FN241: Session 6
Statistical Concepts and Risk Management

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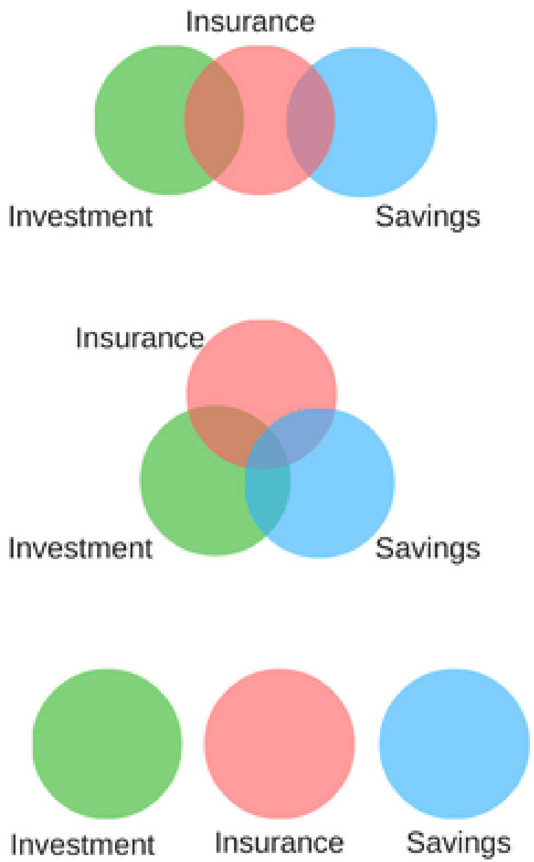
Background: Which one is correct?

