Solution: Exercise 2

Part I (*Derivative Definition, Power/Sum/Product/Quotient Rules*) .

1. Find the derivative $f'(x)$ of the following functions $f(x)$ by using the definition of the derivative:
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

(a) $f(x) = 2x^2 + x + 1$
Solution

$$\begin{aligned} f(x+h) - f(x) &= 2(x+h)^2 + (x+h) + 1 - (2x^2 + x + 1) \\ &= 2(x^2 + 2xh + h^2) + (x+h) + 1 - 2x^2 - x - 1 \\ &= 4xh + h^2 + h \end{aligned}$$

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4xh + h^2 + h}{h} = \lim_{h\to 0} 4x + h + 1 = 4x + 1.$$

(b) $f(x) = \cos(x)$
Hint: Use the identity: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, and the fact that $\lim\limits_{h\to 0} \frac{\sin(h)}{h} = 1, \lim\limits_{h\to 0} \frac{\cos(h)-1}{h} = 0$.
Solution

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$$

$$\begin{aligned} \frac{d}{dx}\cos(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h\to 0} \frac{[\cos(x)\cos(h) - \sin(x)\sin(h)] - \cos(x)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)\,[\cos(h) - 1] - \sin(x)\sin(h)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)\,[\cos(h)-1]}{h} - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h} \\ &= \cos(x) \underbrace{\lim_{h\to 0} \frac{[\cos(h)-1]}{h}}_{=0} - \sin(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{=1} \\ &= -\sin(x). \end{aligned}$$

2. Differentiate each of the following functions to find $f'(x)$.

(a) $f(x) = \pi^{15}$
Solution: $f'(x) = \frac{d}{dx}(\pi^{15}) = 0$.

(b) $f(x) = x^8 - \frac{1}{3x^3} + \sqrt{x^3} + \pi$

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}x^8 - \frac{1}{3}\frac{d}{dx}x^{-3} + \frac{d}{dx}x^{3/2} + \frac{d}{dx}\pi \\ &= 8x^7 - \frac{-3}{3}x^{-3-1} + x^{3/2-1} + 0 \\ &= 8x^7 + x^{-4} + x^{1/2}. \end{aligned}$$

(c) $f(x) = \left(x^2 + x + 1\right)\left(x^8 - x^4 + \pi\right)$
Solution: From the Product rule:

$$\begin{aligned} f'(x) &= \left(x^2+x+1\right)\underbrace{\frac{d}{dx}\left(x^8 - x^4 + \pi\right)}_{8x^7-4x^3} + \left(x^8 - x^4 + \pi\right)\underbrace{\frac{d}{dx}\left(x^2+x+1\right)}_{2x+1}. \\ &= \left(x^2+x+1\right)(8x^7 - 4x^3) + \left(x^8 - x^4 + \pi\right)(2x+1) \\ &= \left(8x^9 + 8x^8 + 8x^7 - 4x^5 - 4x^4 - 4x^3\right) + \left(2x^9 - 2x^5 + 2x\pi + x^8 - x^4 + \pi\right) \\ &= 10x^9 + 9x^8 + 8x^7 - 6x^5 - 5x^4 - 4x^3 + \pi. \end{aligned}$$

(d) $f(x) = \frac{x^8 - x^4 + \pi}{x^2 + x + 1}$
Solution: From the Quotient rule:

$$\begin{aligned} f'(x) &= \frac{\left(x^2+x+1\right)\frac{d}{dx}\left(x^8 - x^4 + \pi\right) - \left(x^8 - x^4 + \pi\right)\frac{d}{dx}\left(x^2+x+1\right)}{(x^2+x+1)^2}. \\ &= \frac{\left(x^2+x+1\right)(8x^7 - 4x^3) - \left(x^8 - x^4 + \pi\right)(2x+1)}{(x^2+x+1)^2}. \end{aligned}$$

3. Determine every point x on the graph $f(x) = \frac{1}{3}x^3 - 16x$ where the tangent line is horizontal.
Solution: The tangent line is horizontal when the slope is 0. Since $f'(x)$ is the slope at the point x, we therefore want to find the point x such that $f'(x) = 0$.

$$f'(x) = \frac{d}{dx}\left(\frac{1}{3}x^3 - 16x\right) = x^2 - 16$$

$$f'(x) = 0 \qquad \Longleftrightarrow \qquad x^2 - 16 = 0 \qquad \Longleftrightarrow \qquad x = \pm 4$$

That is, there are only two points on the graph with horizontal tangent and these points are $x = 4$ and $x = -4$.

Part II (*Derivatives of trigonometric functions, Chain Rule, Implicit Differentiation*) .

1. Find $f'(x)$ for the following functions.

(a) $f(x) = 3\sin(x) + \tan(x)$
Solution: $f'(x) = 3\frac{d}{dx}\sin(x) + \frac{d}{dx}\tan(x) = 3\cos(x) + \sec^2(x)$.

