EE422
Mathematical Economics 2 (1/2013)

Optimal Control theory
(The Maximum Principle)

Read chapter 9

Introduction

- Our dynamic problem is to allocate resources among competing ends over an interval of time (planning period) from initial time to terminal time.
- Equivalently, we choose time paths of choice variables to maximize some objective function subject to some constraints.
- Choices: time, control and state variables which are functions of time.
- The control variables affect movements of the state variables via some differential equation called “equations of motions (or transition equation)”.
- Constraints also specify initial and terminal conditions for state variables.

Introduction

- Ex. Consumption over time or life-cycle theory:
- Objective: maximized life-time utility from consumption
- Three Choice variables for $t \in [0,T]$ are
- (a) consumption flow is a control variable;
- (b) stock of assets is a state variable;
- (c) time variable
- Constraints: (a) Initial condition requires the state variable begins at a given value. (b) Terminal condition. (c) must obey the state equation.

$$Ex1. \quad \max \int_0^T [u(c(t)e^{-\rho t}] dt \qquad [\text{Objective}]$$

$$st. \qquad \dot{a}(t) = ra(t) - c(t), \qquad [\text{State eq.}]$$

$$a(0) = a_0, \qquad [\text{Initial cond.}]$$

$$a(T) = 0. \qquad [\text{Terminal cond.}]$$

$Control : c\,(consumption);\ \ State : a\,(assets). Time\ enters\ via\ the\ discount\ factor (not\ appear\ in\ u, but\ outside).$

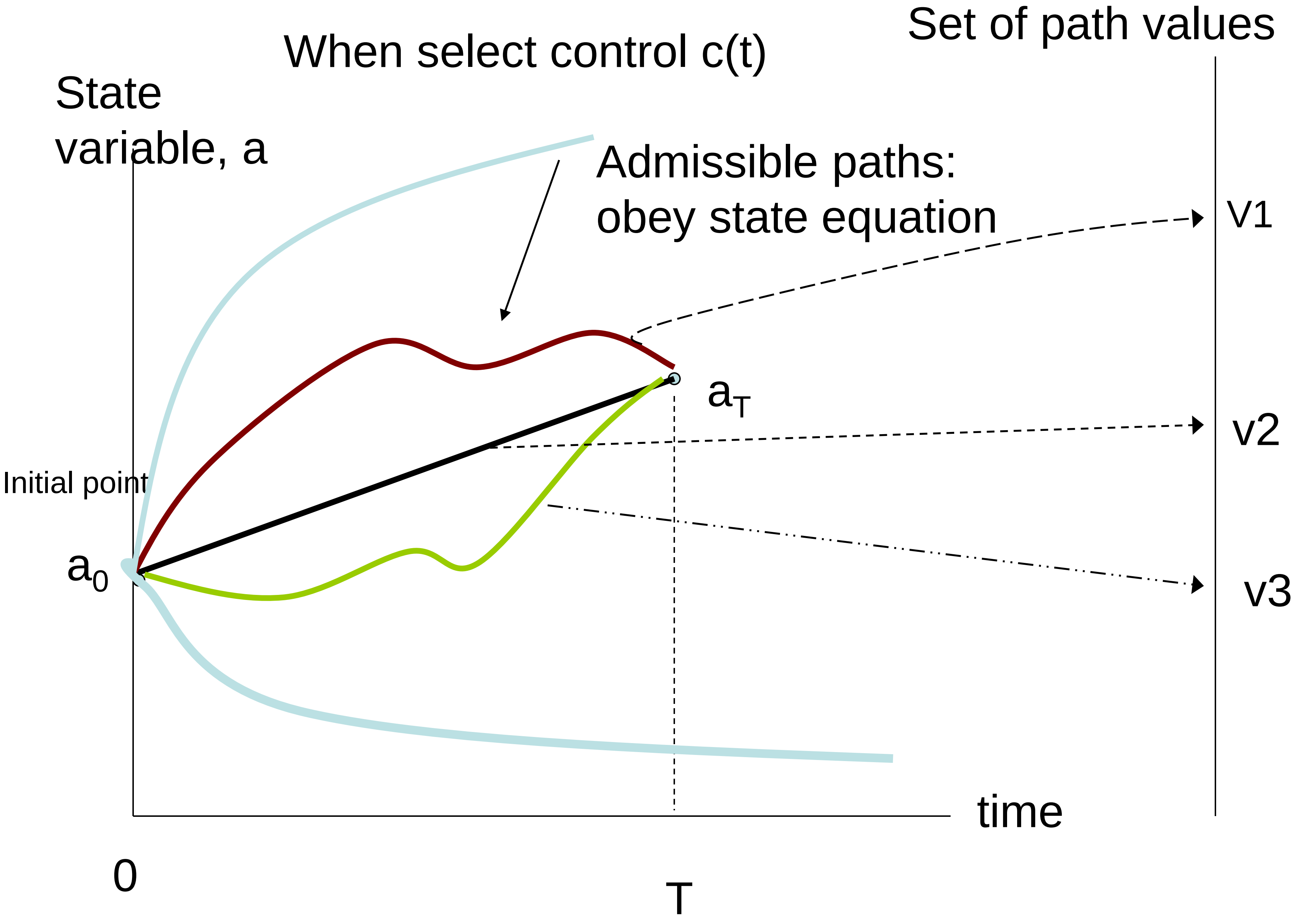
$e^{-\rho t}$ $is\ a\ discount\ factor\ where\ \rho\ is\ a\ subjective\ discount\ rate.$

