EE320 Lecture Note
Chapter 10: Integration and Its Application

1 Terminology

integration = anti-derivative or inverse of differentiation

y = F(x) $\Rightarrow$ F(x) = $\int F'(x)dx + c$

$\frac{dy}{dx} = F'(x)$

✓ indefinite integral

if we integrate f(x) where values of x are not given, we have to integrate without a limit.

(i.e. to find indefinite integral)

A symbol for integrating a function f(x) is $\int f(x)dx = F(x) + c \quad \Leftrightarrow \quad F'(x) = f(x)$

where

$\int$	is	integral sign
f(x)	is	integrand
c	is	constant of integration
dx	indicates	the variable involved in the integration

Note function does not have a unique integral

ex. $f(x) = 3x^2 \quad \Rightarrow \quad$ F(x) = $x^3 + 1$, $x^3 + 7$, $x^3 + c$

2 Basic Rules for Integration

1) $\int x^n dx = \frac{1}{n+1}x^{n+1} + c\ ,\ n \neq -1$

2) $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$$\begin{aligned} \Rightarrow \int a^{bx} dx &= \frac{1}{blna}a^{bx} + c \\ \Rightarrow \int f(x)e^{f(x)} dx &= e^{f(x)} + c \\ \Rightarrow \int \frac{f'(x)}{f(x)} dx &= ln|f(x)| + c, f(x) \neq 0 \end{aligned}$$

3) $\int \frac{1}{x} dx = ln|x| + c$

4) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

5) $\int af(x)dx = a\int f(x)dx$

<u>ex.</u>

1. $\int \frac{1}{x^3} dx = -\frac{1}{2x^2} + c$
2. $\int \sqrt{x\sqrt{x\sqrt{x}}}dx = \frac{8}{15}x^{\frac{15}{8}} + c$
3. $\int (3x^4 + 5x^2 - 2)dx = \frac{3}{5}x^5 + \frac{5x^3}{3} - 2x + c$
4. $\int e^{3x} - e^{2x} + e^x dx = \frac{e^{3x}}{3} - \frac{e^{2x}}{2} + e^x + c$
5. $\int \frac{(y-2)^2}{\sqrt{y}} dy = \frac{2}{5}y^{\frac{5}{2}} - \frac{8}{3}y^{\frac{3}{2}} + 8y^{\frac{1}{2}} + c$
6. $\int 2^x dx = \frac{1}{in2}26x + c$

6) $\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c$

<u>ex.</u> $\int (ax+b)^P dx$: let u = ax + b $\Rightarrow$ du = a dx $\Rightarrow$ $dx = \frac{du}{a}$

$\therefore \int u^P \frac{du}{a} = \frac{u^{P+1}}{(P+1)a} + c = \frac{(ax+b)^{P+1}}{a(P+1)} + c$

7) $\int v\ du = uv - \int u\ dv$ "integration by parts"

<u>ex.</u> $\int xe^x dx$

$$\begin{aligned} \Rightarrow \text{let } v &= x &\Rightarrow dv &= dx \\ u &= e^x &\Rightarrow du &= e^x dx \end{aligned}$$

$$\begin{aligned} \int xe^x du &= \int vdu \\ &= uv - \int udv \\ &= e^x \cdot x - \int e^x dx \\ &= e^x \cdot (x-1) + c \end{aligned}$$

✓ Initial-Value Theorem

$\int f(x)dx = F(x) + c$

If we have an initial condition, we can determine the value of c.

<u>ex.</u> 1. find F(x) if $F'(x) = \frac{1}{2} - 2x$ and $F(0) = \frac{1}{2}$

$\int \frac{1}{2} - 2xdx = \frac{1}{2}x - \frac{2x^2}{2} + c = \frac{x}{2} - x^2 + c = F(x)$

$F(0) = \frac{0}{2} - 0^2 + c = \frac{1}{2} \therefore c = \frac{1}{2}$

2. find F(x) if $F'(x) = x(1-x^2)$ and F(1) = $\frac{5}{12}$

$\int x(1-x^2)dx = \frac{x^2}{2} - \frac{x^4}{4} + c = F(x)$

$F(0) = \frac{0^2}{2} - \frac{0^4}{4} + c = \frac{5}{12}n \qquad \therefore c = \frac{5}{12}$

(Application 1 : Derivation of TR from MR)

TR = $\int$ MR(Q) dQ

<u>ex.</u> MR = 10 - Q , TR = ?