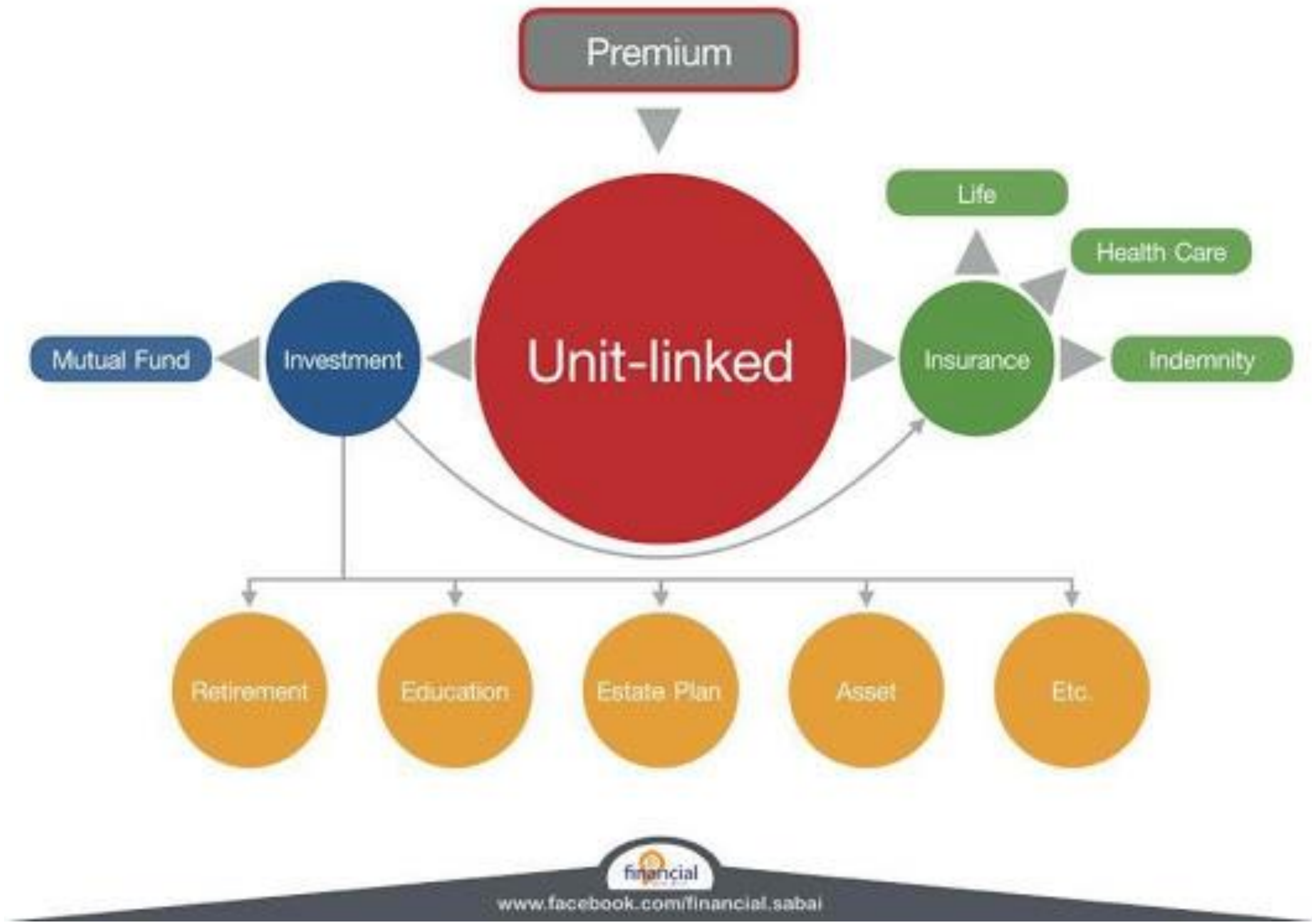


Saving:
= Parking your money in risk free instruments like saving accounts, liquid funds, etc.

Investment:
= Investing in assets like equity, mutual funds, gold, real estate, etc.

Insurance:
= an expense which will help cover your risks
…

Example: Unit Liked Insurance



Outline

- Statistical Concepts
- Risk Management:
  - Are returns predictable?
  - How volatile are stock returns?
  - How does volatility change over time?

Statistical Concepts

Calculating insurance premiums

- **Example:** Kate was born with a rare disease and has a 40% chance of relapse in a year. If relapse occurs, she has to visit a doctor for consultation once every 3 months. Each consultation costs 30$. Calculate the health insurance premium.

Expected value

Random Variable

- The expected value of a random variable is the probability-weighted average of the possible outcomes for that variable.

$$E(X) = \sum_{i=1}^{n} X_i P(X_i)$$

- We anticipate that there is a 15% chance that next year's return on holding Cleveland Corp will be 4%, a 60% chance it will be 6%, and a 25% chance it will be 8%. What is the expected return on Cleveland Corp stock?

$$E(X) = \sum_{i=1}^{n} 0.04(0.15) + 0.06(0.60) + 0.08(0.25)$$

$$E(X) = 0.062$$

Expected value

- **<u>Example:</u>** Further, in the consultation the doctor may either prescribe medication or recommend that she take laboratory tests. Probability of prescription is 80% while that of tests is 20%. Cost of medication is 50$ and cost of tests is 150$. Given $30 consultation costs, calculate the health insurance premium.

Premium = probability of illness in a year x average no. of utilization of services per year x unit cost of each utilization

= 0.4 x 4 x [30$ + (0.8 x 50$) + (0.2 x 150$)]

= 0.4 x 4 x [30$ + 40$ + 30$]

= 0.4 x 4 x 100$ = 160$

Variance

- The variance of a random value is the sum of the squared deviations from the expected value weighted by their associated probabilities.

$$\sigma^2(X) = \sum_{i=1}^{n} [X_i - E(X)]^2 P(X_i) = E\{[X - E(X)]^2\}$$

  - This value is a measure of the dispersion of possible values.
  - Because it has units that are squared, it is not easy to interpret. Accordingly, we use its positive square root, standard deviation, more often because it also measures dispersion but has the same units as expected value.

- The standard deviation of returns for Cleveland Corp. is then:

$$\sigma^2(X) = 0.15(0.04 - 0.062)^2 + 0.6(0.06 - 0.062)^2 + 0.25(0.08 - 0.062)^2$$
$$\sigma(R) = 0.01249$$

Conditional Expectations

- The total probability rule applies to expected values just as it does any mutually exclusive and exhaustive set of possible outcomes across a set of states.

$$E(X) = \sum_{i=1}^{n} E(X|S_i)P(S_i)$$

- This equation allows us to calculate the expected value of a random variable (*X*) as a function of the probabilities of future possible states, $P(S)$, and the conditional value of the expected value of *X* in those states, $E(X|S)$.

Conditional Expected value

Example:

Recall that we have a 60% chance of relaxed trade restrictions and, therefore, a 40% of maintaining them. If we expect the Cleveland Corp. stock to return 6% if trade restrictions are maintained and lose 11% if they are relaxed, what is the expected change in return for Cleveland Corp.?

$$E(X) = \sum_{i=1}^{n} E(X|S_i)P(S_i) = 0.6(-0.11) + 0.4(0.06)$$
$$= -0.042$$

Covariance

Covariance and correlation are both measures of the extent to which two random variables move together.

- Covariance is the expected value of the product of each variable's deviation from its respective mean.

$$\sigma_{X,Y} = E\{[X - E(X)][Y - E(Y)]\}$$

$$\sigma_{R_i,R_j} = \sum_{i=1}^{n} P(R_i)[R_i - E(R_i)][R_j - E(R_j)]$$

Risk Management:

Are returns predictable?

Defining Return and Risk

Return

= Income received on an investment plus any change in market price, usually expressed as a percent of the beginning market price of the investment.

Risk

= The variability of returns from those that are expected.

Key questions …

What characterizes stock returns?

