CHAPTER 8

Optimization without Constraint: More-Than-One Independent Variable Cases

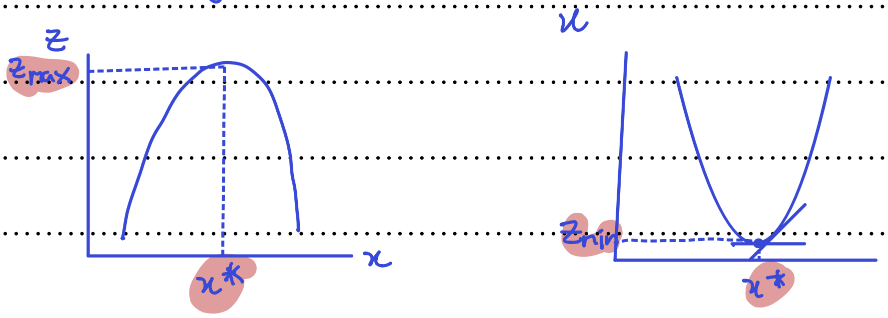
- Conditions for maximum or minimum
- Third degree price discrimination → multi market
- Competitive Firm Input Choices: Cobb-Douglas Technology → multi factor of production
- Multiproduct-firm → competitive mkt → monopoly
- Multiplant-firm

Optimization Condition

One choice variable

The objective function: $z = f(x)$

$\max_{x} \quad z = f(x)$ $\qquad\qquad$ $\min_{x} \quad z = f(x)$



Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	Derivative $\frac{dz}{dx} = f'(x) = 0$ slope must be zero Total differential $dz = \underbrace{f'(x)}_{=0}\underbrace{dx}_{>0} = 0$ z doesn't change for extremely small increase (or decrease) in x	Derivative $f'(x) = 0$ Total differential $dz = f'(x)\,dx = 0$
Second-order sufficient (SOSC, SOC)	Derivative $f''(x) < 0$ slope is decreasing $(0 \rightarrow -)$ Total differential $d^2z < 0$	Derivative $f''(x) > 0$ slope is increasing $(0 \rightarrow +)$ Total differential $d^2z > 0$

$$\begin{aligned} d^2z &= d(dz) \\ &= d(f'(x)\overbrace{dx}^{\text{constant}}) \\ &= dx\, d(f'(x)) \\ &= dx(f''(x)\,dx) \\ &= f''(x)(dx)^2 \end{aligned}$$

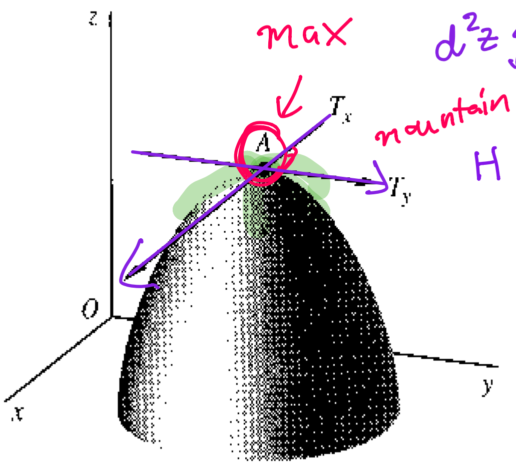
$$\boxed{d^2z = f''(x)\underbrace{dx^2}_{>0}}$$

2nd order total differential of z

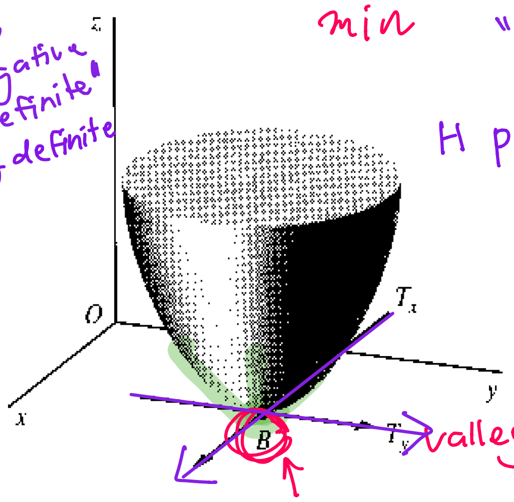
Extreme Value of a Function of Two Variables