TR = $\int 10 - QdQ = 10Q - \frac{Q^2}{2} + c$

Suppose at Q = 0 , TR = 0 $\Rightarrow$ c = 0 $\therefore$ TR = 10Q-$\frac{Q^2}{2}$

(Application 2: Derivation of TC from MC)

TC = $\int$ MC(Q) dQ

<u>ex.</u> MC = $2e^{0.2Q}$, C_F = 90 , TC =?

TC(Q) = $\int 2e^{0.2Q}dQ = \frac{2}{0.2}e^{0.2Q} + c = 10e^{0.2Q} + c$

TC(0) = 90 = $10 \cdot e^{0.2(0)} + c \Rightarrow c = 80 \quad \therefore TC = 10e^{0.2Q} + 80$

(Application 3: Derivation of Profit function from MR-MC)

$\Pi'(Q)$ = Marginal profit = MR-MC

$\Pi = \int \Pi'(Q)dQ$

<u>ex.</u>

MR = 50 - 2Q , MC = 10+Q , find total profit when Q =10.

Assure there is no fixed cost.

$\Pi(10)$ = ?

$\Pi'(Q)$ = 50-2Q-10-Q = 40-3Q

$\int \Pi'(Q)$ dQ = 40Q - $\frac{3Q^2}{2}$ + c

If Q = 0 , TR = 0 , TC = 0 $\Rightarrow$ $\Pi(Q) = 0$ $\Rightarrow$ $c = 0$

$\therefore \Pi(Q) = 40Q - \frac{3}{2}Q^2$

(Application 4: Derivation of Utility function from MU)

U(x)= $\int MU(x)dx$

ex. MU(x) = $\frac{5}{3\sqrt{x}}$, U(x) = ?

U(x) = $\int \frac{5}{3\sqrt{x}}dx = \frac{10}{3}\sqrt{x} + c$

(Application 5: Derivation of Consumption/Saving function from marginal propensity function)

ex.

MPS: $S'(Y) = 0.3 - 0.1Y^{-\frac{1}{2}}$

S(81) = 0

Find saving and consumption function

S(Y) = $\int S'(Y)dY = 0.3Y - \frac{0.1Y^{\frac{1}{2}}}{\frac{1}{2}} + c$

S(Y) = $0.3y - 0.2\sqrt{Y} + c$

S(81) = $0.3(81) - 0.2\sqrt{81} + c = 0$ $\Rightarrow$ $c = -22.5$

$\therefore$ S(Y) = $0.3Y - 0.2\sqrt{Y} - 22.5$

For C(Y) , $C = Y - S = Y - (0.3Y - 0.2\sqrt{Y} - 22.5) = 0.7Y + 0.2\sqrt{Y} + 22.5$

3 Definite Integrals

$\int_a^b f(x)\,\mathrm{d}x = F(x) \mid_a^b = F(b) - F(a)$

<u>ex.</u> $\int_0^2 5x^2\,\mathrm{d}x = \frac{5x^3}{3} \mid_0^2 = \frac{5(2)^3}{3} - \frac{5(0)^3}{2} = \frac{40}{3}$

$\int_0^1 ax^b\,\mathrm{d}x = \frac{ax^{b+1}}{b+1} \mid_0^1 = \frac{a(1)^{b+1}}{b+1} - \frac{a(0)^{b+1}}{b+1} = \frac{a}{b+1}$

✓Area and definite integral

The area under the graph of a continuous and nonnegative function f(x) over the interval [a,b] is $\int_a^b f(x)\,\mathrm{d}x$ or

$$\text{Area } A = \lim_{\Delta x \to 0} {}_a^b[f(x) - \Delta x] = \int_a^b f(x)\,\mathrm{d}x$$

✓Properties of Definite Integrals

1. $\int_a^b f(x)\,\mathrm{d}x = \int_b^a -f(x)\,\mathrm{d}x$
2. $\int_a^a f(x)\,\mathrm{d}x = 0$
3. $\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x \ , \ (a<b<c)$
4. $\int_a^b -f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$
5. $\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$
6. $\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
7. $\int_{x=a}^{x=b} v\,\mathrm{d}u = uv \mid_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,\mathrm{d}u$

ex. 1. $\int_0^5 (x + x^2)\,\mathrm{d}x = \frac{325}{6}$

2. $\int_2^4 x^2(\frac{1}{3}x^3 + 1)\,\mathrm{d}x = \frac{1{,}184}{9}$

3. $\int_{-2}^2 e^x - e^{-x}\,\mathrm{d}x = 0$

4. $\int_e^6 (\frac{1}{x} + \frac{1}{1+x})\,\mathrm{d}x = (lnx + ln|1+x|)_e^6 = [ln6 + ln7] - [1 - ln(e+1)]$