(b) $f(x) = \frac{\cos(x)\cot(x)}{x+\csc(x)}$
Solution: First note that, from the product rule:

$$\frac{d}{dx}(\cos(x)\cot(x)) = \cos(x)\frac{d}{dx}\cot(x) + \cot(x)\frac{d}{dx}\cos(x) = -\cos(x)\csc^2(x) - \cot(x)\sin(x).$$

By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{[x+\csc(x)]\frac{d}{dx}[\cos(x)\cot(x)] - \cos(x)\cot(x)\frac{d}{dx}[x+\csc(x)]}{[x+\csc(x)]^2} \\ &= \frac{[x+\csc(x)][-\cos(x)\csc^2(x) - \cot(x)\sin(x)] - \cos(x)\cot(x)[1-\cot(x)\csc(x)]}{[x+\csc(x)]^2}. \end{aligned}$$

2. Use the Chain Rule to find the derivatives $f'(x)$ of the functions below.

(a) $f(x) = (x^3-1)(x^2+1)^{100}$
Solution: By the product rule and chain rule:

$$\begin{aligned} f'(x) &= (x^3-1)\frac{d}{dx}(x^2+1)^{100} + \frac{d}{dx}(x^3-1)(x^2+1)^{100} \\ &= (x^3-1)\left[100(x^2+1)^{99}\frac{d}{dx}(x^2+1)\right] + 3x^2(x^2+1)^{100} \\ &= 100(x^3-1)(x^2+1)^{99}2x + 3x^2(x^2+1)^{100}. \end{aligned}$$

(b) $f(x) = \sin^3(\cos(\sqrt{x^3-2}))$
Solution: By the product rule and chain rule:

$$\begin{aligned} f'(x) &= 3\sin^2(\cos(\sqrt{x^3-2}))\frac{d}{dx}\sin(\cos(\sqrt{x^3-2})) \\ &= 3\sin^2(\cos(\sqrt{x^3-2}))\cos\left(\cos(\sqrt{x^3-2})\right)\frac{d}{dx}\cos(\sqrt{x^3-2}) \\ &= 3\sin^2(\cos(\sqrt{x^3-2}))\cos(\cos(\sqrt{x^3-2}))\left[-\sin(\sqrt{x^3-2})\frac{d}{dx}\sqrt{x^3-2}\right] \\ &= -3\sin^2(\cos(\sqrt{x^3-2}))\cos(\cos(\sqrt{x^3-2}))\sin(\sqrt{x^3-2})\frac{1}{2}(x^3-2)^{-1/2}\frac{d}{dx}(x^3-2) \\ &= -3\sin^2(\cos(\sqrt{x^3-2}))\cos(\cos(\sqrt{x^3-2}))\sin(\sqrt{x^3-2})\frac{1}{2}(x^3-2)^{-1/2}3x^2 \\ &= \frac{-9x^2\sin^2(\cos(\sqrt{x^3-2}))\cos(\cos(\sqrt{x^3-2}))\sin(\sqrt{x^3-2})}{2\sqrt{x^3-2}}. \end{aligned}$$

3. Use *implicit differentiation* to find $\frac{dy}{dx}$ for the graphs of the following equations.

(a) $y^3 - 2y = x + \sec(3y)$
Solution: We can find $\frac{dy}{dx}$ by (i) differentiating both sides of the equation and (ii) solving for $\frac{dy}{dx}$ in terms of x and y.

$$\begin{aligned}
\frac{d}{dx}\left(y^3 - 2y\right) &= \frac{d}{dx}\left(x + \sec(3y)\right) \\
3y^2\frac{dy}{dx} - 2\frac{dy}{dx} &= \frac{d}{dx}x + \sec(3y)\tan(3y)\frac{d}{dx}3y \\
3y^2\frac{dy}{dx} - 2\frac{dy}{dx} &= 1 + 3\sec(3y)\tan(3y)\frac{dy}{dx} \\
\left[3y^2 - 2 - 3\sec(3y)\tan(3y)\right]\frac{dy}{dx} &= 1
\end{aligned}$$