$$z = f(x, y)$$

max $d^2z < 0$ "negative definite" mountain H neg definite



min $d^2z > 0$ "positive definite" H positive definite



First-Order Necessary Condition $z = f(x, y)$

$dz = 0$ for every dx and dy

$dz = 0$

$dz = f_x \, dx + f_y \, dy = 0$; $dx, dy \neq 0$

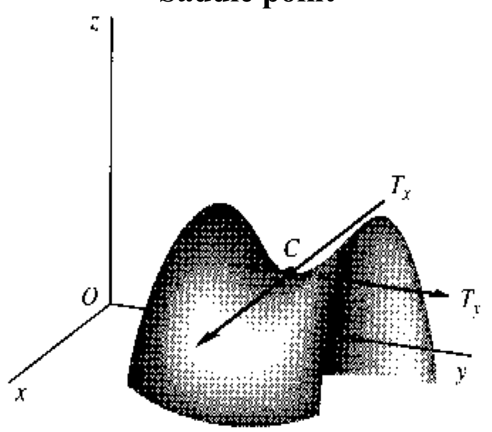
($dx \neq 0$, $dy \neq 0$)

$f_x = 0$, $f_y = 0$

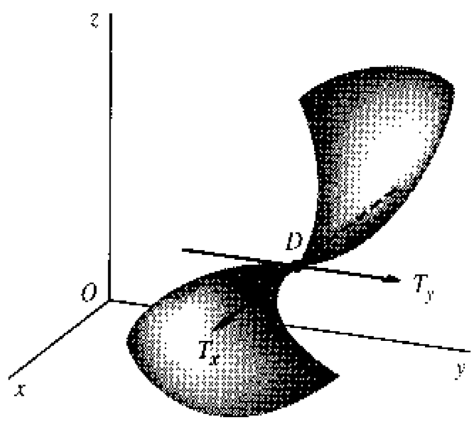
the slope of the tangent line T_x, T_y are zero

The first order necessary condition is necessary, but not sufficient to establish a maximum or a minimum.

Saddle point



Inflection point



Second order sufficient condition

The satisfaction of the first-order condition earmarks certain values of z as the stationary values of the objective function. If at a stationary value of z we find that d^2z is positive definite, i.e., $d^2z > 0$, this will suffice to establish that value of z as a minimum. Analogously, the negative definiteness of d^2z, $d^2z < 0$, is a sufficient condition for the stationary value to be a maximum.

$[f_{ij}]_{n\times n} \Rightarrow H_{n\times n}$

$y = f(x_1, x_2, \ldots, x_n)$

$H_{n\times n}$

Second-order total differential, Hessian Matrix and Definiteness of Hessian Matrix

$A_{n\times n}$

📖 **About "H_k"**

Definition : Let A be an $n \times n$ matrix. A $k \times k$ submatrix, deriving from deleting the last $n-k$ columns and the last $n-k$ rows of matrix A, can be called the k^{th} order leading principal submatrix of matrix A. The corresponding determinant of this $k \times k$ submatrix is called the k^{th} order leading principal minor of matrix A. A_k

The k^{th} order leading principal submatrix of matrix A is denoted by A_k.

The k^{th} order leading principal minor of matrix A is denoted by $|A_k|$

$$A = \begin{array}{ccc} A_1 & A_2 & A_3 \\ \end{array}$$
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

Leading principal submatrices are:

The 1^{st} order leading principal submatrix $\Rightarrow$ delete the last $3-1$ rows & columns

$$A_1 = [a_{11}]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

2^{nd}

$\Rightarrow$ $3-2$

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

3^{rd}

$\Rightarrow$ $3-3$

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

Leading principal minors are:

$|A_1| = \underset{\text{1st order}}{|a_{11}|}$

2nd order

$|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

3rd order

$|A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

For Hessian Matrix of function $y = f(x_1, x_2, \dots, x_n)$

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{bmatrix}_{n \times n}$$

$\underline{H_1} = [f_{11}]$

$\underline{H_2} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ ... delete the last n-2 rows & column