5. $\int_{-2}^3 |x+1|\,\mathrm{d}x = \int_{-2}^{-1} -(x+1)\,\mathrm{d}x + \int_{-1}^3 (x+1)\,\mathrm{d}x = 8.5$

(Application 1 : Capital Formation and Investment Functions)

Definition : K(t) = capital stock at t

$\frac{d}{dt}K(t)$ = rate of capital formation

$I(t)$ = rate of net investment flow at t

$$\frac{d}{dt}K(t) \equiv I(t) \quad \Rightarrow \quad \int I(t)dt = \int \frac{dK}{dt}dt = K(t)$$

Gross investment $\equiv I_g(t) = I(t) + \delta K(t)$

Capital formation during a time interval [a,b] $= \int_a^b I(t)\,\mathrm{d}t = K(t)\mid_a^b$

ex. Suppose net investment flow is I(t) = $3t^{\frac{1}{2}}$ and the initial capital stock at t = 0 is K = 25. What is K(t) during [1,4]?

Ans $K(t) = \int I(t)dt = \int 3t^{\frac{1}{2}}dt = \frac{3t^{\frac{3}{2}}}{\frac{3}{2}} + c = 2t^{\frac{3}{2}} + c$

$K(0) = 2(0)^{\frac{3}{2}} + c = 25 \quad \Rightarrow \quad c = 25 \qquad \therefore K(t) = 2t^{\frac{3}{2}} + 25$

$K(t)\mid_1^4 = K(4) - K(1) = 2(4)^{\frac{3}{2}} - 2(1)^{\frac{3}{2}} = 14$

(Application 2 : Consumer and Producer Surpluses)

CS $= \int_0^{Q^*} D(Q) - P^* \, \mathrm{d}Q = \int_{P^*}^{P \ intercept} D(Q) \, \mathrm{d}P$

PS $= \int_0^{Q^*} P^* - S(Q) \, \mathrm{d}Q = \int_{P \ intercept}^{P^*} S(Q) \, \mathrm{d}P$

ex. S(P) $= -\frac{1}{2} + \frac{1}{2}P$

D(P) $= \frac{25}{2} - \frac{1}{2}P$

Find PS , CS and total welfare

$\Rightarrow \quad Q^s = frac12 + \frac{1}{2}P \quad \Rightarrow \quad -\frac{1}{2} + \frac{1}{2}P^* = \frac{25}{2} - \frac{1}{2}P^* \quad \Rightarrow \quad P^* = 13$

$Q^d = -\frac{25}{2} - \frac{1}{2}P \qquad Q^* = 6$

or $CS \;=\; \int_{13}^{25} \frac{25}{2} - \frac{1}{2}P \, \mathrm{d}P \;=\; \frac{25}{2}P - \frac{P^2}{4} \,|_{13}^{25} \;=\; \frac{625}{4} - \frac{13}{2}(\frac{50}{2} - \frac{13}{2})$

$= \frac{625}{4} - \frac{481}{4} = \frac{144}{4} = 36$

$PS \;=\; \int_1^{13} -\frac{1}{2} + \frac{1}{2}P \, \mathrm{d}P = -\frac{1}{2}P + \frac{P^2}{4} \,|_1^{13} \;=\; -\frac{26}{4} - \frac{13^2}{4} - (-\frac{2}{4} + \frac{1}{4})$

$= \frac{143}{4} + \frac{1}{4} = \frac{144}{4} = 36$

Find inverse functions : $Q^s = -\frac{1}{2} + \frac{1}{2}P \quad \Rightarrow \quad P = 1 + 2Q^s$

$Q^d = -\frac{25}{2} - \frac{1}{2}P \quad \Rightarrow \quad P = 25 - 2Q^d$

CS $= \int_0^6 25 - 2Q - 13 \, \mathrm{d}Q \;=\; 12Q - \frac{2Q^2}{2} \,|_0^6 = 36$