That is,

$$\frac{dy}{dx} = \frac{1}{3y^2 - 2 - 3\sec(3y)\tan(3y)}.$$

(b) $\frac{x+y}{x-y} = x^3$

Solution: We can find $\frac{dy}{dx}$ by (i) differentiating both sides of the equation and (ii) solving for $\frac{dy}{dx}$ in terms of x and y.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x+y}{x-y}\right) &= \frac{d}{dx}\left(x^3\right) \\
\frac{(x-y)\frac{d}{dx}(x+y) - (x+y)\frac{d}{dx}(x-y)}{(x-y)^2} &= 3x^2\frac{d}{dx}x \\
\frac{(x-y)\left(1 + \frac{dy}{dx}\right) - (x+y)\left(1 - \frac{dy}{dx}\right)}{(x-y)^2} &= 3x^2 \\
\frac{(x-y) + (x-y)\frac{dy}{dx} - (x+y) + (x+y)\frac{dy}{dx}}{(x-y)^2} &= 3x^2 \\
\frac{-2y + 2x\frac{dy}{dx}}{(x-y)^2} &= 3x^2 \\
-2y + 2x\frac{dy}{dx} &= 3x^2(x-y)^2 \\
\frac{dy}{dx} &= \frac{3x^2(x-y)^2 + 2y}{2x}
\end{aligned}$$

4. Find $\frac{dy}{dx}$ for $y = \sin(xy)$ at the point $(x, y) = (\pi/2, 1)$.

Solution: We can find $\frac{dy}{dx}$ by (i) differentiating both sides of the equation and (ii) solving for $\frac{dy}{dx}$ in terms of x and y. Then we can obtain the final answer by substituting $x = \pi/2$ and $y = 1$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sin(xy) \\
&= \cos(xy)\frac{d}{dx}(xy) \\
&= \cos(xy)\left[x\frac{d}{dx}y + y\frac{d}{dx}x\right] \\
&= \cos(xy)\left[x\frac{dy}{dx} + y\right] \\
[1 - x\cos(xy)]\frac{dy}{dx} &= y\cos(xy) \\
\frac{dy}{dx} &= \frac{y\cos(xy)}{1 - x\cos(xy)}
\end{aligned}$$

For $x = \pi/2$ and $y = 1$,

$$\left.\frac{dy}{dx}\right|_{(\pi/2,1)} = \frac{\cos(\pi/2)}{1 - (\pi/2)\cos(\pi/2)} = \frac{0}{1-0} = 0.$$

5. Find an equation of the tangent line to the graph of $\tan(y) = x$ at $y = \pi/4$.
   Solution: First note that when $y = \pi/4$, $x = \tan(\pi/4) = 1$. That is, we want to find a tangent line equation of this graph at $(x, y) = (1, \pi/4)$. By using *implicit differentiation*, the slope is $\frac{dy}{dx}$:

$$\frac{d}{dx}\tan(y) = \frac{d}{dx}x \qquad \Longrightarrow \qquad \sec^2(y)\frac{dy}{dx} = \frac{d}{dx}x \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{1}{\sec^2(y)}.$$

   The slope at $(x, y) = (1, \frac{\pi}{4})$ is

$$\left.\frac{dy}{dx}\right|_{(1,\frac{\pi}{4})} = \frac{1}{\sec^2(\frac{\pi}{4})} = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$$

   and an equation of the tangent line:

$$\frac{1}{2} = \frac{y - \frac{\pi}{4}}{x-1} \qquad \Longrightarrow \qquad y = \frac{\pi}{4} + \frac{1}{2}(x-1) \qquad \text{or} \qquad y = \frac{1}{2}x + (\frac{\pi}{4} - \frac{1}{2}).$$

6. Find an equation of the normal line to the graph of $x^4 + y^3 = 24$ at $(-2, 2)$.
   Solution: We first find the slope at $(-2, 2)$ by computing $\frac{dy}{dx}$ from *implicit differentiation*:

$$\begin{aligned}
\frac{d}{dx}\left(x^4 + y^3\right) &= \frac{d}{dx}24 \\
4x^3 + 3y^2\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-4x^3}{3y^2}.
\end{aligned}$$

At $(-2,2)$, the slope of the tangent line on this graph is

$$\left.\frac{dy}{dx}\right|_{(-2,2)} = \frac{-4(-2)^3}{32^2} = \frac{8}{3}.$$

and therefore the slope m_n of the normal line at $(-2,2)$ on this graph is

$$m_n = \frac{-1}{dy/dx|_{(-2,2)}} = \frac{-1}{8/3} = -3/8.$$

That is, the equation of the normal line at $(-2,2)$ is:

$$\frac{-3}{8} = \frac{y-2}{x-(-2)} \quad \Longrightarrow \quad y-2 = -\frac{3}{8}(x+2) \quad \Longrightarrow \quad y = -\frac{3}{8}x + \frac{5}{4}.$$

7. Find the point(s) on the graph of $x^2+y^2=8$ at which the slope of the tangent is 1.
Solution: First we find the slope of the graph or $\frac{dy}{dx}$ by using implicit differentiation:

$$\begin{aligned}\frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}8\\ 2x+2y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y}.\end{aligned}$$

The slope of the tangent line is 1 when

$$\frac{dy}{dx} = 1 \quad \Longrightarrow \quad -\frac{x}{y} = 1 \quad \Longrightarrow \quad y = -x.$$

To find every point (x,y) on the graph whose tangent lines have slope $= 1$, we substitute $y=-x$ into the equation of the graph $x^2+y^2=8$:

$$x^2+(-x)^2 = 8 \quad \Longrightarrow \quad 2x^2 = 8 \quad \Longrightarrow \quad x^2 = 4 \quad \Longrightarrow \quad x = \pm 2.$$

By using $y=-x$,

- when $x=2$, $y=-2$
- when $x=-2$, $y=-(-2)=2$.