$\underline{H_3} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$ ... " " n-3 " "

.... $H_n = H$

Leading principal minors are:

this is not the absolute sign

$\underline{|H_1|} = |f_{11}|$

$\underline{|H_2|} = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$

$\underline{|H_3|} = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$ ..... $|H_n| = |H|$

🕮 About "Definiteness of *H*"

Let ***H*** be a symmetric matrix with dimension $n \times n$

$d^2z > 0$ (a) ***H*** is **positive definite** if all n leading principal minors are positive

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, \ldots$$ all positive

$d^2z < 0$ (b) ***H*** **is negative definite if** n leading principal minor duly alternate in sign with the first one being negative.

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, \ldots >, <, >, <$$

starting with negative, alternate in sign

If (a) or (b) is not met, H is indefinite.

Example: Find definiteness of matrix $B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}_{2\times 2}$

$B_1 = [-3]$, $|B_1| = -3 < 0$

$B_2 = B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$, $|B_2| = 18 - 16 = 2 > 0$

B is neg. definite

Homework: Find definiteness of the following symmetric matrices ต่อไปนี้

(a) $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

Second-Order Sufficient Condition (SOSC, SOC)

For the maximum of objective function $z = f(x, y)$

the slope is decreasing … $d^2z < 0$, d^2z neg def.

$\Rightarrow$ H is neg def.

$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$|H_1| = f_{xx} < 0$

$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$

For the minimum of objective function $z = f(x, y)$

the slope is increasing … $d^2z > 0$, d^2z positive def.

$\Rightarrow$ H is positive def.

$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$|H_1| = f_{xx} > 0$

$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$

Example: Find the maximum/minimum of function $z = x + 2ey - e^x - e^{2y}$

FOC: $\frac{\partial z}{\partial x} = 1 - e^x = 0 \Rightarrow x^* = 0$

$\frac{\partial z}{\partial y} = 2e - 2e^{2y} = 0 \Rightarrow e = e^{2y}$

$2y = 1$

note: $\frac{d}{dx} e^{kx} = ke^{kx}$ $\qquad y^* = \frac{1}{2}$

SOC: $f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial (1-e^x)}{\partial x} = -e^x$

$f_{xy} = f_{yx} = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}(1-e^x) = 0$

$f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y}(2e - 2e^{2y}) = -4e^{2y}$

$H = \begin{bmatrix} -e^x & 0 \\ 0 & -4e^{2y} \end{bmatrix}$ (H_1, H_2)

$|H_1| = |-e^x| = -e^x < 0$

$|H_2| = 4e^{x+2y} = 4e^{0+2\cdot\frac{1}{2}} = 4e > 0$

H is neg def. $\therefore (x^*, y^*)$ gives z_{max}

$z_{max} = 0 + e - 1 - e = -1$ #

Objective functions with more three choice variables

$$z = f(x_1, x_2, x_3)$$

F.O.N.C :

$dz = 0$

$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0;\ dx_1, dx_2, dx_3 > 0$

$\boxed{f_1 = f_2 = f_3 = 0}$ $\quad f_1 = f_2 = f_3 = \cdots = f_n = 0$

We can use $f_1 = f_2 = f_3 = 0$ to find the stationary values/critical values

S.O.S.C

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}_{3\times 3}$$

All leading principal minor

$|H_1| = |f_{11}|$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

z is at the $\begin{bmatrix} \text{maximum} \\ \text{minimum} \end{bmatrix}$ if $\begin{cases} d^2z < 0 \\ d^2z > 0 \end{cases}$ if $\begin{cases} |H_1| < 0, |H_2| > 0, |H_3| < 0 \\ |H_1| > 0, |H_2| > 0, |H_3| > 0 \end{cases}$ if $\begin{cases} H \text{ is negative definite} \\ H \text{ is positive definite} \end{cases}$

Objective functions with more n −choice variables

$$z = f(x_1, x_2, x_3, \dots, x_n)$$

F.O.N.C :

$$dz = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n = 0$$

$$\frac{\partial^2 f_i}{\partial x_i} = 0, for\ all\ i, i = 1,2, \dots n$$

S.O.S.C:

$$\therefore\ |H| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

All principal minors $|H_1|$, $|H_2|$,… $|H_n|$

If $z\ is\ at$ is the maximum when $d^2z < 0$, H is negative definite:

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, |H_4| > 0, |H_5| < 0, \dots$$

If $z\ is\ at$ is the minimum when $d^2z > 0$, H is positive definite:

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, |H_4| > 0, |H_5| > 0, \dots$$

Homework:Find extreme values of the following functions and check for the sufficient condition

$z = 8x^3 + 2xy - 3x^2 + y^2 + 1$

$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$

$z = -3x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$

$z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$

$z = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$

$z = e^x + e^y + e^{w^2} - 2e^w - (x + y)$

$z = x^4 + x^2 - 6xy + 3y^2$

$w = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

$w = (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$

Summary of optimal conditions

The objective function: $z = f(x, y)$

Condition	**Maximum**	**Minimum**
First-order necessary (FONC, FOC)	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient (SOSC, SOC)	$\lvert H_1 \rvert = f_{xx} < 0$ $\lvert H_2 \rvert = f_{xx}f_{yy} - f_{xy}^2 > 0$	$\lvert H_1 \rvert = f_{xx} > 0$ $\lvert H_2 \rvert = f_{xx}f_{yy} - f_{xy}^2 > 0$

H =

The objective function: $z = f(x_1, x_2, \dots, x_n)$

Condition	**Maximum**	**Minimum**
First-order necessary (FONC, FOC)	$f_1 = f_2 = \cdots = f_n = 0$	$f_1 = f_2 = \cdots = f_n = 0$
Second-order sufficient (SOSC, SOC)	$d^2z < 0$, neg def. H is negative definite. $(-1)^k \lvert H_k \rvert > 0$ $k = 1,2, \dots, n$	$d^2z > 0$, positive def. H is positive definite. $\lvert H_k \rvert > 0$ $k = 1,2, \dots, n$