$$\cong \left(\frac{1}{1+\rho}\right)^t \ or\ \beta^t \ in\ discrete\ time.$$

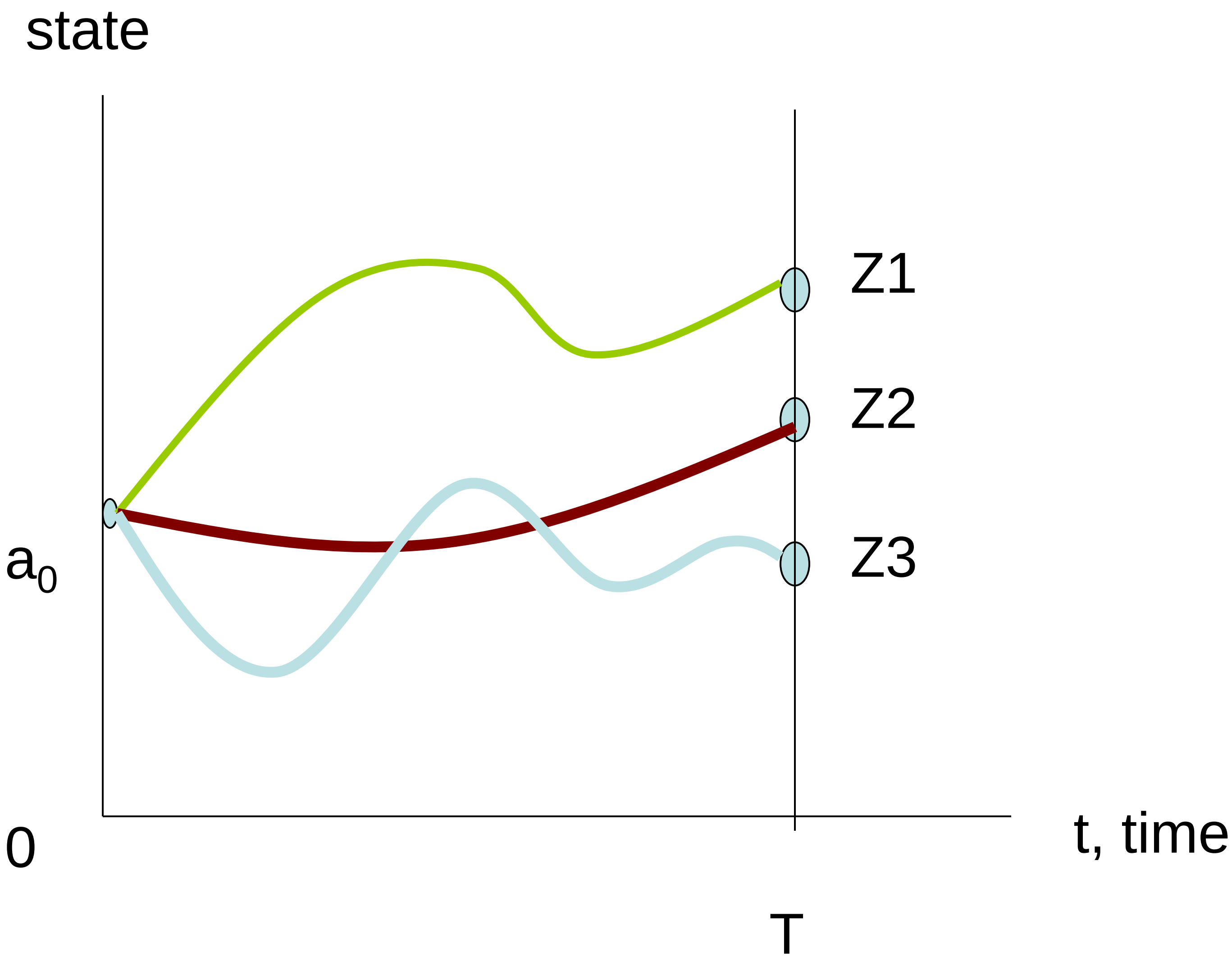
$\dot{a}(t) \equiv da \,/\, dt, change\ in\ a\ at\ time\ t. [differential\ eq.]$

$If\ we\ replace\ \dot{a}(t)\ with\ a_{t+1} - a_t, we\ get\ a_{t+1} = (1+r)a_t - c_t.$

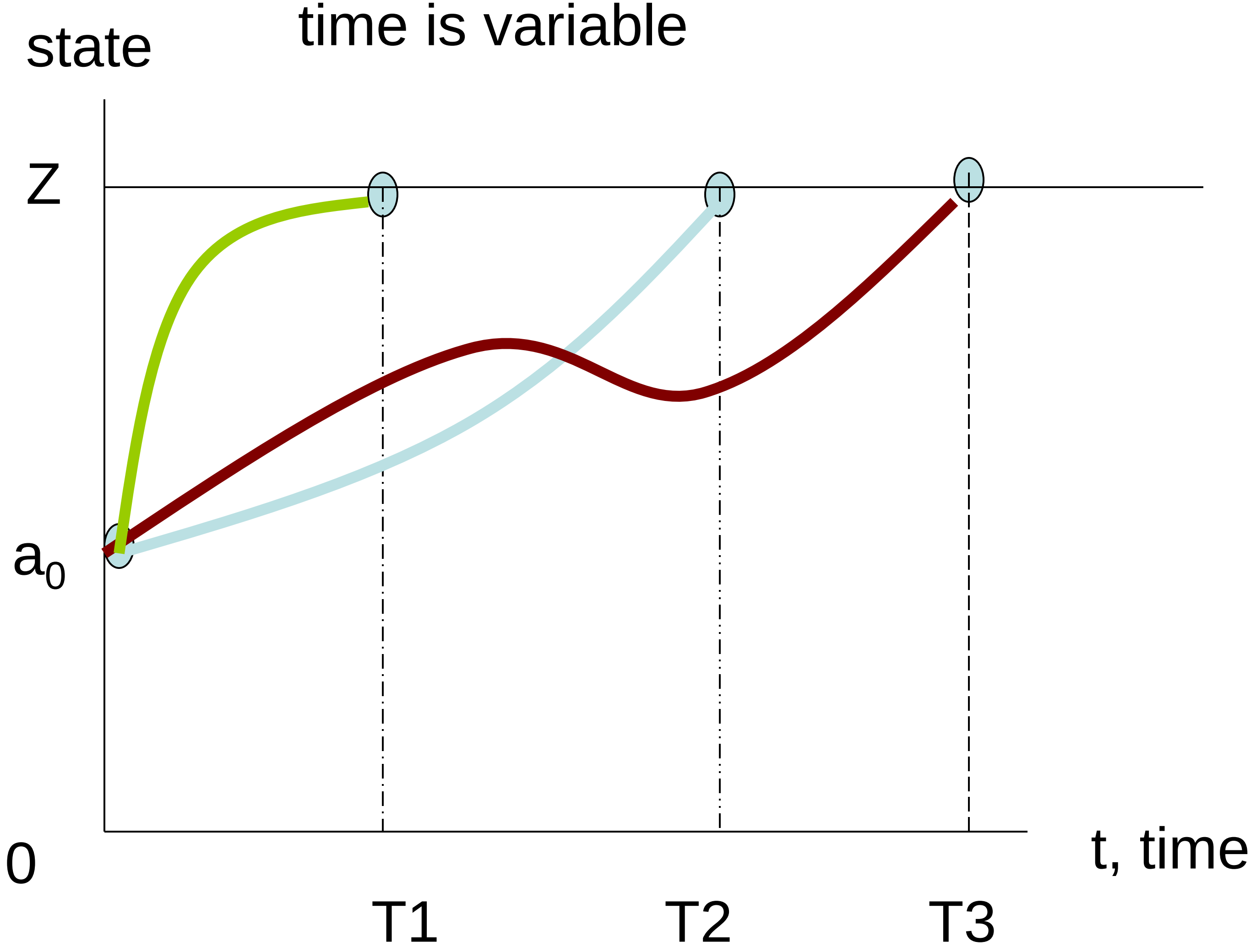
When select control c(t)