- Are returns predictable?
- How volatile are stock returns?
- How does volatility change over time?
- What types of stocks have the highest returns?

Capital Asset Pricing Model (CAPM)

- **CAPM** is a model that describes the relationship between **risk and expected (required) return**; in this model, a security's expected (required) return is **the risk-free rate <u>plus</u> a premium** based on the **systematic risk** of the security.

$$R_j = R_f + \beta_j(R_M - R_f)$$

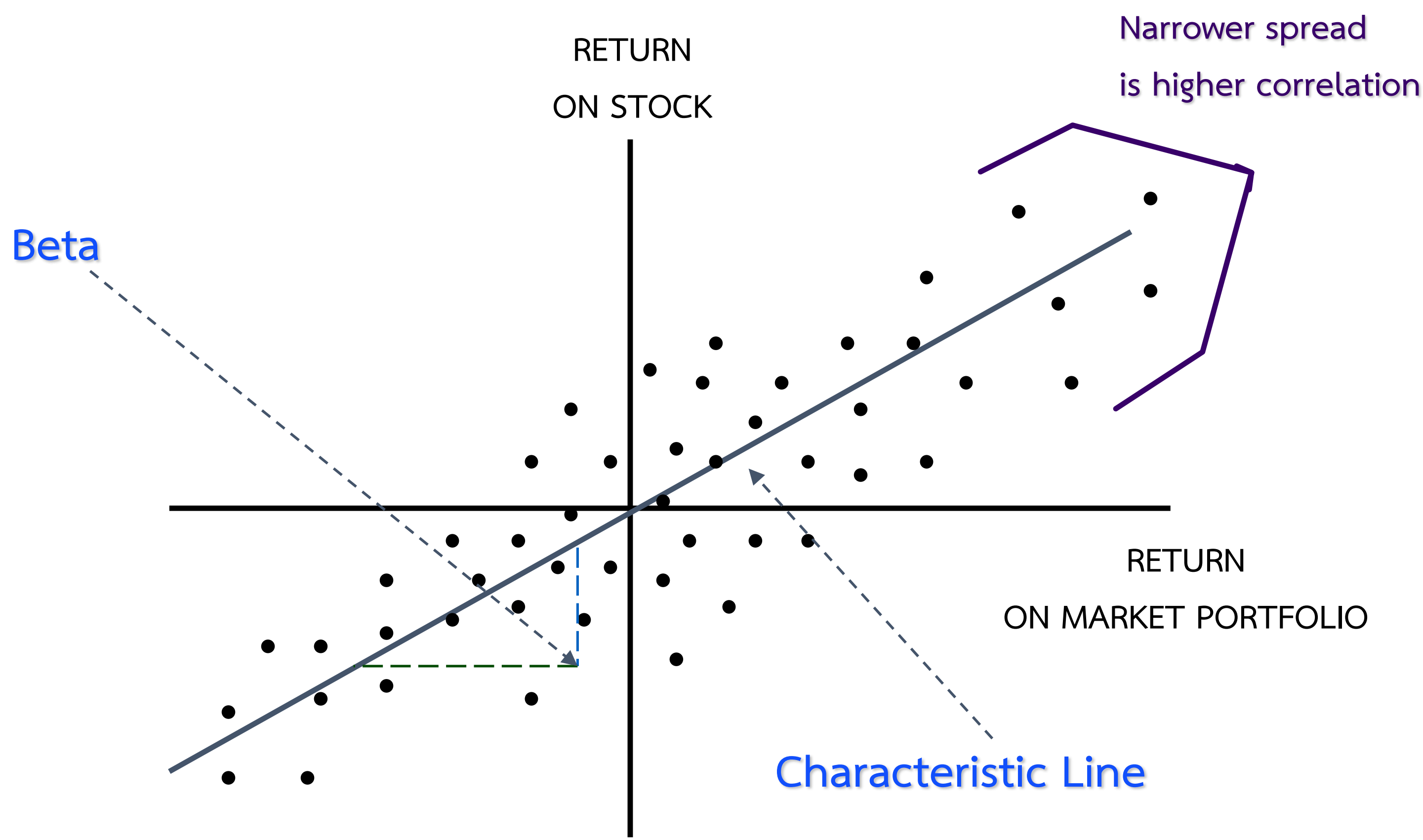
R_j is the required rate of return for stock j,

R_f is the risk-free rate of return,

β_j is the beta of stock j (measures systematic risk of stock j),

R_M is the expected return for the market portfolio.

