

### 6.2.1c) Comparative Static Analysis using partial derivative

- Model  $\rightarrow$  solve for reduced-form equations of endogenous variables.
  - Endo as a function of (i) exogenous variables and (ii) parameters
  - We can apply partial derivative to the derived endogenous equilibrium solution.
  - This allows us to see how endogenous variables respond to the changes in exogenous variables and parameters.

**Example 6. G:** Recall that in a simple macroeconomic model, the solution for equilibrium output can be given by  $y^* = \frac{I_0 + G_0}{1 - (1-t)b}$ , where  $t$  is level of income tax and  $b$  is marginal propensity to consume. What about the effect of an increase in “ $b$ ” and an increase in “ $t$ ” on output?

$$\frac{\partial y^*}{\partial b}$$

$$\frac{\partial y^*}{\partial t}$$

$$y^* = \frac{I_0 + G_0}{1 - (1-t)b} = (I_0 + G_0) (1 - (1-t)b)^{-1}$$

quotient rule

$$\frac{dy^*}{db} = \frac{(1 - (1-t)b)(0) - (I_0 + G_0)(-(1-t))}{(1 - (1-t)b)^2}$$

chain rule

$$\frac{dy^*}{db} = -(I_0 + G_0) (1 - (1-t)b)^{-2} (-(1-t))$$

Both get  $\frac{dy^*}{db} = \frac{(I_0 + G_0)(1-t)}{(1 - (1-t)b)^2}$

## 6.2.3) Total differential

- Total differential in "y" = Total change in "y"
- Recall the concept of single variable differential

$$y = f(x)$$

- $dy = f'(x)dx$
- $f'(x) =$  marginal change in "y" per a unit of change in x.
- $dx =$  units of change in "x".

- How about total differential in multivariate relationship.

**Definition:** Suppose  $y = f(x_1, x_2, x_3, \dots, x_n)$

$$dy = \sum_{i=1}^n f_i dx_i$$

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 + \dots$$

**Tips:** When you are taking the total differential, you are just taking all the partial derivatives and adding them up.

**Example 6.H:** Suppose  $C(y, r) = \sqrt{\frac{y}{r}}$ . Derive total differential of C.

$$dC = f'_y dy + f'_r dr$$

$$f'_y = \frac{\partial C}{\partial y} = \frac{1}{2} y^{-\frac{1}{2}} r^{-\frac{1}{2}}$$

$$f'_r = \frac{\partial C}{\partial r} = -\frac{1}{2} y^{\frac{1}{2}} r^{-\frac{3}{2}}$$

$$dC = \frac{1}{2} y^{-\frac{1}{2}} r^{-\frac{1}{2}} dy - \frac{1}{2} y^{\frac{1}{2}} r^{-\frac{3}{2}} dr$$

$$z = 3x + 2y$$

$z$  can change when  $x$  or  $y$  or both change.

Total Differential of  $z$

$$dz = f'_x dx + f'_y dy$$

↑ change in  $z$       ↑ change in  $x$       ↑ change in  $y$

$\frac{dz}{dx}$ : effect of  $x$  on  $z$

$\frac{dz}{dy}$ : effect of  $y$  on  $z$

$$dz = 3dx + 2dy$$

e.g. if  $x = 1$  &  $y = 1 \rightarrow z = 5$

what happens to  $z$  if

$x$  increases by 4 &  $y$  increases by 5?

$$dz = 3(4) + 2(5) = 22$$

Thus,  $z$  will increase by 22.

Proof

if  $x = 1$  &  $y = 1 \rightarrow z = 5$

↓  $dx = 4$

↓  $dy = 5$

↓  $dz = 22$

if  $x = 5$  &  $y = 6 \rightarrow z = 27$

**Example 6.J:** Consider the equilibrium function of real income  $y^* = f(I_0, G_0, t, b) = \frac{I_0 + G_0}{1 - (1-t)b}$ . Find total differential of  $y^*$ .