When terminal state is free



when terminal state is fixed but terminal time is variable



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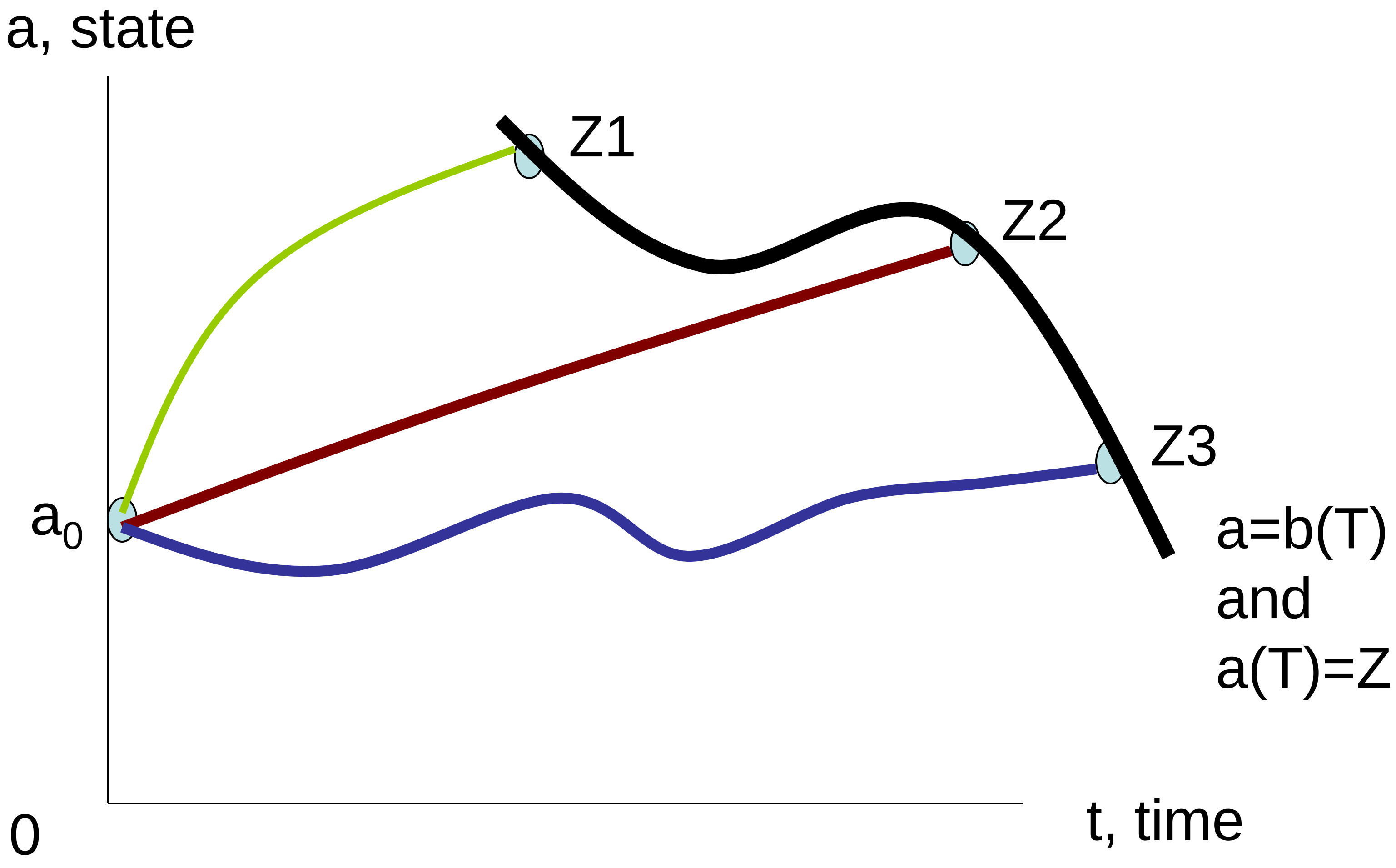
When both T and state are tied together by a=b(T)



Introduction

- In the last picture, to select the optimal path from other admissible paths, we need to condition how the optimal path crosses the terminal line or terminal surface.
- We also call this “GO across condition” as “Transversality condition”.
- For example, $a(T)e^{-rT} \geq 0$. You cannot die with debt. For e^{-rT} is positive, your assets at T, a(T), is nonnegative.