$(-1)^1 \lvert H_1 \rvert > 0$

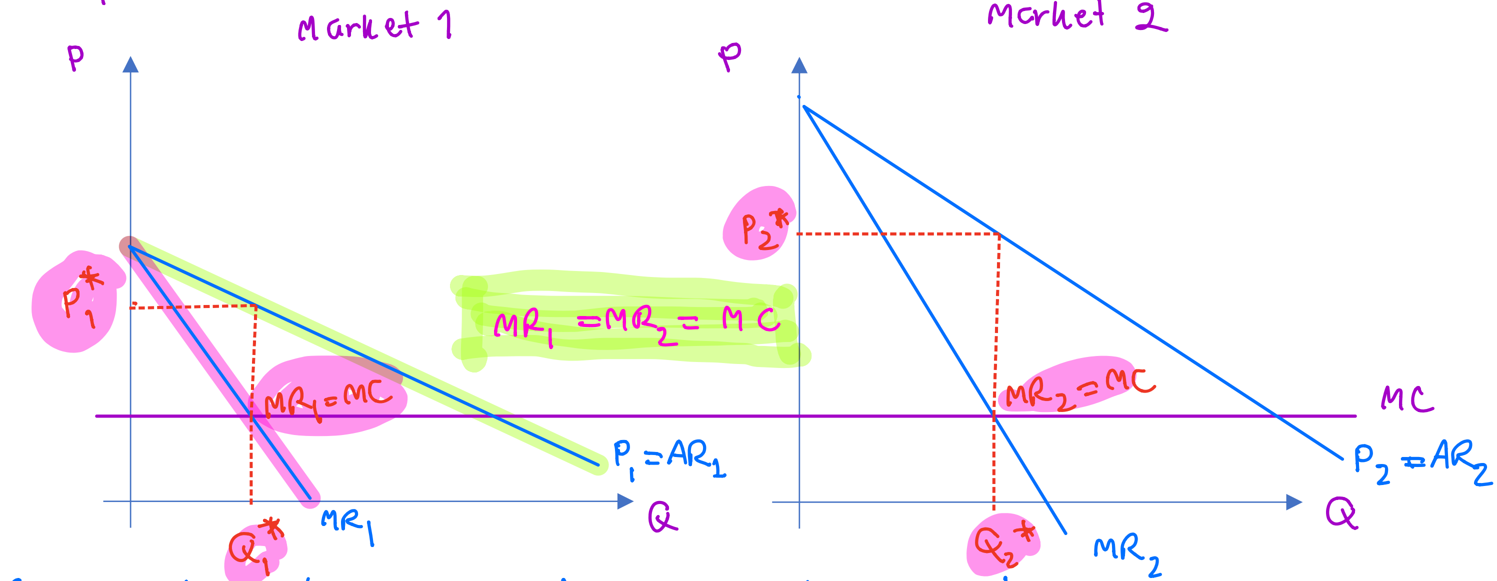
$(-1)^2 \lvert H_2 \rvert > 0$

Third- Degree Price Discrimination

Monopolist

Third-Degree Price Discrimination involves charging different prices to different segments of the market for the same good. This is to maximize profit within each segment of the market. For example, a theater may divide cinema goers into seniors, adults, and children, each paying a different price when seeing the same movie.

Example : assume MC = constant



Suppose that there are three segmented markets

Total Revenue of a company is:

$$TR = R_1(Q_1) + R_2(Q_2) + R_3(Q_3),$$

R_i is a total revenue function of market i. segment market according to the nature of consumer's demand in each market

Total cost of the company is:

$$C = C(Q)$$

total quantity produced

$$Q = Q_1 + Q_2 + Q_3$$

Q_i is quantity of product sold in market i.

Find the quantity and price in each market i that maximizes firm's profit.

Step 1 State the objective Function: Profit function

$\pi = TR - TC$

$\max_{Q_1, Q_2, Q_3} \pi = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q)$; $Q = Q_1 + Q_2 + Q_3$

$$\max_{Q_1,Q_2,Q_3} \pi = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q) \quad ; \quad Q = Q_1+Q_2+Q_3$$

Step 2 Find FONC and use FONC to find critical values

$$\frac{\partial(Q_1+Q_2+Q_3)}{\partial Q_1} = 1$$

FOC

$$\frac{\partial \pi}{\partial Q_1} = \pi_1 = R_1'(Q_1) - C'(Q)\frac{\partial Q}{\partial Q_1} = 0 \quad (1)$$

$$\frac{\partial \pi}{\partial Q_2} = \pi_2 = R_2'(Q_2) - C'(Q)\frac{\partial Q}{\partial Q_2} = 0 \quad (2)$$

$$\frac{\partial \pi}{\partial Q_3} = \pi_3 = R_3'(Q_3) - C'(Q)\frac{\partial Q}{\partial Q_3} = 0 \quad (3)$$

use these to derive Q_1^*, Q_2^*, Q_3^* → P_1^*, P_2^*, P_3^*

(to get P^* we substitute Q^* into each segmented mkt. demand)

$$C'(Q) = R_1'(Q_1) = R_2'(Q_2) = R_3'(Q_3)$$

$$\boxed{MC = MR_1 = MR_2 = MR_3}$$

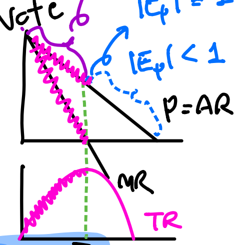
From $R_i = P_i Q_i = P_i(Q_i)\, Q_i$ ← inverse mkt. demand of market i

$$MR_i = \frac{dR_i}{dQ_i} = P_i(Q_i) + Q_i \frac{dP_i(Q_i)}{dQ_i}$$

$$\vdots$$

$$MR_i = P_i\left[1 + \frac{1}{E_{p_i}}\right]$$

Note



$$\boxed{MC = P_1\left[1 + \frac{1}{E_p^1}\right] = P_2\left[1 + \frac{1}{E_p^2}\right] = P_3\left[1 + \frac{1}{E_p^3}\right]}$$

more   less   less   more

Mkt 1 is less price elastic than Mkt 2

Conclusion: In the market with lower price elasticity of demand (lower $|\varepsilon_{di}|$), the maximizing-profit price in that market will be.........more