$$dy^* = \frac{\partial y^*}{\partial I_0} dI_0 + \frac{\partial y^*}{\partial G_0} dG_0 + \frac{\partial y^*}{\partial t} dt + \frac{\partial y^*}{\partial b} db$$

$$\frac{\partial y^*}{\partial I_0} = \frac{1}{1 - (1-t)b}$$

$$\frac{\partial y^*}{\partial G_0} = \frac{1}{1 - (1-t)b}$$

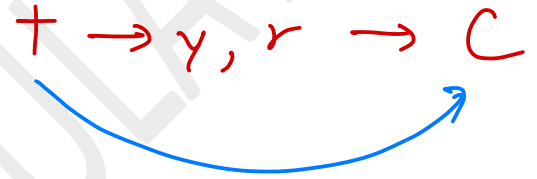
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## 6.2.4) Total derivative (aka. Chain rule!)

$$C = f(y, r)$$

- For an illustrative purpose, consider the consumption function in **example 6.H**. Suppose that “y” and “r” grows over time.
- That is,  $y = g(t)$  and  $r = h(t)$  where t is the numbers of period from now.
- By the composite function, consumption will be *ultimately* determined by “t”.
- Then, how do we calculate the derivative of “c” with respect to “t”?
- This requires the concept of total derivative.



- Total differential in “c” can be given by

$$dC = \frac{\partial c}{\partial y} dy + \frac{\partial c}{\partial r} dr$$

$\underbrace{\hspace{100px}}$   
effect of y on c
 $\underbrace{\hspace{100px}}$   
effect of r on c

- Given this assumption, one obtains that

$$dy = g'(t)dt$$

$$dr = h'(t)dt$$

$$\frac{dy}{dt} = g'(t)$$

$$\frac{dr}{dt} = h'(t)$$

- Applying the two expressions above, we obtain that

$$dC = \frac{\partial c}{\partial y} g'(t)dt + \frac{\partial c}{\partial r} h'(t)dt$$

- As a result, total derivative of C with respect to t is given by

$$\frac{dC}{dt} = \frac{\partial c}{\partial y} g'(t) + \frac{\partial c}{\partial r} h'(t)$$

$$z = f(x(t), y(t))$$

→  $z$  depends on  $x(t)$  &  $y(t)$   
 $x$  depends on  $t$  &  $y$  depends on  $t$

Thus,  $z$  indirectly depends on  $t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

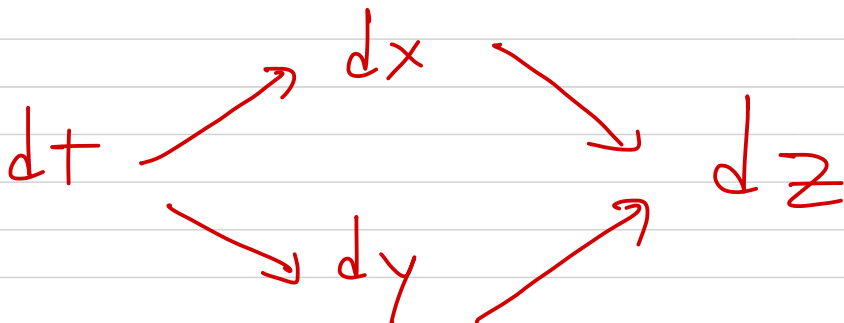
↑ effect of  $t$  on  $z$

↑ effect of  $t$  on  $x$

↑ effect of  $t$  on  $y$

↑ effect of  $x$  on  $z$

↑ effect of  $y$  on  $z$



**Example 6.K** Consider a consumption function with  $C = t^2 + \sqrt{y} - r^3$  and  $y = 2t^3$  and  $r = t + 1$ . Derive the total derivative of  $C$  with respect to  $t$ . Calculate the value of  $dc$  when  $t = 0$

$$C = f(t, y(t), r(t))$$

$$\frac{dc}{dt} = \underbrace{\frac{dc}{dt}}_{\text{direct effect}} + \underbrace{\frac{dc}{dy} \frac{dy}{dt} + \frac{dc}{dr} \frac{dr}{dt}}_{\text{indirect effect}}$$

$$\frac{dy}{dt} = 6t^2 \quad \frac{dr}{dt} = 1$$

$$\frac{dc}{dt} = 2t + \frac{1}{2} y^{-\frac{1}{2}} (6t^2) - 3r^2 (1)$$

$$\frac{dc}{dt} = 2t + \frac{3t^2}{\sqrt{2t^3}} - 3(t+1)^2$$

## 6.2.5 Partial total derivative (aka. Chain rule for multivariate!)

- From the example above, what if both  $y$  and  $r$  depend on “ $t$ ” and “ $s$ ”?
- Consumption would then be ultimately driven by “ $t$ ” and “ $s$ ”.
- The concept is changed to *partial total derivative*:  $\frac{\partial c}{\partial t}$  and  $\frac{\partial c}{\partial s}$