Introduction

- Popular technique to solve dynamic optimization problem with a continuous time is Optimal control.
- Optimal control theory or Maximum principle is developed by a group of Russian mathematicians. (See Pontryagin et al.)
- Maximum principle gives necessary conditions for optimality in a wide range of dynamic optimization problems.
- Can handle corner solutions, suggest the nature of solution (compared to DP), handle more general constraints (compared to calculus of variation).

Discussion topics

- 1. A Simple Case (The Basic problem)
- 2. Standard Problem and Maximum Principle
- 3. Economic Interpretation of the Maximum principle
- 4. Current Value Formulations and Infinite Horizon.

1. A Simple Case

- Consider a basic system with one state and one control
- Control variable $u(t)$: flow variable
- State variable $x(t)$: stock variable, assumed to be a continuous function of time,
- Assume that at initial point, the value of state is known or predetermined: $x(0)= x_0$.

1. The Basic Problem

- Rate of change of *x(t)* is described by a differential equation

$$\dot{x}(t) = g(t, x(t), u(t))$$

- This diff eq. relates the change in state variable to control variable, state variable and time.
- This equation is called *Equation of Motion* or *Transition equation* or *state equation*.
- By choosing u(t) and substituting into the state equation, we know how *x(t)* will evolve.
- Ex. $\dot{x}(t) = x(t) + u(t) \; or \; \dot{x}(t) = u(t)$.
- Different *u(t)* gives us different paths of the system.

1. The Basic Problem

- Suppose we can measure benefits associated to each path of the system as

$$J = \int_0^T f(t, x(t), u(t))\, dt$$

- J is the objective function. Assuming the integration or summation is converged.
- f is the immediate function that contain u as an argument.
- Our problem is to maximize J by choosing among all admissible pairs $(x(t), u(t))$ that obey the differential equation with $x(0) = x_0$ and the constraint imposed on $x(T)$.

1. Basic Problem

$$
\begin{aligned}
\max \; & \int_0^T f(t, x(t), u(t))\, dt, \; u(t) \in (-\infty, \infty) \\
st. \quad & \dot{x}(t) = g(t, x(t), u(t)), \\
& x(0) = x_0, \;\; x(T) \;\, free. \qquad\qquad (2)
\end{aligned}
$$

- We search for an optimal pair $(x^*(t), u^*(t))$, that maximize the objective fn .
- Here, we maximize the objective with respect to u subject to constraints (2).

1. Basic Problem

- Notice that our constraint is a differential equation, one for each time t, thus it can be viewed as an infinite number of equality constraints.
- For each constraint, we can handle this equality constraint by associating each constraint with a Lagrange multiplier as in static optimization.
- Thus, we need lots of Lagrange multipliers.
- Let define $p=p(t)$ as the adjoint or co-state variables associated with the differential equation.
- The costate variables vary over time, and is assumed to be nonzero continuous function of time.
- These costate variables can be interpreted as shadow prices: value of additional unit of state at time t.

1. Basic Problem

- Corresponding to Lagrangian equation, we define the Hamiltonian function H for each time.
- Our problem is to find $(x,\ u,\ p)$ that maximize H for each time t in $[0,\ T]$:

$$H(t,x,u,p) = f(t,x,u) + pg(t,x,u)$$

1. Basic Problem

- A set of necessary conditions for optimality is :

$$\max_{u} H(t,x,u,p) \ \ for\ \ all\ \ t,\ \ or$$

$$\frac{\partial H}{\partial u} = 0 \quad [\text{Maximum principle}]$$

$$\frac{\partial H}{\partial x} = -\dot{p}(t) \quad [\text{costate or Euler equ}ation]$$

$$\frac{\partial H}{\partial p} = \dot{x}(t) \qquad [\text{state equation}]$$

$$p(T) = \ 0 \qquad [\text{Transversality}]$$

1. Basic Problem

- The first Maximum principle condition as stated allows for corner solutions when the control region is a closed bounded set. If we have interior solution, we have usual condition $\partial H/\partial u = 0$. We choose control variable to max H at each time t without worrying about control at any other date than t.
- Notice that there is no differential equation for u in the H system.
- The second condition Costate or Euler equation deals with forward planning or intertemporal problem. Since $u(t)$ affects the future $x(t)$, we need to plan ahead. All future planning can be summarized by the shadow price. Thus, the shadow price must follow some appropriate path.

1. Basic Problem

- The third condition is from the problem restated to make it similar to equation for the costate. (not part of necessary condition)
- Two differential equations for x and p are called as the Hamiltonian system.
- The last condition concerns what should happen at the terminal time. This condition is called a "Transversality condition". It requires $p(T)=0$ since $x(T)$ is free (means $x(T)$ needs not to be binding or can hold with inequality).
- When constraint is binding, the associated shadow price must be positive, reflecting scarcity of resource.
- Initial and terminal conditions for x and transversality condition for p will help to pin down some constant terms when solving differential equation for x and p.

1. Basic Problem

- These Maximum Principle conditions are necessary but not sufficient for optimality.
- For sufficient conditions, we need H or both f and g to be concave in x and u. Ex. f concave and g linear. Linear fn. is a concave function. Sums of concave functions is concave.

1. Basic Problem

$$Ex\,1.\max\int_{o}^{T}[1-tx(t)-u(t)^2]dt$$

$$st.\quad \dot{x}(t)=u(t)$$

$$x(0)=x_0$$

$$x(T)\,free,\; u\in\mathbb{R}$$

$where\,x_0\;and\,T\,are\,positive\,\mathrm{constants.}$

$Find\,u*,x\,*\,.$

$[refresh\,your\,calculus\,!]$

1. Basic Problem

$$Solution: H(t,x,u,p) = 1 - tx - u^2 + pu$$

$$FOCs.$$

$$H'_u = 0 \quad : -2u + p(t) = 0 \Rightarrow u*(t) = 0.5p(t).$$

$$H'_x = -\dot{p}(t) : -t = -\dot{p}(t) \quad \Rightarrow \dot{p}(t) = t.$$

$$p(T) = 0.$$

$$and\, state\, equation.$$

$$Integrating\, \dot{p}(t) = t\, gives\, p(t) = \frac{t^2}{2} + C = 0.$$

$$For\, p(T) = 0.5T^2 + C = 0 \Rightarrow C = -0.5T^2.$$

$$thus\, p(t) = 0.5t^2 - .5T^2 = -.5(T^2 - t^2).$$

$$and \boxed{u^*(t) = -0.25(T^2 - t^2).}$$

1. Basic Problem

Substituting into state eq. gets

$$\dot{x}^*(t) = u^*(t) = -0.25(T^2 - t^2).$$

Integrating and using $x*(0) = x_0$ yields

$$x^*(t) = x_0 - 0.25T^2 t + (1/12)t^3. \qquad \square$$

Thus, we find one pair $x^*(t), u^*(t)$ along with $p(t)$ satisfies FOCs.