$|E_p^1| < |E_p^2|$

$|-1.5| < |-2|$

Step 3 Check SOSC whether $Q_1^*, P_1^*, Q_2^*, P_2^*, Q_3^*, P_3^*$ indeed give the maximum profit

$$\frac{1}{E_p^1} < \frac{1}{E_p^2}$$

$$-\frac{1}{1.5} < -\frac{1}{2}$$

$$-0.67 < -0.5$$

$$1 + \frac{1}{E_p^1} < 1 + \frac{1}{E_p^2}$$

$$1 - 0.67 < 1 - 0.5$$

Hessian Matrix:

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}$$

$P_1 > P_2$

$\pi_1 = MR_1 - MC$, $\pi_{11} = MR_1' - MC'\frac{\partial Q}{\partial Q_1}$ (=1)

$\pi_2 = MR_2 - MC$, $\pi_{12} = \frac{\partial \pi_2}{\partial Q_1} = \frac{\partial(MR_2 - MC)}{\partial Q_1} = -\frac{dMC}{dQ}\frac{\partial Q}{\partial Q_1}$ (=1)

$\pi_3 = MR_3 - MC$ $= -MC'$

H_1 H_2 $H_3 = H$

$$H = \begin{bmatrix} MR_1' - MC' & -MC' & -MC' \\ -MC' & MR_2' - MC' & -MC' \\ -MC' & -MC' & MR_3' - MC' \end{bmatrix}_{3\times 3}$$

Check all leading principal minors

H is neg def.

$|H_1| = |\pi_{11}| = MR_1' - MC' < 0$

$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = (MR_1' - MC')(MR_2' - MC') - (MC')^2 > 0$

$|H_3| = \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{vmatrix} = \ldots\ldots < 0$

$\rightarrow$ at $P_1^*, Q_1^*, P_2^*, Q_2^*, P_3^*, Q_3^*$, π is maximized.

Example: Suppose a monopolist firm sells its product in three markets with corresponding market demand:

Domestic market:	$P_1 = 63 - 4Q_1$	price $= f(Q \text{ demanded})$
Asia market:	$P_2 = 105 - 5Q_2$	
Europe market:	$P_3 = 75 - 6Q_3$	

Total cost function of this firm is:

$$C = 20 + 15Q;$$
$$Q = Q_1 + Q_2 + Q_3$$

Find total quantity produced by the monopolist and price and quantity in each market that maximize profit.

<u>**Step 1**</u> State the objective Function: Profit function

$\pi = TR - TC$

$= TR_1 + TR_2 + TR_3 - TC$

$= P_1Q_1 + P_2Q_2 + P_3Q_3 - TC$

$\max\limits_{Q_1, Q_2, Q_3} \pi = (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - 6Q_3)Q_3 - (20 + 15Q)$; $Q = Q_1 + Q_2 + Q_3$

$\max\limits_{Q_1, Q_2, Q_3} \pi = (63Q_1 - 4Q_1^2) + (105Q_2 - 5Q_2^2) + (75Q_3 - 6Q_3^2) - 20 - 15Q_1 - 15Q_2 - 15Q_3$

Step 2 Find FONC and use FONC to find critical values

FOC : $\frac{\partial \pi}{\partial Q_1} = 63 - 8Q_1 - 15 \overset{FOC}{=} 0 \quad (1.) \Rightarrow Q_1^* = 6$

$\frac{\partial \pi}{\partial Q_2} = 105 - 10Q_2 - 15 \overset{FOC}{=} 0 \quad (2.) \Rightarrow Q_2^* = 9$

$\frac{\partial \pi}{\partial Q_3} = 75 - 12Q_3 - 15 \overset{FOC}{=} 0 \quad (3.) = Q_3^* = 5$

$Q = 6+9+5 = 20$

$P_1^* = 63 - 4(6) = 39$

$P_2^* = 105 - 5(9) = 60$

$P_3^* = 75 - 6(5) = 45$

Step 3: SOSC

$$H = \begin{bmatrix} -8 & 0 & c \\ 0 & -10 & 0 \\ 0 & 0 & -12 \end{bmatrix}$$

Check all leading principal minors

$|H_1| =$ $|-8| = -8$ < 0

$|H_2| =$ 80 > 0

$|H_3| =$ $-8 \times -10 \times -12 = -960$ < 0

$\therefore H$ is neg def., Q_1^*, Q_2^*, Q_3^* maximize π

P_1^*, P_2^*, P_3^* #

Competitive Firm Input Choices: Cobb-Douglas Technology

Let a firm in competitive market have Cobb-Douglas Production Function:

$$Q = L^{\alpha} K^{\beta}$$

Let wage be w, rent be r baht per unit.