PS $= \int_0^6 13 - (1 + 2Q) \, \mathrm{d}Q \;=\; 12Q - \frac{2Q^2}{2} \,|_0^6 = 36$

Total welfare = CS+PS = 36+36 = 72

ex. (cont'ed) if the demand changes to D(P) = $\frac{30}{2} - \frac{1}{2}P$, ΔCS and ΔPS?

$$\left.\begin{aligned} Q^d = 15 - \tfrac{1}{2}P \quad &\Rightarrow \quad P = 30 - 2Q^d \\ Q^s = -\tfrac{1}{2} + \tfrac{1}{2}P \quad &\Rightarrow \quad P = 1 + 2Q^s \end{aligned}\right\} P^* = 15.5 \ , \ Q^* = 7.25$$

$$\begin{aligned} CS' &= \int_0^{7.25} 30 - 2Q - 15.5 \ \mathrm{d}Q \\ &= 14.5Q - Q^2 \,|_0^{7.25} = 52.5625 \\ \therefore \Delta CS &= 52.5625 - 36 = 16.5625 \end{aligned}$$

$$\begin{aligned} PS' &= \int_0^{7.25} 15.5 - 12Q \ \mathrm{d}Q \\ &= 14.5Q - Q^2 \,|_0^{7.25} = 52.5625 \\ \therefore \Delta PS &= 52.5625 - 36 = 16.5625 \end{aligned}$$

Total Δ welfare = Δ PS + Δ CS = 33.125

ex. If the demand does not change and government imposes \$ 4 per unit tax on producer instead,

calculate total welfare loss.

$$\Rightarrow \quad P^s = P^d - 4$$

$$\left.\begin{aligned} Q^s &= -\tfrac{1}{2} + \tfrac{1}{2}(P - 4) \\ Q^d &= \tfrac{25}{2} - \tfrac{1}{2}P \end{aligned}\right\} P^* = 15 \ , \ Q^* = 5$$

$$CS'' = \int_0^5 (25 - 2Q) - 15 \, \mathrm{d}Q = 10Q - Q^2 \,|_0^5 = 25$$

$$PS'' = \int_0^5 11 - (1 + 2Q) \, \mathrm{d}Q = 10Q - Q^2 \,|_0^5 = 25$$

$$\left.\begin{aligned} \Delta CS = CS - CS'' = 36 - 25 = 11 \\ \Delta PS = PS - PS'' = 36 - 25 = 11 \end{aligned}\right\} \Delta\text{total welfare} = 22$$

T = 4 · 5 = 20

$\therefore$ DWL = 22 − 20 = 2

$$= \int_5^6 (25-2Q)-(1+2Q)\,\mathrm{d}Q = \int_5^6 24-4Q\,\mathrm{d}Q$$

$$= 24Q - 2Q^2 \mid_5^6 = \ 24(6-5)-2(6^2-5^2) = 2$$

(Application 3: First-Degree Price discrimination or Perfect Price discrimination)

Monopolist charges max. price for each unit of output sold.

<u>ex.</u> Suppose that a monopolist faces a demand function P = 24 -Q, and MC = 4 + 3Q

⇒ Find CS and PS at profit-max. Q

$$\begin{aligned}\Pi &= TR - TC \\ &= P\cdot Q - \int MC\,dQ \\ &= 24Q - Q^2 - \int 4+3Q\,dQ \\ &= 24Q - Q^2 - 4Q - \tfrac{3Q^2}{2} + c \\ \tfrac{d\Pi}{dQ} &= 24-2Q-4-3Q \quad \Rightarrow \quad Q^* = 4\end{aligned}$$

$$P^* = 20$$

$$PS = \int_0^4 20-(4+3Q)\,\mathrm{d}Q = 16Q - \tfrac{3Q^2}{2} \mid_0^4 = \ 40$$

$$CS = \int_0^4 (24-Q)-20\,\mathrm{d}Q = 4Q - \tfrac{Q^2}{2} \mid_0^4 = \ 8$$

⇒ Find PS

$$\begin{aligned}PS' &= \int_0^5 (24-Q)-(4+3Q)\,\mathrm{d}Q \\ &= \int_0^5 20-4Q\,\mathrm{d}Q \\ &= 20Q - \tfrac{4Q^2}{2} \mid_0^5 \\ &= 50\end{aligned}$$

$$\begin{aligned}24-Q &= 4+3Q \\ Q^* &= 5\end{aligned}$$

Reference

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