Lastly, we can see that H is concave in (x, u) for it is a sum of concave functions.

2.The Standard Problem

Standard End Constrained Problem

$$\max \int_0^T f(t,x(t),u(t))\,dt,\ u(t)\in U \subseteq \mathbb{R} \quad ....(1)$$

$$st. \quad \dot{x}(t) = g(t,x(t),u(t)),\ x(0) = x_0 \quad ....(2)$$

and with one of the following terminal conditions

$$(a)\ x(T) = x_1;(b)\ x(T) \geq x_1;(c)\ x(T)\ free..(3).$$

2. Standard end constraints Problem: Necessary conditions

$(A)\quad u^*(t)\,\mathrm{maximizes}\,H:\forall u\ in\ U,$

$$H(t,x^*(t),u,p(t))\le H(t,x^*(t),u^*(t),p(t)).$$

$(B)\ \partial H/\partial x=-\dot{p}(t).$

$(C)\ Transversality$

$$for\,(3b):\quad p(T)\ge 0\ \ (with\,p(T)=0\,if\,x^*(T)>x_1);$$

$$for\,(3c):\quad p(T)=0;$$

$$for(3a):\quad no\ condition\,for\,p(T).$$

Ex1.(optimal consumption). Let y be predicted income, r constant interest rate, a wealth and c consumption, and ρ is a constant discount rate.

$$\max \quad \int_0^T \ln(c(t))e^{-\rho t}dt$$

$$st. \quad \dot{a}(t) = y(t) + ra(t) - c(t)$$

$$a(0) = a_0 > 0 \ and \ \ a(T) \geq 0.$$

$Find\, c^*, a^*, p.$

$First, set\ up\ Hamiltonian\ as$

$$H(t,a,c,p) = \ln(c)e^{-\rho t} + p(y + ra - c).$$

$FOCs:$

$$\partial H / \partial c = \frac{1}{c} e^{-\rho t} - p = 0 \quad or \quad c = p^{-1} e^{-\rho t} \quad ..(A)$$

$$\partial H / \partial a = pr = -\dot{p} \qquad or\, \dot{p} = -rp \qquad ....(B)$$

$$p(T) \geq 0. \qquad\qquad [TVC]$$

$and\, state\, equation.$

$Notice\, that\, \dot{a}\, depends\, on\, c; c\, on\, p;\ \dot{p}\, depends\, on\, p.$

$[step : from\, p\, to\, c\, then\, a]$

$From\,(A), for\, positive\, c, p\, cannot\, be\, zero. Thus, p(T) > 0$

$implies\, a(T) = 0.$

$\text{Initial condition}, a(0) = a_0, \text{ and terminal condition}$

$a(T) = 0 \text{ to find two constants in } a(t) \text{ and } p(t).$

$(i)\, From\, (B), solution\, is\, p = p_0 e^{-rt}$

$where\, p_0$ is a constant to be determined.

$(ii)\, Sub\, p\, \text{int}\, o\, (A) :\, c = p_0^{-1} e^{(r-\rho)t}.$

$(iii)\, Substitute\;\; c\, in\, the\, state\, eq.\;\, yields$

$$\dot{a} = y + ra - p_0^{-1} e^{(r-\rho)t}. \qquad\qquad (C)$$

$To\, solve\, for\, a(t), we\, use\, this\, fact :$

$$\frac{d(ae^{-rt})}{dt} = (e^{-rt}\dot{a} - rae^{-rt}) = (\dot{a} - ra)e^{-rt} \;\; thus$$

$$\frac{d(ae^{-rt})}{dt} = \left(y - p_0^{-1}e^{(r-\rho)t}\right)e^{-rt} \quad [from\,(C)]$$

$$= ye^{-rt} - p_0^{-1}e^{-\rho t}$$

Integrates $both\ \ sides\ \ from\, 0\, to\, any\, t\, yields$

$$ae^{-rt} - a(0).1 = \frac{y}{r}[1 - e^{-rt}] - \frac{p_0^{-1}}{\rho}[1 - e^{-\rho t}]$$

$$ae^{-rt} - a_0 = \frac{y}{r}[1 - e^{-rt}] - \frac{p_0^{-1}}{\rho}[1 - e^{-\rho t}].$$

$For\ given\ a(T) = 0, we\ can\ solve\ for\ p_0 :$

$$0 - a_0 = \frac{y}{r}[1 - e^{-rT}] - \frac{p_0^{-1}}{\rho}[1 - e^{-rT}]$$

$Thus\ p_0 = ..$

$Then\ we\ \ know\ p(t)\ and\ a^*(t). Use\ p(t)\ in\,(A), we\ get\ c^*(t).$

Some intersting facts without knowing the complete solution.

$(i)\, From\ c^* = p^{-1}e^{-\rho t} = p_0^{-1}e^{rt}e^{-\rho t} = p_0^{-1}e^{(r-\rho)t}.$

This shows that

- if $r > \rho$, the optimum consumption grows over his lifetime . He will save early on, $c < y$, builds up assets, and in the last year of life runs down his saving.

-if $r < \rho$, the opposite happens .

-if $r = \rho, c^*$ is constant and independent of t.