Characteristic Line

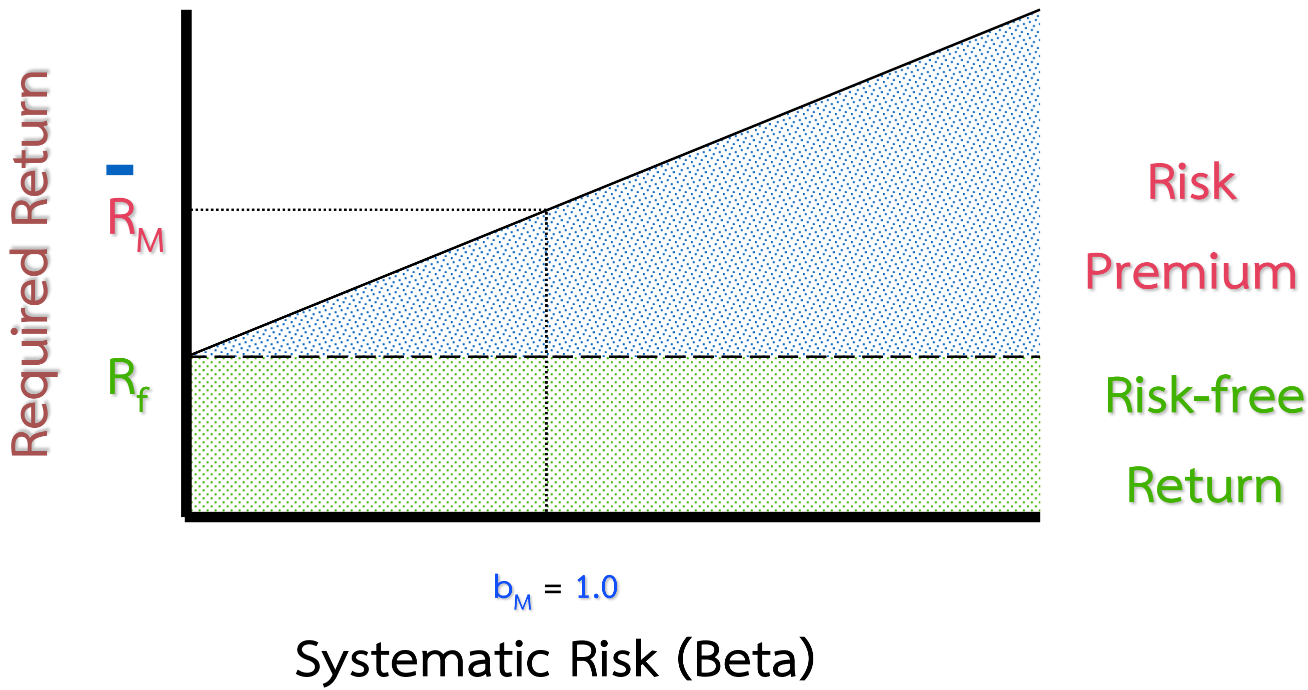


What is Beta?

- An index of systematic risk.
- It measures the sensitivity of a stock’s returns to changes in returns on the market portfolio.
- The beta for a portfolio is simply a weighted average of the individual stock betas in the portfolio.

Security Market Line

$$R_j = R_f + \beta_j(R_M - R_f)$$



Is CAPM appropriate in all situations?

Case Study:

H. Levy, "Equilibrium in an Imperfect Market: A constraint on the number of securities in the portfolio," American Economic Review, 68 (4), 1978, pp. 643–658.

Risk Management:

How volatile are stock returns?

Definition of Volatility

- Suppose that S_i is the value of a variable on day *i*. The volatility per day is the **standard deviation** of:

$$\ln\left(\frac{S_i}{S_{i-1}}\right) \qquad \text{or} \qquad \frac{S_i - S_{i-1}}{S_{i-1}}$$