That is, these points are $(x,y)=(2,-2)$ and $(-2,2)$.

8. Find the point(s) on the graph of $x^2-xy+y^2=3$ where the tangent line is horizontal.
Solution:

$$\begin{aligned}\frac{d}{dx}\left(x^2-xy+y^2\right) &= \frac{d}{dx}3\\ 2x-\left[x\frac{dy}{dx}+y\right]+2y\frac{dy}{dx} &= 0\\ \left[2y-x\right]\frac{dy}{dx} &= y-2x\\ \frac{dy}{dx} &= \frac{y-2x}{2y-x}.\end{aligned}$$

The tangent line is horizontal when the slope $\frac{dy}{dx} = 0$:

$$\frac{dy}{dx} = 0 \iff \frac{y-2x}{2y-x} = 0 \iff y - 2x = 0 \iff y = 2x.$$

To make sure that the points are on the graph, we will substitute $y = 2x$ and solve for x

$$x^2 - x(2x) + (2x)^2 = 3 \iff (1-2+4)x^2 = 3 \iff x^2 = 1 \iff x = \pm 1.$$

By using $y = 2x$,

- when $x = 1$, $y = 2(1) = 2$
- when $x = -1$, $y = 2(-1) = -2$.

That is, these points are $(x, y) = (1, 2)$ and $(-1, -2)$.

9. Find the point(s) on the graph of $x^2 - xy + y^2 = 27$ at which the tangent line is parallel to the the line $y = 1$.

Solution: Since the line $y = 1$ has slope 0,then to be parallel to this line $y = 1$ only when the tangent line also has slope 0. From implicit differentiation, we can find the slope $\frac{dy}{dx}$ from $\frac{d}{dx}\left(x^2 - xy + y^2\right) = \frac{d}{dx}3$ gives

$$2x - \left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{y-2x}{2y-x}.$$

The slope is zero only when

$$\frac{dy}{dx} = 0 \iff \frac{y-2x}{2y-x} = 0 \iff y - 2x = 0 \iff y = 2x.$$

To make sure that the points are on the graph, we will substitute $y = 2x$ and solve for x

$$x^2 - x(2x) + (2x)^2 = 27 \iff (1-2+4)x^2 = 27 \iff x^2 = 9 \iff x = \pm 3.$$

By using $y = 2x$,

- when $x = 3$, $y = 2(3) = 6$
- when $x = -3$, $y = 2(-3) = -6$.

That is, these points are $(x, y) = (3, 6)$ and $(-3, -6)$.

10. Find $\frac{d^2y}{dx^2}$ for

(a) $x^2 - y^2 = 25$

Solution: Using implicit differentiation gives

$$\begin{aligned} \frac{d}{dx}\left(x^2 - y^2\right) &= \frac{d}{dx}25 \\ 2x - 2y\frac{dy}{dx} &= 0 \implies \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}. \end{aligned}$$

To find the second derivative $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{1}{y^2}\left(y\frac{d}{dx}x - x\frac{dy}{dx}\right) = \frac{1}{y^2}\left(y - x\frac{x}{y}\right) = \frac{1}{y^2}\left(\frac{y^2-x^2}{y}\right) = -\frac{25}{y^3},$$

where we have used the fact that $x^2 - y^2 = 25$.

(b) $x + y = \sin(y)$.
Solution: Using implicit differentiation gives

$$\begin{aligned} \frac{d}{dx}(x+y) &= \frac{d}{dx}\sin(y) \\ 1 + \frac{dy}{dx} &= \cos(y)\frac{dy}{dx} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{1}{\cos(y)-1}. \end{aligned}$$

To find the second derivative $\frac{d^2y}{dx^2}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\cos(y)-1\right)^{-1} = -\left(\cos(y)-1\right)^{-2}\frac{d}{dx}\cos(y) \\ &= -\left(\cos(y)-1\right)^{-2}\left(-\sin(y)\frac{dy}{dx}\right) = \left(\cos(y)-1\right)^{-2}\left(\sin(y)\left(\cos(y)-1\right)^{-1}\right) \\ &= \frac{\sin(y)}{(\cos(y)-1)^3} = \frac{x+y}{(\cos(y)-1)^3} \end{aligned}$$

where we have used the fact that $x + y = \sin(y)$.