$In\ the\ last\ case, let\ c^* = \overline{c}.\ From\ state\ equation$

$$\dot{a}(t) = ra(t) + y(t) - \overline{c},\ which\ can\ be\ solved\ as$$

$$a^*(t) = e^{rt}[a_o + \int_0^t e^{-rs}y(s)ds - \frac{\overline{c}}{r}(1 - e^{-rt})].$$

$$And\,that\,a^*(T) = 0, \text{this can determine}\,\overline{c}$$

$$as \quad \overline{c} = \frac{r}{1 - e^{-rT}}[a_0 + \int_0^T e^{-rs} y(s)ds].$$

$$(ii)\,Euler : from\,(\text{A}), \text{take ln} : \ln c = -\ln p - \rho t$$

$$\text{and time derivates :} \qquad \dot{c}\,/\,c = -(\dot{p}\,/\,p) - \rho.$$

$$From\,(B)\,which - (\dot{p}\,/\,p) = r : \qquad \boxed{\dot{c}\,/\,c = r - \rho.}$$

c grows at a rate equal to diff. betw. interest rate and discount rate.

Homework.

Use $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta}\, for\, \theta > 0, \neq 1$ in this example.

(1) show Elasticity of marginal utility : $\dfrac{cu''}{u'}$=-θ.

(2) show $\boxed{\dot{c} \,/\, c = \dfrac{1}{\theta}(r - \rho).}$

$note\, that\, as\, \theta \to 1, u(c) \to \ln(c).$

3.Economic interpretation of Maximum principle

- Consider a firm that seeks to maximize its profits over time interval $[0,\ T]$.
- Control: business decision at each time, say inventory, affecting the rate of change in K over time and thus profits.
- State variable: K, capital stock,
- $K(0)$=K_0 given; $K(T)$ is free
- Our problem is

3.Economic interpretation of Maximum principle

$$V^*(K_o, K^*(T), 0, T) = \max \int_0^T f(t, K, u) dt$$

$$st.\ \dot{K}(t) = g(t, K, u)$$

$$K(0) = K_0,\ K(T)\ free.$$

V^* *is the optimal value function.*

We can write

$$H(t, u, K, p) = f(t, K, u) + p(t) g(t, K, u)$$

3.Economic interpretation of Maximum principle

$$H(u,K,p,t) = f(t,K,u) + p(t)g(t,K,u)$$

$$FOC$$

$$\frac{\partial H}{\partial u} = 0 \quad [select\,u\,to\max H\,at\,each\,t]$$

$$[\text{Find the optimal balance betw. current and future profit}]$$

$$\frac{\partial H}{\partial K} = -\dot{p}(t).$$

$$[\text{margnial value of K to H is the falling rate of marginal}$$

$$\text{contribution of K}]$$

$$p(T) \geq 0.$$

$$[\text{Transversality must hold}]$$

3.Economic interpretation of Maximum principle

$$H = f(t, K, u) + p(t)g(t, K, u)$$

- H function combines the current (immediate) profit (first term) due to u, and the future-profit effect of u.
- So, H shows overall profit prospect of various policies.
- The second term captures how u in the current period affecting the future profit. The g function shows the rate of change in K due to u. This change is valued by its shadow price $p(t)$. So the shadow price is value of extra profit stream created by an additional unit of capital. (value of investment)
- Generally, these two effects are competing.
- Thus, optimal u must balance current gains against sacrifice of future profit at each t.

3.Economic interpretation of Maximum principle

$$\frac{\partial H}{\partial u} = \frac{\partial f}{\partial u} + p(t)\frac{\partial g}{\partial u} = 0, or$$

$$\boxed{\frac{\partial f}{\partial u} = -p(t)\frac{\partial g}{\partial u}}. \qquad [nb. g = \dot{K}]$$

- The optimal choice u^* must balance LHS (<u>marginal increase</u> in current profit) against RHS (<u>marginal decline</u> in the future profit) induced by the change in K.
- Nice thing about this is that we choose u at any date t without worrying about other dates than t. The difficulties about intertemporal choice must be solved by selecting the right values of the costate variables instead.

- Next maximum principle condition is Euler equation:

$$\boxed{-\dot{p}(t) = \frac{\partial H}{\partial K} = \frac{\partial f}{\partial K} + p(t)\frac{\partial g}{\partial K}}$$

- The shadow price $p(t)$ is the marginal value of the state variable, $\partial V/\ \partial K$ (which is hard to evaluate directly).
- LHS is the falling rate of price or the rate at which value of stock depreciates.
- The first term of RHS is marginal contribution of K to current profit, and the second term is marginal change in value of K on future profit
- Overall, this requires that the shadow price of a unit of capital depreciate at the rate in which K contributes to current and future profits.
- Value of K at the beginning problem is equal to the sum of contributions of K over time. Thus, as we move across time, ability of K to contribute to V is used up.

3.Economic interpretation of Maximum principle

- Last TVC for a fixed terminal time is $p(T)=0$ if $K(T)$ is free.
- This means the shadow price of capital should be driven to zero at T.
- At T, whatever K that exists will have no value for firm to make use of it because it is too late to be put to use.

3.Economic interpretation of Maximum principle

- *p(t)* is the shadow price of capital at time t. It measures the contribution to optimal profits from an additional unit of k at time t.

$$\boxed{\begin{aligned}&\partial V^* / \partial K_0 = p^*(0)\\ &\partial V^* / \partial K^*(T) = -p^*(T)\end{aligned}}$$

- So, *p*(0)* is a value of a unit of initial capital. It shows a change in optimal total profits due to an increase in a unit of initial capital. At t=0, we try to accumulate.
- *p*(T)* is the shadow price of a unit of terminal capital stock. *At T, we try to use all of them.* Thus, if we want to keep one more unit (use less)of *K(T)*, then we have to sacrifice our profit by the amount -*p*(T).*
- *In all, this is the marginal valuation of the state variable at time t.*

- Summary of procedure to find solution

1. Max H wrt u. If lucky, we get $u^*(t,x,p)$.
2. Insert u^* function into two diff equations:

   x-dot and p-dot to solve for $x(t)$ and $p(t)$.
3. The constants in the general solution found by initial condition $x(0)=x_0$, terminal conditions and Transversality conditions.
4. u^* and x^* then a candidate for optimality.
5. Check for sufficient condition.