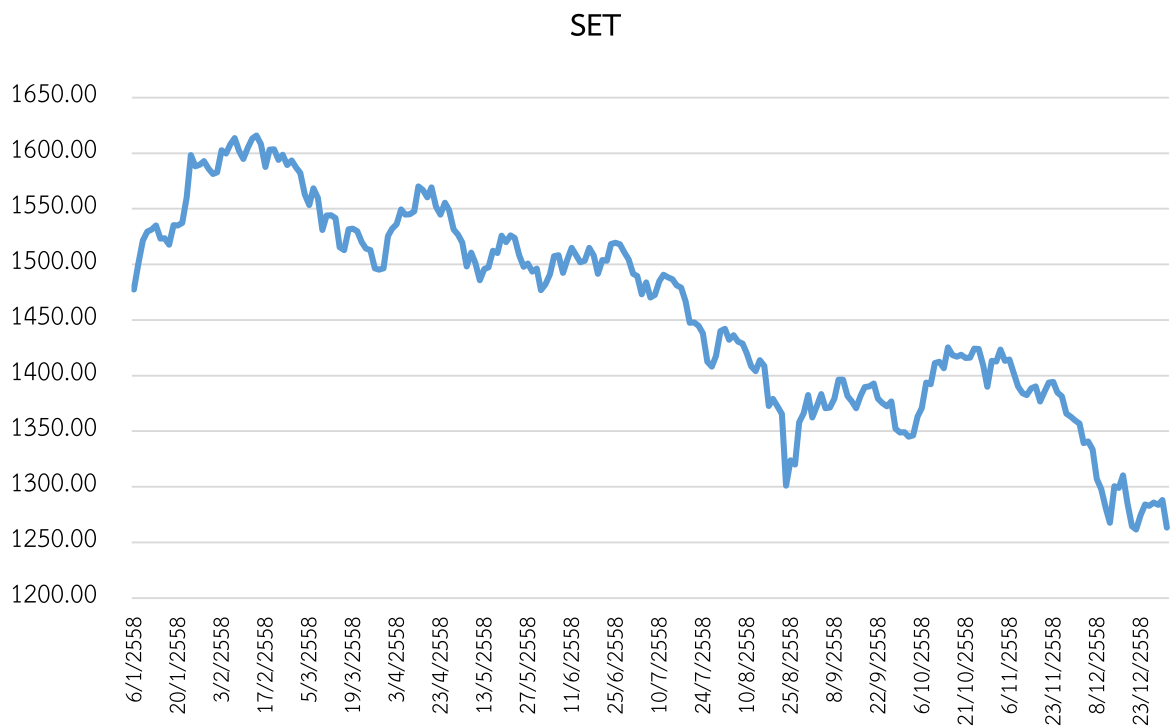
- Normally, **days when markets are closed are ignored** in volatility calculations. Thus, the volatility per year is number of business days times the daily volatility:

$$\sigma_{yr} = \sigma_{day}\sqrt{252} \qquad \text{or} \qquad \sigma_{day} = \frac{\sigma_{yr}}{\sqrt{252}}$$

- **Variance rate** is the square of volatility

Implied Volatilities

- Since volatility cannot be directly observed, we can therefore imply volatilities from **market prices** and vice versa