Characteristics of the production function:

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<u>**Step 1**</u> State the objective Function: Profit function

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<u>**Step 2**</u> Find FONC and use FONC to find critical values

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The Economic interpretation

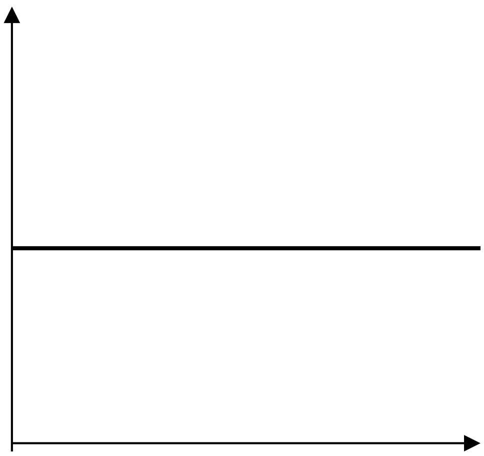
<u>Step 3</u>: SOSC

$$H = \begin{bmatrix} \quad \end{bmatrix}$$

Check all leading principal minors

$|H_1| =$

$|H_2| =$



Multiproduct Firm

Multiproduct Firm in competitive market

Let a firm produce and sell two products. Each product is sold in competitive market, hence the firm is price taker. Total Revenue is:

$TR =$

P_1 is market price of good 1.
P_2 is market price of good 2.
Q_1 is quantity of good 1 sold in the market.
Q_2 is quantity of good 2 sold in the market.

Total cost of this firm is:

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of π, $d^2\pi$, is ....definite. Hence, the profit is………………..

Multiproduct Firm in monopoly market

Let market demand of good 1 is $Q_1 = 40 - 2P_1 + P_2$
Let market demand of good 2 is $Q_2 = 15 + P_1 - P_2$

From market demand, good 1 and good 2 are ....

The market demands are given as functions of P_1, P_2. Since we would like to choose Q_1, Q_2, we need to convert the above market demands to be in the form of $P_1 = f(Q_1, Q_2)$, and $P_2 = g(Q_1, Q_2)$.

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Then, we can use the inverse market demand functions to get total revenue function:

$TR =$……………………………………………………………………………………………

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Let the cost function of the monopolist be

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

Profit Function:

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of π, $d^2\pi$, is ....definite. Hence, the profit is………………..

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Multiplant Firm Problem

Consider a firm with many plants. Each plant i has total cost TC_i.

$$TC_i = C_i(q_i),\ i = 1,2,\ldots,n$$

q_i is the level of output produced by factory i.

Assume that this firm sells its product in one market. Firm's total revenue function is:

$TR =$………………………………………………………………………………………

$Q =$…………………………………

If output market is competitive market,

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If output market is monopoly market,

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: …………………………………………………

NOTE: Third-degree price discrimination vs. Multiplant firm

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First Order Necessary Condition:

π_i = ………………………………………………………………for all $i = 1,2,\ldots,n$

SOSC:

$$H = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$|H_1| = R'' - C_1'' < 0$

$|H_2| =$

Economic interpretation:

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Example: Consider a monopolist firm with two plants:

Factory 1: $C_1(Q_1) = 10Q_1^2$

Factory 1: $C_2(Q_2) = 20Q_2^2$

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Firm's market demand is $P = 700 - 5Q$, $Q = Q_1 + Q_2$

Draw MC_1, MC_2, MC_T, AR, MR and indicate the maximizing level of output from each plant and the profit-maximizing price.

Find Q_1, Q_2, Q_{Total}, P that maximize profit of this firm.

Profit Function:

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of π, $d^2\pi$, is ....definite. Hence, the profit is………………..

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Summary:

3rd price discrimination:

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Choosing levels of factor of production:

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Multiproduct:

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Multiplant:

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