4. Current Value Formulations

- In economics, we often add the discount factor to the objective function. Thus, we try to maximize total present discounted values.
- We can simplify FOCs by redefining H to current-value H.
- We will show new conditions.

$$\max \int_0^T f(t,x,u)e^{-rt}dt$$

$$st. \qquad \dot{x} = g(t,x,u), \quad x(t_o) = x_0,$$

$$x(T) = \left\{ \begin{bmatrix} x \quad \ldots(a) \\ \geq x_1 \ldots(b) \\ free\ldots(c) \end{bmatrix} \right\}$$

4. Current Value Formulations

The *present − value* Hamiltonian is

$$H = f(t,x,u)e^{-rt} + pg(t,x,u).$$

The costate is also *a* present- value shadow price.

Multiply H *by* e^{rt}, we get the current-value Hamiltonian

$$H^c = He^{rt}.$$

$$H^c = f(t,x,u) + e^{rt} p g(t,x,u).$$

Let $\lambda = e^{rt} p$ *denotes the current − value shadow price*

$$\boxed{H^c = f(t,x,u) + \lambda g(t,x,u). \qquad (*)}$$

Notice that no discount factor.

4. Current Value Formulations

$$FOCs\, can\, also\, be\, rewriten\, in\, terms\, of\, \lambda\, and\, H^c :$$

$$(A)\, \partial H / \partial u = 0 \Rightarrow \boxed{\partial H^c / \partial u = (\partial H / \partial u)e^{rt} = 0}$$

$$(B) \qquad \partial H / \partial x = -\dot{p}$$

$$RHS : from\, \lambda = e^{rt} p \Rightarrow \dot{\lambda} = re^{rt} p + e^{rt}\dot{p},$$

$$so \quad \dot{p} = e^{-rt}(\dot{\lambda} - r\lambda).$$

$$LHS : H^c = He^{rt} \Rightarrow \partial H^c / \partial x = e^{rt}(\partial H / \partial x)$$

$$or \qquad \partial H / \partial x = e^{-rt}(\partial H^c / \partial x).$$

$$\Rightarrow e^{-rt}(\partial H^c / \partial x) = -e^{-rt}(\dot{\lambda} - r\lambda)$$

$$\Rightarrow \quad \boxed{(\partial H^c / \partial x) = -(\dot{\lambda} - r\lambda) \qquad (*)}$$

$$(B)\, For\, TVC : p(T) = 0 \Rightarrow \lambda(T)e^{-rT} = 0$$

$$\Rightarrow \boxed{\lambda(T) = 0. \quad (*)}$$

4. Current Value Formulation: Necessary conditions

$$For\ H^C = f(t,x,u) + \lambda g(t,x,u)$$

$$(A)\quad u = u^*(t)\max\ H^C$$

$$(B)\quad \frac{\partial H^C}{\partial x} = r\lambda - \dot{\lambda}$$

$$(C)\, Transversality$$

$$(a')\quad \lambda(T)\ no\ condition$$

$$(b')\quad \lambda(T) \geq 0 \quad (\lambda(T) = 0\ if\ x^*(t) > x_1)$$

$$(c')\quad \lambda(T) = 0.$$

4. Current Value Formulation: Necessary conditions

- Current value formulations of H is usually more interesting than discounted one for economists.
- Ex. The market price of capital stock will equal the current-value costate variable, λ
- Intrepretation of costate equation (B) now is more difficult but interesting.
- Euler equation (B) can be interpreted as asset pricing formula or equilibrium condition for asset prices.

4. Current Value Formulation: Necessary conditions

- State variable x can be thought as capital or asset in the agent's portfolio. The costate λ is the price of capital.
- So, λ-dot is capital gain and $\partial H^c/\partial x$ is the dividend.
- Let r be rate of return of an alternative asset (say consumption)

$$\frac{\partial H^c}{\partial x} = r\lambda - \dot{\lambda} \Rightarrow \frac{\left(\frac{\partial H^c}{\partial x} + \dot{\lambda}\right)}{\lambda} = r$$

- Thus, Euler equation means the agent is indifferent between the two types of investments because the overall rate of return to capital equals to the return to consumption.

4. Current Value Formulation:

$$Ex.\ \max \int_0^{20} (4K - u^2) e^{-0.25t} dt$$

$$st.\quad \dot{K} = -0.25K + u$$

$$K(0) = K_0,\ \ K(20)\, free.$$

$$where\, K(t)\, is\, the\, value\, of\ a\, machine, u(t)\, is$$

$$the\, repair\, effort;\ f(.)\, is\, the\, profit.$$

$$H^c = 4K - u^2 + \lambda(-0.25K + u).$$

4. Current Value Formulation:

$$FONC : assume\, that\, u^*(t) > 0$$

$$\frac{\partial H^c}{\partial u} = -2u + \lambda = 0 \Rightarrow u = 0.5\lambda.$$

$$\frac{\partial H^c}{\partial K} = r\lambda - \dot{\lambda} \Rightarrow 4 - 0.25\lambda = 0.25\lambda - \dot{\lambda}$$

$$\dot{\lambda} - 0.5\lambda = -4$$

$$\lambda(20) = 0.$$

$$From\, \dot{\lambda} - 0.5\lambda = -4 \;\; and \;\; \lambda(20) = 0$$

$$\Rightarrow \lambda(t) = 8(1 - e^{0.5t-10})\, and\, use\, this\, in\, u^*(t)\, get$$

$$u^*(t) = 0.5\lambda = 4(1 - e^{0.5t-10}).\square$$

4. Current Value Formulation:

$$Sub\, u * in\, \dot{K} :$$

$$\dot{K} = -.25K^* + u^* = -0.25K^* + 4(1 - e^{0.5t-10})$$

$$Solve\, this\, with\, K(0) = K_0\, and\, get$$

$$K^*(t) = (K_0 - 16 + \tfrac{16}{3}e^{-10})e^{-0.25t} + 16 - \tfrac{16}{3}e^{0.5t-10}.$$

$$check\, that\, H^c\, is\, concave\, in\, (K, u), we're\, done.$$

4. Current Value and Infinite Horizon

- In economics, we often assume that time horizon is infinite.
- Reasons:

  -avoid specifying the end-of-horizon stocks or scrap value function;

  -simplified formulas and appealing results;

  -can use for long-run equilibrium interpretation; not very different from large finite problem;

  -the very distant future has no big influence on the optimal path for the near future which we are interested in.