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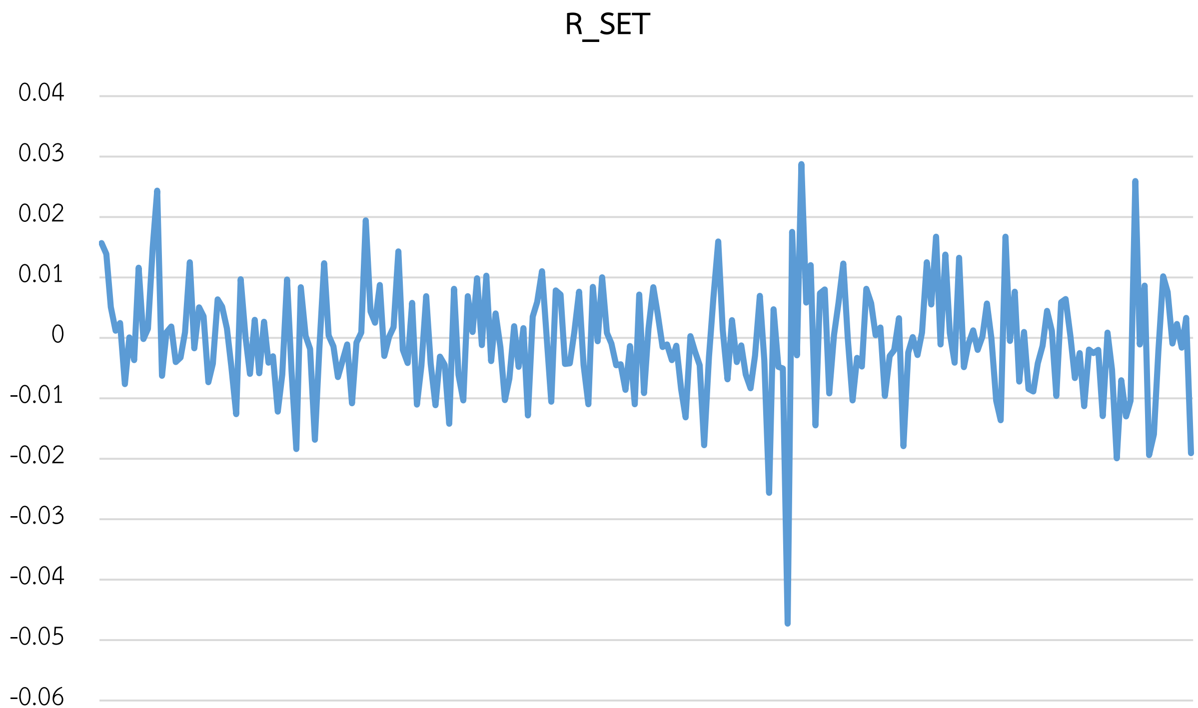
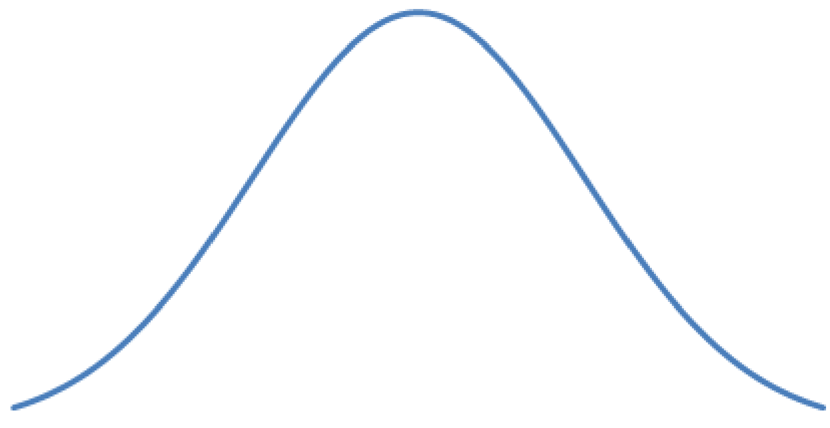
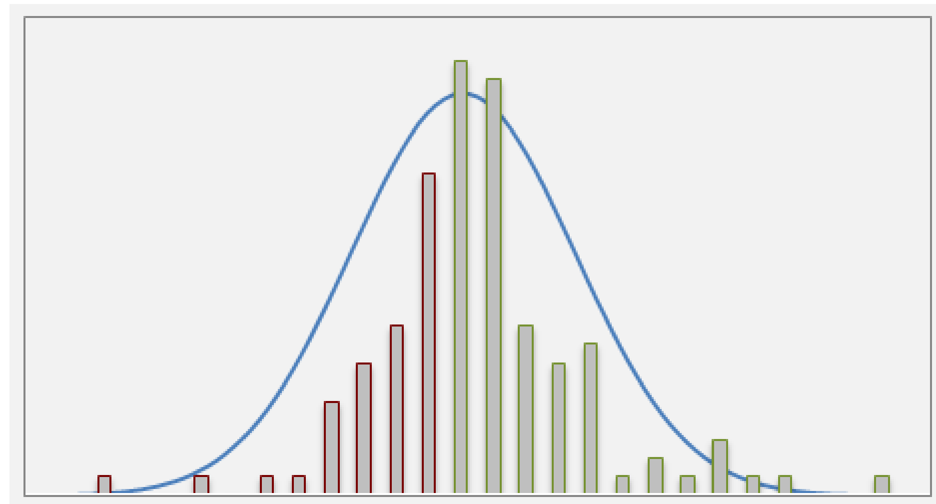
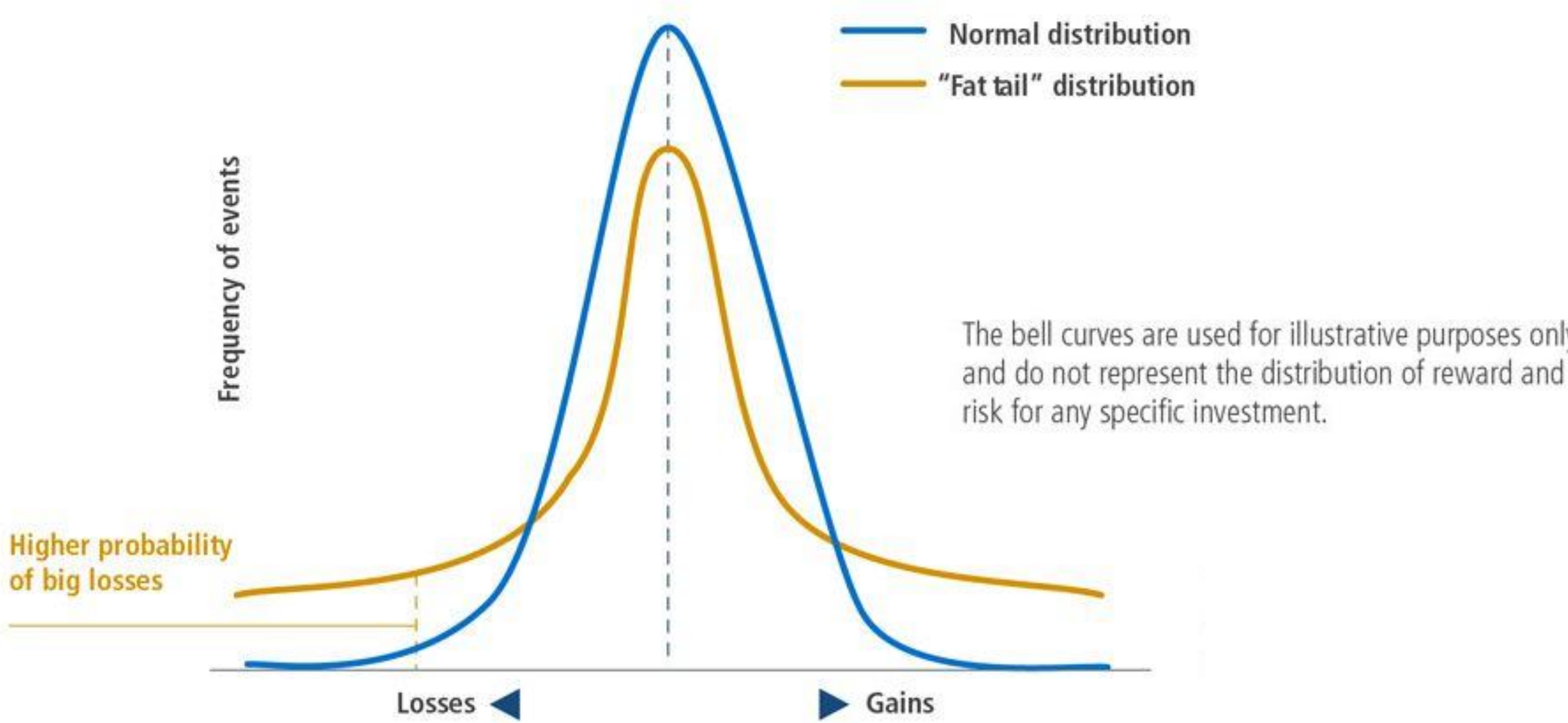