Following the same method as we applied to total derivative, we know that

$$dC = \frac{\partial c}{\partial y} \left( \frac{\partial y}{\partial t} dt + \frac{\partial y}{\partial s} ds \right) + \frac{\partial c}{\partial r} \left( \frac{\partial r}{\partial t} dt + \frac{\partial r}{\partial s} ds \right)$$

As a result,

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial y} \left( \frac{\partial y}{\partial t} \right) + \frac{\partial c}{\partial r} \left( \frac{\partial r}{\partial t} \right)$$

Meanwhile, we would yield that

$$\frac{\partial c}{\partial s} = \frac{\partial c}{\partial y} \left( \frac{\partial y}{\partial s} \right) + \frac{\partial c}{\partial r} \left( \frac{\partial r}{\partial s} \right)$$

## 6.3 The Implicit derivative

### 6.3.1 Implicit derivative: General concepts

- Derivative method applied to an *implicit function*
- Let's first discuss what implicit functions are.

Explicit function	Implicit function
$y = f(x)$ <ul style="list-style-type: none"> <li>• <math>y = 2x - 1</math></li> <li>• <math>y = x^2 + 2x + 1</math></li> <li>• <math>y = \ln(2x + 1) + 2x + 1</math></li> <li>• <math>z = \ln(2x + 1) + 2y^2x</math></li> </ul>	$y = f(x)$ <ul style="list-style-type: none"> <li>• <math>x^2 + y^2 = 3</math></li> <li>• <math>xw + yw - w^2 + zx^2 = 9</math></li> </ul>

For an implicit function, how do we obtain the derivative of  $y$  with respect to  $x$ ?

**Method 1: (brute force) → Rewrite  $y$  in terms of  $x$ .**

- Comments:
  - Tedious... and NOT applicable for all the cases.
  - In many cases, the function doesn't admit any closed-form solutions: impossible to rewrite " $y$ " in term of " $x$ ".

**Example 6.L** Suppose that  $x^2 + y^2 = 3$ , find  $dy/dx$ .

$$y^2 = 3 - x^2$$

$$y = (3 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \dots \text{ using chain rule}$$

$$x^2 + y^2 = 3$$

**Method 2:** Apply the implicit function theorem

$$z = f(x, y) = x^2 + y^2 - 3 = 0$$

**Theorem:** Suppose a given function is written in the form of  $F(x, y) = 0$ ,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

partial derivative  
of  $z$  wrt.  $x$

partial derivative  
of  $z$  wrt.  $y$

**Proof:** Use the total differentials

$$z = f(x, y) = 0$$

**Theorem:** Suppose a given function is written in the form of  $F(x, y) = 0$ ,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

**Proof:** Use the total differentials

$$dz = \frac{dz}{dx} dx + \frac{dz}{dy} dy$$

Since  $z$  is constant as  $z = 0$ ,  $dz = 0$

$$0 = \frac{dz}{dx} dx + \frac{dz}{dy} dy$$

$$0 = f'_x dx + f'_y dy$$

$$\frac{dy}{dx} = -\frac{f'_x}{f'_y}$$

**Example 6.M** Suppose that  $\ln(x+2) - y^2 - e^{x+y} = 3$ , Find  $\frac{dy}{dx}$

impossible to write  $y = f(x)$

instead, we write  $z = f(x, y) = 0$

$$z = f(x, y) = \ln(x+2) - y^2 - e^{x+y} - 3 = 0$$

use formula  $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$f'_x = \frac{\partial z}{\partial x} = \frac{1}{x+2} - e^{x+y}$$

$$f'_y = \frac{\partial z}{\partial y} = -2y - e^{x+y}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x+2} - e^{x+y}}{-2y - e^{x+y}}$$