4. Current Value and Infinite Horizon

$$\max \int_0^{\infty} f(t, x(t), u(t)) e^{-rt} dt,$$

$$st. \quad \dot{x}(t) = g(t, x(t), u(t)),$$

$$x(0) = x_0, \quad u(t) \in U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$and \ no \, condition \, imposed \, on \, x(t) \, as \, t \rightarrow \infty.$$

$$Often, many \, economic \, problems \, impose$$

$$\lim_{t \to \infty} x(t) \geq k \, (\text{a constant}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

4. Current Value and Infinite Horizon

- If the integral or J converges, the FOC in the maximum principle will hold, except the TVC.
- Without TVC, we get too many solution candidates.
- What should TVC be when t goes to infinity?
- Remind that in a finite time problem with $x(T)$ free, TVC is $p(T)=0$.
- A natural answer for infinite horizon is to have $p(t)$ goes to zero as t goes to infinity. (finite anology)
- But this is not a correct necessary condition in general. Some problem does not require this. Only occur in a special case. The right one is too strong for many economics application.

4. Current Value and Infinite Horizon

- In Economics literature, we pay less attention to TVC for infinite horizon problem by assuming that $p(t)$ goes to zero as t tends to infinity.
- For $p(t) = \lambda(t)e^{-rt}$
- Thus, often we use the following TVC:

$$\lim_{t \to \infty} [\lambda(t)e^{-rt}x(t)] \geq 0.$$

- This condition tries to rule out the kind of inefficiency involved in accumulating capital forever without using it.
- This holds if $x(T)=0$, or its marginal value $\lambda(T)e^{-rT}$ must be zero.

4. Current Value and Infinite Horizon

Example. Ramsey's Optimal Growth Model.

This is also a problem of optimal consumption but from the point of view of the economy as a whole.

New features:

(a) return on saving is not exogenously given, but must be the endogenous MPK.

(b) we assume infinite time.

$$\max \quad \int_0^\infty \ln(c)e^{-\rho t}dt$$

$$st. \quad \dot{k} = F(k) - \delta k - c$$

$$k(0) = b > 0 \ and \lim_{t\to\infty} k(t) = 0.$$

4. Current Value and Infinite Horizon

$Assume\, F\, is\, concave\, with\, F(0) = 0\, and\, F'(0) = \infty.$

$Assmume\, \rho > r \geq 0, and\, \delta\, is\, the\, rate\, of\, capital\, depreciation.$

$First, define\, H^c = U(c) + \lambda(F(k) - \delta k - c)$

$FOCs:$

$$\partial H^c / \partial c = \boxed{U'(c) - \lambda = 0 .....(a)}$$

$$\partial H^c / \partial k = \lambda[F'(k) - \delta] = \rho\lambda - \dot{\lambda}$$

$$or \qquad \boxed{\dot{\lambda} / \lambda - \rho = -(F'(k) - \delta). \quad ......(b)}$$

$$and\, TVC: \lim_{t\to\infty} \lambda e^{-rt} k = 0 ....(c).$$

$[(b) \left(\dot{\lambda} - \rho\lambda\right) + \left(\lambda[F'(k) - \delta]\right) = 0; if\, k$ is an asset which price movement is given by $\lambda, (b)$ means capital gain *plus* marginal value product=0 at *optimal*.

4. Current Value and Infinite Horizon

Here, we shall work with c and k instead.

(i) differentiation of (a) gives $U''(c)\dot{c} = \dot{\lambda}$

(ii) use (i) in (b) : $\dfrac{U''(c)\dot{c}}{U'(c)} - \rho = -[F'(k) - \delta]$

$$\frac{cU''(c)}{U'(c)}\frac{\dot{c}}{c} - \rho = -[F'(k) - \delta]$$

$Let\, \eta = -cU'' / U'$

$thus,$ $\quad \eta\ (\dot{c} / c) + \rho \quad = [F'(k) - \delta]$

$$\frac{\dot{c}}{c} = \frac{F'(k) - (\rho + \delta)}{\eta}.$$

The Ramsey rule : $c \uparrow\downarrow or$ constant depends on whether MPK $>, <, =$ rate of time preference $+$ rate of capital depreciation, or net MPK $>, <, =$ rate of time preference.

4. Current Value and Infinite Horizon

- We can solve for stationary state of this growth model where both capital K and c are constant.
- Similarly, we can use the Hamiltonian system in K and shadow price.
- Using the phase diagram, we can analyze the behavior of the system.
- Given the initial value of K and shadow price, we can find the trajectory of the economy.

Some First-order Ordinary Differential equation (FODE)

$$1.\ \dot{x} + ax = b \qquad \Rightarrow x = Ce^{-at} + b\,/\,a.$$

$$2.\ \dot{x} + ax = b(t) \ \ \Rightarrow x = Ce^{-at} + e^{-at}\int e^{at} b(t)dt.$$

$$3.\ \dot{x} + a(t)x = b(t)$$

$$\Rightarrow x = e^{-\int a(t)dt}\left(C + \int e^{\int a(t)dt} b(t)dt\right).$$

$$we\,solve\,for\,x(t), its\,paths\,depend\,on\,time. Thus,$$

$$optimal\,control\,gives\,answers\,that\,are\,open - looped.$$

Some integration

- One important note

$$\int_{o}^{T} De^{-rt}dt = D\int_{o}^{T} e^{-rt}dt = D[\frac{-1}{r}e^{-rt}]_0^T$$

$$= \frac{-D}{r}[e^{-rt}]_0^T = \frac{-D}{r}(e^{-rT} - 1)$$

$$= \frac{D}{r}(1 - e^{-rT}).$$