^*_{T-1} :

$$\delta^{T-1} C_{T-1}^{*-\gamma} = E_{T-1}[\delta^T (S_{T-1}R_{T-1})^{-\gamma} R_{T-1}]$$

$$C_{T-1}^{*-\gamma} = \delta (W_{T-1} - C_{T-1})^{-\gamma} E_{T-1}[(R_{T-1})^{1-\gamma}]$$

$$C^*_{T-1} = \delta^{-\frac{1}{\gamma}}(W_{T-1} - C_{T-1})\left(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}]\right)$$

$$\frac{C^*_{T-1}}{W_{T-1}} - 1 = \delta^{-\frac{1}{\gamma}}\left(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}]\right)$$

$$C^*_{T-1} = \left\{1 + \delta^{-\frac{1}{\gamma}}\left(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}]\right)\right\} W_{T-1}$$

$$= \frac{(\delta E_{T-1}[R^\gamma_{T-1}])^{\frac{1}{\gamma}}}{1 + (\delta E_{T-1}[R^\gamma_{T-1}])^{\frac{1}{\gamma}}} \cdot W_{T-1}$$

$$C^{*}_{T-1} = \left\{ 1 + \delta^{-\frac{1}{\gamma}} \left(E_{T-1}\left[(R_{T-1})^{\frac{\gamma-1}{\gamma}} \right] \right) \right\} W_{T-1}$$

$$\max_{C_{T-1}, W_{T-1}} E_{T-1}\left[\delta^{T-1} \left(\frac{C_{T-1}^{(1-\gamma)}}{1-\gamma} \right) + \delta^{T} \left(\frac{W_T^{(1-\gamma)}}{1-\gamma} \right) \right.$$

Score......................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1}, T-1) = \delta^{T-1} \left(\frac{\left[\left\{ 1 + \delta^{-\frac{1}{\gamma}} \left(E_{T-1}\left[(R_{T-1})^{\frac{\gamma-1}{\gamma}} \right] \right) \right\} W_{T-1} \right]^{(1-\gamma)}}{1-\gamma} \right)$$

$$+ \delta^{T} \left(\frac{\left(\left[W_{T-1} - \left\{ 1 + \delta^{-\frac{1}{\gamma}} \left(E_{T-1}\left[(R_{T-1})^{\frac{\gamma-1}{\gamma}} \right] \right) \right\} W_{T-1} \right] (R_{T-1}) \right)^{(1-\gamma)}}{1-\gamma} \right.$$

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score......................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

Optimal condition is

$$U_c(C^*_{T-2}, T-2) = E_{T-2}[J_W(W_{T-1}, T-1)R_{T-2}]$$

$$\delta^{T-2} C^{*\,-\gamma}_{T-2} = E_{T-2}\Big[\delta^{T-1}\Big[1+\delta^{-\frac{1}{\gamma}}\big(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}]\Big]\Big]^{-\gamma} + \Big\{\delta^T\Big[1-\Big\{1+\delta^{-\frac{1}{\gamma}}\big(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}]\big)\Big\}\Big]R_{T-1}\Big\}^{-\gamma}\Big]$$

$$C^{*\,-\gamma}_{T-2} = \delta E_{T-2}\Big[\big[1+\delta^{-\frac{1}{\gamma}}\big(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}]\big]^{-\gamma} + \delta\big[\delta^{-\frac{1}{\gamma}} \cdot E_{T-1}[R_{T-1}^{\frac{\gamma-1}{\gamma}}]R_{T-1}\big]^{-\gamma}\Big]$$

$$C^*_{T-2} = \delta^{-\frac{1}{\gamma}} E_{T-2}\Big[\big[1+\delta^{-\frac{1}{\gamma}}\big(E_{T-1}[R_{T-1})^{\frac{\gamma-1}{\gamma}}\big]^{-\gamma} + \delta^{\frac{\gamma-1}{\gamma}} \cdot E_{T-1}[R_{T-1}^{\frac{\gamma-1}{\gamma}}]R_{T-1}\big]^{-\gamma}\Big]$$

Note: $C^*_{T-1} = \Big\{1+\delta^{-\frac{1}{\gamma}}\big(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}]\big)\Big\}W_{T-1}$

Score......................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = U(c^*_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)]$$

$$= \delta^{T-2} \frac{c^{*(1-\gamma)}_{T-2}}{1-\gamma} + E_{T-1}\Big[\delta^{T-1} \Big(\frac{[\{1 + \delta^{-\frac{1}{\gamma}}(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}])\} W_{T-1}]^{(1-\gamma)}}{1-\gamma} \Big)$$

$$+ \delta^{T} \frac{([W_{T-1} - \{1 + \delta^{-\frac{1}{\gamma}}(E_{T-1}[(R_{T-1})^{\frac{\gamma-1}{\gamma}}])\} W_{T-1}](R_{T-1}))^{(1-\gamma)}}{1-\gamma} \Big]$$

Note: $c^{*}_{T-2} = \delta^{-\frac{1}{\gamma}} E_{T-2}\Big[[1 + \delta^{-\frac{1}{\gamma}}(E_{T-1}[R_{T-1})^{\frac{\gamma-1}{\gamma}}]^{-\gamma} + \delta^{\frac{\gamma-1}{\gamma}}$

$\cdot E_{T-1}[R_{T-1}{}^{\frac{\gamma-1}{\gamma}}] R_{T-1}]^{-\gamma}\Big]$