Normal Distribution





Normal Distribution vs. Heavy-Tailed



Heavy Tails

- Daily returns or changes are **NOT normally distributed**
  - The distribution has **heavier tails** than the normal distribution
  - It is **more peaked** than the normal distribution
- This means that **small changes and large changes are more likely** than the normal distribution would suggest
- Many market variables have this property, known as **EXCESS KURTOSIS.**

Alternatives to Normal Distributions: The Power Law

- An alternative to assuming normal distributions.
- It is approximately true that the value of the variable, *v*, has the property that when *x* is large:

$$Prob(v > x) = Kx^{-\alpha}$$

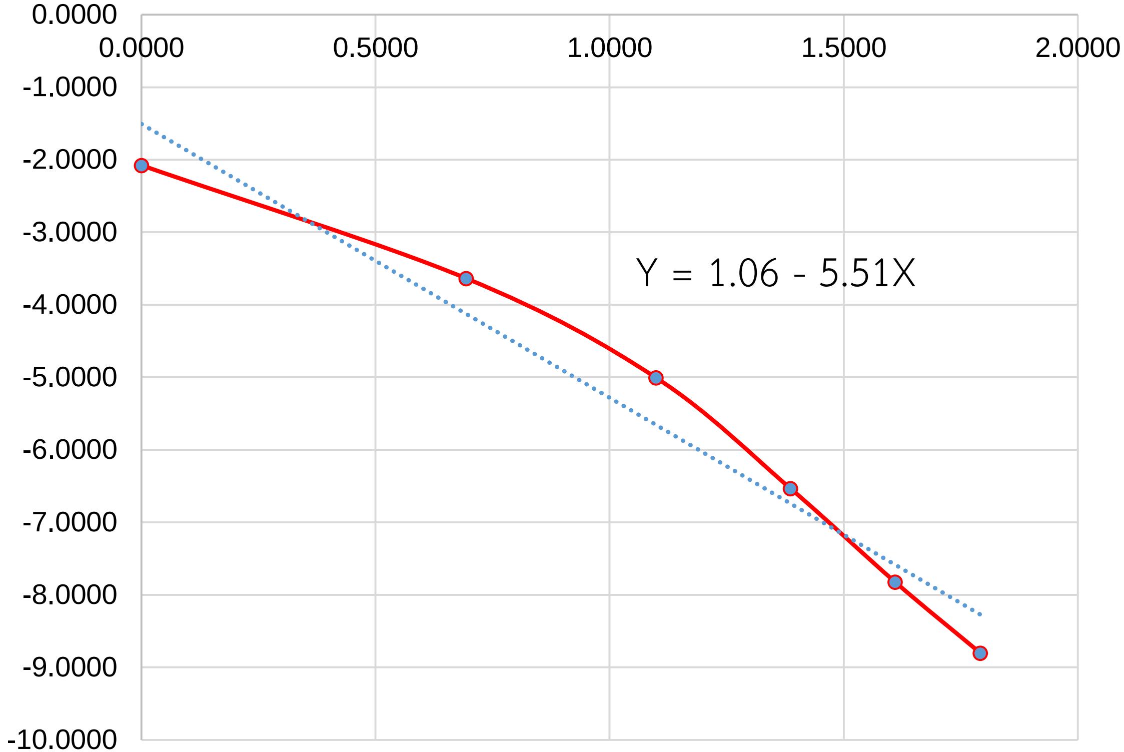
$$\ln\left[Prob(v > x)\right] = \ln K - \alpha \ln x$$

- This seems to fit the behavior of the returns on many market variables better than the normal distribution

Alternatives to Normal Distributions: The Power Law

	State	Real World (%)	Normal Model (%)	ln(x)	Prob(v>x)	ln[Prob(v>x)]
>1 SD	1	25.04	31.73	0.0000	**0.1252**	**-2.0778**
>2SD	2	5.27	4.55	0.6931	**0.02635**	**-3.6363**
>3SD	3	1.34	0.27	1.0986	**0.0067**	**-5.0056**
>4SD	4	0.29	0.01	1.3863	**0.00145**	**-6.5362**
>5SD	5	0.08	0	1.6094	**0.0004**	**-7.8240**
>6SD	6	0.03	0	1.7918	**0.00015**	**-8.8049**

Alternatives to Normal Distributions: The Power Law



What are K and alpha?

Risk Management:

How does volatility change over time?

Standard Approach to Estimating Volatility

- Define σ_n as the volatility per day (on day n) between day n-1 and day n, as estimated at end of day n-1
- Define S_i as the value of market variable at end of day i
- Define

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

then, we have

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2$$

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m}u_{n-i}$$

Standard Approach to Estimating Volatility

- Simplifications Usually Made in Risk Management
  - Define $$u_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$
  - Assume that the mean value of u_i is zero
  - Replace m-1 by m
- This gives:

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^{m} u_{n-i}^2$$

Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \qquad \text{where} \qquad \sum_{i=1}^{m} \alpha_i = 1$$

In an ARCH(m) model we also **assign some weight to the long-run variance rate**, V_L:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \qquad \text{where} \qquad \gamma + \sum_{i=1}^{m} \alpha_i = 1$$

Generalized Autoregressive Conditional Heteroskedasticity: GARCH (p,q)

Note for general form of the Generalized Autoregressive Conditional Heteroskedasticity:

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

GARCH (1,1)

- In GARCH (1,1) we assign some **weight to the long-run average variance rate**

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

Since weights must sum to 1: $\gamma + \alpha + \beta = 1$

- Setting $\omega = \gamma V_L$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and $V_L = \dfrac{\omega}{1-\alpha-\beta}$

GARCH (1,1)

Example: Suppose

$$\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

What are the long-run variance rate and the long-run volatility per day?

Suppose that the estimate of the volatility on day is 1.6% per day and that on day n-1 the market variable decreases by 1%, what is the new estimated volatility?

Forecasting Future Volatility

A few lines of algebra shows that:

$$
\begin{aligned}
\sigma_n^2 &= (1-\alpha-\beta)V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2 \\
\sigma_n^2 - V_L &= \alpha(u_{n-1}^2 - V_L) + \beta(\sigma_{n-1}^2 - V_L) \\
\sigma_{n+t}^2 - V_L &= \alpha(u_{n+t-1}^2 - V_L) + \beta(\sigma_{n+t-1}^2 - V_L) \\
E\left[\sigma_{n+t}^2 - V_L\right] &= (\alpha+\beta)E\left[\sigma_{n+t-1}^2 - V_L\right] \\
E\left[\sigma_{n+t}^2 - V_L\right] &= (\alpha+\beta)E\left[\sigma_{n+t-1}^2 - V_L\right]
\end{aligned}
$$

To estimate the volatility for an option lasting T days we must integrate this from 0 to T

Volatility Term Structures

Assume that

$$V(t) = E[\sigma_{n+t}^2] \qquad \text{and} \qquad a = \ln \frac{1}{\alpha + \beta}$$

Thus $E[\sigma_{n+t}^2] = V_L + (\alpha + \beta)^t (\sigma_n^2 - V_L)$ becomes:

$$V(t) = V_L + e^{-at}\left[V(0) - V_L\right]$$

The average variance rate per day between today and time *T* is:

$$\frac{1}{T}\int_0^T V(t)dt = V_L + \frac{1 - e^{-aT}}{aT}\left[V(0) - V_L\right]$$

Assuming 252 business days per year, the variance rate per day is?

Forecasting Future Volatility

The volatility per year lasting T days is

$$\sigma(T) = \sqrt{252\left\{V_L + \frac{1-e^{-aT}}{aT}\left[V(0) - V_L\right]\right\}}$$

where $$a = \ln\frac{1}{\alpha+\beta}$$

Forecasting Future Volatility

- Suppose that GARCH(1,1) parameters have been estimated as w= 0.000003, alpha=0.04, and beta=0.94. The current daily volatility is estimated to be 1%. Estimate **the daily volatility** in 30 days.
- Suppose that GARCH(1,1) parameters have been estimated as w= 0.000002, alpha=0.04, and beta=0.94. The current daily volatility is estimated to be 1.3%. Estimate **the volatility per annum** that should be used to price a 20-day option.

Question?